Mathematics G4402. Modern Geometry  
Assignment 5  
Due on Monday, October 19, 2009


(1) Define a smooth function $Q$ on $\mathbb{R}^{n+1}$ by
$$Q(x_0, x_1, \ldots, x_n) = -x_0^2 + x_1^2 + \cdots + x_n^2.$$ 
Define a smooth $(0,2)$ symmetric tensor $q$ on $\mathbb{R}^{n+1}$ by
$$q = -dx_0^2 + dx_1^2 + \cdots + dx_n^2.$$ 

(a) (Hyperbolic space) Note that $-1$ is a regular value of the smooth function $Q$, so $H^n = \{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid Q(x_0, \ldots, x_n) = -1, x_0 > 0 \}$ is an $n$-dimensional submanifold of $\mathbb{R}^{n+1}$. Let $i: H^n \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion map, and define $g = i^*q \in C^\infty(H^n, S^2T^*H^n)$. Show that $g$ is positive definite, so it is a Riemannian metric on $H^n$.

(b) (Poincaré disk) Show that $(x_0, \ldots, x_n) \mapsto \frac{1}{x_0 + 1}(x_1, \ldots, x_n)$ defines a diffeomorphism $f$ from $H^n$ onto the unit disk $D^n = \{ (y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_1^2 + \cdots + y_n^2 < 1 \}$. Show that $(f^{-1})^*g = \rho \sum_{i=1}^n dy_i^2$ for some smooth, positive function $\rho$ on $D^n$, and find $\rho(y_1, \ldots, y_n)$.

(2) Let $T^2$ be embedded in $\mathbb{R}^3$ as image of $\mathbb{R}^2$ by the map $\Phi$ defined by
$$\Phi(\theta, \phi) = ((a + b \cos \theta) \cos \phi, (a + b \cos \theta) \sin \phi, b \sin \theta)$$
where $a > b > 0$. Let $g$ the Riemannian metric induced on $T^2$ by the Euclidean metric of $\mathbb{R}^3$.

(a) Write $g$ in the form $g = E d\theta^2 + F(d\theta d\phi + d\phi d\theta) + Gd\phi^2$.

(b) Compute the volume of $(T^2, g)$.

(3) Show that any isometry of the Euclidean space $\mathbb{R}^n$ must take straight lines to straight lines. Show that the only isometries of $\mathbb{R}^n$ are those of the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ for constant $A \in O(n)$, $\mathbf{b} \in \mathbb{R}^n$.

(4) Let $G \times M \rightarrow M$ be a properly discontinuous action of a group $G$ on a smooth manifold $M$. Let $\varphi_g, g \in G$, and the smooth manifold $M/G$, be defined as in [dC] page 22 and 23, 4.8 Example.

(a) Prove that $M/G$ is orientable if and only if there exists an orientation on $M$ that is preserved by all $\varphi_g, g \in G$.

(b) Prove that $P^n(\mathbb{R})$ is orientable if and only if $n$ is odd.