Assignment 1

Due on Monday, September 21, 2009

In this assignment:

- smooth structure = $C^\infty$ differentiable structure
- smooth map = $C^\infty$ map

1. (Stereographic projection) Let $S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$ be equipped with the subset topology. Let $N = (0, \ldots, 0, 1)$ and $S = (0, \ldots, 0, -1)$ be the north and south poles of $S^n$, respectively.

Define $\pi_1 : S^n - \{N\} \to \mathbb{R}^n$ (resp. $\pi_2 : S^n - \{S\} \to \mathbb{R}^n$) such that $\pi_1(p), 0)$ (resp. $(\pi_2(p), 0)$) is the point at which the line through $N$ (resp. $S$) and $p$ intersects the hyperplane $\{x_{n+1} = 0\}$. (See do Carmo page 19 Example 4.6.)

(a) Prove that $\Phi = \{(S^n - \{N\}, \pi_1), (S^n - \{S\}, \pi_2)\}$ is a $C^\infty$ atlas on $S^n$.

(b) Prove that the inclusion map $i : S^n \to \mathbb{R}^{n+1}$ is a smooth embedding, where $S^n$ is equipped with the smooth structure defined by the $C^\infty$ atlas $\Phi$. Therefore the smooth structure defined by $\Phi$ coincides with the smooth structure on $S^n$ as a submanifold of $\mathbb{R}^{n+1}$.

2. (Embedding of $P^2(\mathbb{R})$ in $\mathbb{R}^4$) Let $F : \mathbb{R}^3 \to \mathbb{R}^4$ be given by $F(x, y, z) = (x^2 - y^2, xy, zx, yz)$.

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Observe that $f = F|_{S^2}$ satisfies $f(x, y, z) = f(-x, -y, -z)$, so that it descends to a map $\tilde{f} : P^2(\mathbb{R}) = S^2/\{\pm 1\} \to \mathbb{R}^4$. Prove that $\tilde{f}$ is a smooth embedding.

[Hint: A bijective continuous map from a compact topological space to a Hausdorff topological space is a homeomorphism.]

3. Let $(x, y, z)$ be coordinates on $\mathbb{R}^3$. Let $Y_r$ be the set of points in $\mathbb{R}^3$ at distance $r > 0$ from the circle $C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, z = 0\}$.

(a) Let $A = \{r \in (0, \infty) \mid Y_r$ is a smooth submanifold of $\mathbb{R}^3\}$. Find $A$.

(b) Let $S^1$ be equipped with the smooth structure in (1), and let $S^1 \times S^1$ be the product manifold (see do Carmo page 31-32 Exercise 1). Prove that $Y_r$ is $C^\infty$ diffeomorphic to $S^1 \times S^1$ for any $r \in A$.

4. Prove that the tangent bundle $TM$ of a smooth manifold $M$ is an orientable manifold (even though $M$ may not be).