Artin-Verdier duality

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This exposition of Artin-Verdier duality follows Mazur [2].

1 Cohomology of DVRs and $\operatorname{Spec}\mathcal{O}_K$

1.1 The localization exact sequence

Given $U \xhookrightarrow{j} X \xleftarrow{i} Z$ a decomposition of X into an open immersion j and a closed immersion i, we have the following exact sequence of sheaves on $X_{\text{ét}}$.

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0.$$

Indeed, one can check this on geometric stalks, dividing into cases whether the underlying point is in the image of U or Z. This allows us to decompose sheaves on $X_{\text{ét}}$ into sheaves on $U_{\text{ét}}$ and $Z_{\text{ét}}$.

Proposition 1.1. *The* $\mathrm{Sh}(X_{\acute{e}t})$ *is equivalent to the category of triples* $(\mathcal{G}, \mathcal{H}, \phi)$ *where* $\mathcal{G} \in \mathrm{Sh}(Y_{\acute{e}t}), \mathcal{H} \in \mathrm{Sh}(Z_{\acute{e}t})$*,* $\phi : \mathcal{G} \to i^* j_* \mathcal{H}$*, by associating*

$$\mathcal{F} \mapsto (\mathcal{F}_Z, \mathcal{F}_U, \phi).$$

Indeed, the quasi-inverse is given by taking the following fiber product.

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & j_*\mathcal{H} \\ \downarrow & & \downarrow \\ i_*\mathcal{G} & \xrightarrow[i_*\phi]{} & i_* i^* j_* \mathcal{H} \end{array}$$

Example 1.2. Let $(A, \mathfrak{m}, k)$ be a DVR with fraction field K where the residue field k is perfect. In fact, for simplicity, let us just take $A = \mathbb{Z}_p$. Then in this scenario we have

$$U = \operatorname{Spec}\mathbb{Q}_p \xhookrightarrow{j} \operatorname{Spec}\mathbb{Z}_p \xleftarrow{i} \operatorname{Spec}\mathbb{F}_p = Z.$$

Thus $\mathrm{Sh}(\operatorname{Spec}(\mathbb{Z}_p)_{\text{ét}})$ consists of a continuous $G_{\mathbb{F}_p}$-module M, a continuous $G_{\mathbb{Q}_p}$-module N, and some $G_{\mathbb{F}_p}$-morphism $\phi : M \to i^* j_* N = N^{I_p}$.

1.2 Cohomology of a DVR

We are mainly interested in the global case, so we will only briefly sketch the local results. In local Tate duality we need compactly supported cohomology, which we define here. Let A be a DVR. We define

$$H^r_c(\operatorname{Spec} A, \mathcal{F}) \coloneqq \operatorname{Ext}^r_{\mathrm{Sh}((\operatorname{Spec} A)_{\text{ét}})}(i_*\mathbb{Z}, \mathcal{F}).$$

Indeed, this is nothing but cohomology with support on a closed subset. Recall that if $Z \subset X$ is a closed immersion, then we can define cohomology with support on Z as the right derived functors of the functor

$$\Gamma_Z(X, -) = \ker(\Gamma(X, -) \to \Gamma(X \backslash Z, -)).$$

In our scenario, we are taking cohomology with support on the closed point $x : \operatorname{Spec} k \xrightarrow{i} \operatorname{Spec} A$. From the localization exact sequence we have

$$\operatorname{Hom}_A(i_*\mathbb{Z}, -) = \Gamma_x(\operatorname{Spec} A, -) \Rightarrow H^r_x(\operatorname{Spec} A, \mathcal{F}) = H^r_c(\operatorname{Spec} A, \mathcal{F}).$$

In our scenario though, we want to compute the cohomology of $\operatorname{Spec} A$ where A is a DVR and we have

$$U = \operatorname{Spec} K \overset{j}{\hookrightarrow} \operatorname{Spec} A \overset{i}{\leftarrow} \operatorname{Spec} k = Z.$$

We will assume that A is complete with perfect residue field, so that the results of local class field theory may be used.

Proposition 1.3. *There is an exact sequence of sheaves on* $(\operatorname{Spec} A)_{ét}$ *given by*

$$0 \to \mathbb{G}_m \to j_*\mathbb{G}_{m,K} \to i_*\mathbb{Z} \to 0.$$

This can be checked on stalks. Then one uses the long exact sequence in cohomology, along with Leray spectral sequences and local class field theory, to compute that

$$H^0_{\text{ét}}(\operatorname{Spec} A, \mathbb{G}_m) = A^*, \quad H^r_{\text{ét}}(\operatorname{Spec} A, \mathbb{G}_m) = 0 \text{ for } r > 0.$$

For the compactly supported version, one must make use of the adjoint pair $(i_*, i^!)$, where $i^!\mathcal{F}$ is the subsheaf of $\mathcal{F}$ with support in Z. Since i_* is exact, $i^!$ preserves injectives and we have the spectral sequence

$$\operatorname{Ext}^p_{S_k}(\mathbb{Z}, R^q i^! \mathbb{G}_m) \Rightarrow \operatorname{Ext}^{p+q}_{S_A}(i_*\mathbb{Z}, \mathbb{G}_m) = H^{p+q}_c(\operatorname{Spec} A, \mathbb{G}_m).$$

One shows that this collapses and again uses local class field theory to compute the following.

$$H^1_c((\operatorname{Spec} A)_{\text{ét}}, \mathbb{G}_m) = \mathbb{Z}, \quad H^3_c((\operatorname{Spec} A)_{\text{ét}}, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}, \quad H^r_c((\operatorname{Spec} A)_{\text{ét}}, \mathbb{G}_m) = 0 \text{ otherwise.}$$

1.3 Cohomology of $\operatorname{Spec} \mathcal{O}_K$

Let $X = \operatorname{Spec} \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a number field. For 2-torsion reasons, we will assume that K is totally imaginary.

Proposition 1.4. *There is an exact sequence of sheaves on* $(\operatorname{Spec} \mathcal{O}_K)_{ét}$ *given by*

$$0 \to \mathbb{G}_m \to j_*\mathbb{G}_{m,K} \to \bigoplus_{\mathfrak{p}} i_*\mathbb{Z} \to 0.$$

Proposition 1.5. *We have*

$$H^q(X, \mathbb{G}_m) = \begin{cases} \mathcal{O}_K^* & q = 0 \\ \operatorname{Pic}(X) & q = 1 \\ 0 & q = 2 \\ \mathbb{Q}/\mathbb{Z} & q = 3 \\ 0 & q > 3. \end{cases}$$

Proof. Taking the long exact sequence, we have So we have

$$\begin{aligned} 0 &\to \mathcal{O}_K^* \to K^* \to Div(X) \\ &\to \operatorname{Pic}(\mathcal{O}_K) \to H^1(X_{\text{ét}}, j_*\mathbb{G}_{m,K}) \to H^1(X_{\text{ét}}, \operatorname{Div}_X) \\ &\to H^2(X_{\text{ét}}, \mathbb{G}_m) \to H^2(X_{\text{ét}}, j_*\mathbb{G}_{m,K}) \to H^2(X_{\text{ét}}, \operatorname{Div}_X) \\ &\to H^3(X_{\text{ét}}, \mathbb{G}_m) \to H^3(X_{\text{ét}}, j_*\mathbb{G}_{m,K}) \to H^3(X_{\text{ét}}, \operatorname{Div}_X) \\ &\to H^4(X_{\text{ét}}, \mathbb{G}_m) \to \cdots \end{aligned}$$

Let us compute $H^r(X_{\text{ét}}, \mathrm{Div}_X)$. Well, this is just

$$H^r(X_{\text{ét}}, \coprod_{\mathfrak{p}} i_*\mathbb{Z}) = \bigoplus_{\mathfrak{p}} H^r((\operatorname{Spec}\mathbb{F}_q)_{\text{ét}}, \mathbb{Z}).$$

Thus we are reduced to computing the Galois cohomology $H^q(G_k, \mathbb{Z})$. We have $H^1(G_k, \mathbb{Z}) = \mathrm{Hom}_c(\widehat{\mathbb{Z}}, \mathbb{Z}) = 0$. For higher q, one can show that G_k has cohomological dimension 1 for torsion modules. Thus looking at the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0,$$

the only positive nonzero one occurs in dimension 2:

$$H^2(G_k, \mathbb{Z}) \cong H^1(G_k, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\widehat{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}.$$

Next, we again have a Leray spectral sequence associated to

$$\mathrm{Sh}(\operatorname{Spec} K) \xrightarrow{j_*} \mathrm{Sh}(\operatorname{Spec} \mathcal{O}_K) \xrightarrow{\Gamma} \mathbf{Ab}.$$

Since the higher direct images $R^q j_* \mathbb{G}_m$ vanish, the spectral sequence degenerates to give

$$H^r(X_{\text{ét}}, j_*\mathbb{G}_{m,K}) \cong H^n((\operatorname{Spec} K)_{\text{ét}}, \overline{K}^*) = H^n(G_K, \overline{K}^*).$$

Putting these together, we currently have the exact sequence

$$0 \to H^2(X_{\text{ét}}, \mathbb{G}_m) \to H^2(G_K, \overline{K}^*) \to \bigoplus_{\mathfrak{p}} \mathbb{Q}/\mathbb{Z} \to H^3(X_{\text{ét}}, \mathbb{G}_m) \to H^3(G_K, \overline{K}^*) \to 0$$

and the isomorphism

$$H^r(X_{\text{ét}}, \mathbb{G}_m) \cong H^r(G_K, \overline{K}^*)$$

for $r > 3$.

Now we use results from global class field theory:

$$0 \to H^2(G_K, \overline{K}^*) \to \bigoplus_{\mathfrak{p}} \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0, \quad H^r(G_K, \overline{K}^*) = 0 \text{ for } r \geq 3.$$

The maps to $\bigoplus_{\mathfrak{p}} \mathbb{Q}/\mathbb{Z}$ are the same, so at last we obtain

$$H^3(X_{\text{ét}}, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}, \quad H^r(X_{\text{ét}}, \mathbb{G}_m) = 0 \text{ for all other } r > 1.$$

□

2 Duality statements

2.1 Tate duality and Artin-Verdier duality

Proposition 2.1. *Let k be a finite field. Then setting $\tilde{M} = \mathrm{Hom}_c(M, \mathbb{Q}/\mathbb{Z})$, for finite M we have a perfect pairing*

$$H^r(G_k, M) \times H^{1-r}(G_k, \tilde{M}) \to \mathbb{Q}/\mathbb{Z}$$

Proof. For $r \neq 0, 1$ this follows form the fact that $G_k \cong \widehat{Z}$ has cohomological dimension 1 for finite modules. For $r = 0, 1$ this is just Pontryagin duality. □

Theorem 2.2 (Local Tate duality). *Let k be a local field and let $\tilde{M} = \operatorname{Hom}_c(M, \overline{k}^*$. Then for finite M of order prime to $\operatorname{char} k$, we have a perfect pairing given by the cup product:*

$$H^r(G_k, M) \times H^{2-r}(G_k, \tilde{M}) \to H^2(G_k, \overline{k}^*) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z}.$$

Note that the last isomorphism comes from local class field theory.

For convenience we recall the local results. Let $X = \operatorname{Spec} A$ be the spectrum of a complete DVR with field of fractions K and finite residue field k. Then compact cohomology is just cohomology supported at the closed point $x : \operatorname{Spec} k \to X$, and we have

$$H^r_c(X, \mathbb{G}_m) = \begin{cases} 0 & r = 0 \\ \mathbb{Z} & r = 1 \\ 0 & r = 2 \\ \mathbb{Q}/\mathbb{Z} & r = 3 \\ 0 & r > 3. \end{cases}$$

Moreover, $H^r_c(X, \mathbb{G}_m)$ was originally defined as $\operatorname{Ext}^r_X(i_*\mathbb{Z}, \mathcal{F})$, so we can feed it into a Yoneda pairing

$$\operatorname{Ext}^r_X(F, \mathbb{G}_m) \times H^{3-r}_x(X, \mathcal{F}) \to H^3_x(X, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}.$$

Theorem 2.3 (Local Artin-Verdier duality). *The Yoneda pairing defined above is a perfect pairing for all constructible sheaves $\mathcal{F}$.*

In fact, this proof follows from duality in the residue field and local Tate duality. Indeed, the localization exact sequence reduces duality of a sheaf $\mathcal{F}$ to sheaves of the form $i_*i^*\mathcal{F}$ and $j_!j^*\mathcal{F}$, which correspond to duality in residue fields and local fields respectively.

Now let $X = \operatorname{Spec} \mathcal{O}_K$ be the spectrum of the ring of integers of a totally imaginary number field.

Theorem 2.4 (Artin-Verdier duality). *The Yoneda pairing*

$$H^r(X, \mathcal{F}) \times \operatorname{Ext}^{3-r}_X(\mathcal{F}, \mathbb{G}_m) \to H^3(X, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$$

is perfect for all constructible sheaves $\mathcal{F}$.

2.2 The Yoneda product

We recall that the Yoneda product is a pairing on Ext groups in some abelian category:

$$\operatorname{Ext}^p(M, N) \otimes \operatorname{Ext}^q(L, M) \to \operatorname{Ext}^{p+q}(L, N).$$

The simplest definition is through viewing Ext groups as equivalence classes of extensions and splicing them together.

$$0 \to N \to E_1 \to \cdots \to E_p \to M$$
$$0 \to M \to F_1 \to \cdots \to F_q \to L$$
$$0 \to N \to E_1 \to \cdots \to E_p \to F_1 \to \cdots \to F_q \to L.$$

Alternatively, this is just composition in the derived category.

3 Proof of Artin-Verdier duality

A full proof is quite long, and can be found in [1].

Let $m^r(\mathcal{F}) : H^r(X, \mathcal{F}) \to \operatorname{Ext}_X^{3-r}(\mathcal{F}, \mathbb{G}_m)$ be the map in the theorem which we wish to show is an isomorphism. We will perform induction on r. Since the base cases are when $r < 0$, so we first need to know that $H^r(X, \mathcal{F})$ and $\operatorname{Ext}_X^r(\mathcal{F}, \mathbb{G}_m)$ vanish for $r > 3$. This is not trivial, but one can in fact show it for all open subschemes $U \subset X$ using a variety of cohomological methods.

The next step is to reduce to the case where $\mathcal{F} = \mathbb{Z}/p$ and $(\operatorname{char} X, p) = 1$. To do this, one first shows that $\mathcal{F}$ may be replaced by something of the form $j_!\mathbb{G}$ where $\mathbb{G}$ is a locally constant sheaf and j is some open immersion into X. Then one embeds $\mathcal{F}$ into the pushforward of some constant sheaf over some finite étale scheme over X. The usual arguments involving the long exact sequence and diagram chasing reduces us to the constant case as desired.

Next, the effaceability of $H^r(X, -)$ in the category of constructible sheaves implies that given the inductive hypothesis up through $r - 1$, then $m^r(\mathcal{F})$ is injective.

Remark. All the arguments so far are also used in a proof of the proper base change theorem.

By some group-theoretic arguments, we can show that $H^r(U, \mathcal{F})$ and $\operatorname{Ext}_U^r(\mathcal{F}, \mathbb{G}_m)$ are finite groups. Thus injectivity means that we just have to show that

$$|H^r(X, \mathbb{Z}/p)| \geq |\operatorname{Ext}_X^{3-r}(\mathbb{Z}/p, \mathbb{G}_m)|$$

for $0 \leq r \leq 3$. This is fairly straightforward for $r = 0, 1$. Indeed, we can calculate $\operatorname{Ext}_X^q(\mathbb{Z}/p, \mathbb{G}_m)$ through the long exact sequence associated to

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0.$$

Indeed, we have already computed $H^q(X, \mathbb{G}_m) = \operatorname{Ext}_X^q(\mathbb{Z}, \mathbb{G}_m)$. We obtain the following results.

$$\operatorname{Ext}_X^q(\mathbb{Z}/p, \mathbb{G}_m) = \begin{cases} \mu_p(\mathcal{O}_K) & q = 0 \\ ??? & q = 1 \\ \operatorname{Pic} X/p & q = 2 \\ \mathbb{Z}/p & q = 3 \\ 0 & q > 3. \end{cases}$$

For $r = 0$ we have $H^0(X, \mathbb{Z}/p) = \mathbb{Z}/p$, which is good. For $r = 1$ we have that $H^1(X, \mathbb{Z}/p) = \operatorname{Hom}_c(\operatorname{Pic} X, \mathbb{Z}/p)$ which has the same cardinality as $\operatorname{Pic} X/p$.

The other calculations are more difficult and we do not explain them here. We simply note that Mazur deduces them using the nondegeneracy of the Hilbert symbol, interpreted cohomologically.

References

[1] J.S. Milne, *Arithmetic Duality Theorems*.
https://www.jmilne.org/math/Books/ADTnot.pdf.

[2] B. Mazur, *Notes on the étale cohomology of number fields*.
http://www.numdam.org/item/10.24033/asens.1257.pdf