Sullivan notes

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0 Outline

1. Algebraic Constructions

   Localization, profinite completion, formal completion. Topological examples.

2. Homotopy Theoretical localization

   Localization of spaces: construction and properties. Postnikov towers. Lots of examples.

3. Completions in homotopy theory

   Profinite completion as a functor. Representability using Brown's axioms. Cohomology and homotopy of the profinite completion. Examples. S-profinite completion and formal completion. The arithmetic square for spaces.

4. Spherical fibrations

   Local spherical fibrations. Rational homotopy and stable homotopy.

5. Algebraic geometry

   Étale homotopy; both the Artin-Mazur approach and the Lubkin approach. Examples. Complete étale homotopy type. The Galois action. Adams operations and the Adams conjecture.

6. The Galois group in geometric topology

   Piecewise linear bundles and signature invariants. K-theory. Periodicity and Galois symmetry.

Notation: Sullivan's notation (e.g. letting l denote a finite set of primes) is truly unacceptable. Here, we let S be a finite set of primes. $\mathbb{Z}_{S^c}$ denotes the localization of $\mathbb{Z}$ at the primes outside of S. We use $\mathbb{Z}_p$ to denote the p-adic integers.

1 Algebraic Constructions

1.1 Equivariant cohomology

We recall that equivariant cohomology of a space X with a group action G is defined by $H^*(X \times EG/G)$, where $X \times EG$ is given the diagonal G-action. This is motivated by the fact that if the action of G is free, we want the equivariant cohomology to just be $H^*(X/G)$, but since it may not be, we can replace X with homotopy equivalent thing whose G-action is free. For example, $E(\mathbb{Z}/2) = S^\infty$, and $B(\mathbb{Z}/2) = \mathbb{RP}^\infty$.

Example 2: Let X be a locally compact polyhedron with a symmetry of order 2 (involution), T. The equivariant cohomology of (X, T) is given by

$$H^*(X_T; \mathbb{Z}/2) = H^*(X \times S^\infty/T; \mathbb{Z}/2).$$

If F denotes the fixed points and $R = \mathbb{Z}/2[x]$, S is the multiplicative set generated by x, then

$$H^*(F; R_x) \equiv H^*(X_T; \mathbb{Z}/2) \otimes_R R_x.$$

Let $R = H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[x]$. The natural map $(X \times S^\infty)/T \to S^\infty/T = \mathbb{RP}^\infty$ gives an R-module structure on $H^*(X_T; \mathbb{Z}/2)$.

This is, I believe, a special case of the famous localization theorem of Atiyah and Bott [3]. Usually we work with a torus instead; this would correspond (in the $n = 1$ case) to looking at $BS^1 = \mathbb{CP}^\infty$ rather than $B\mathbb{Z}/2 = \mathbb{RP}^\infty$. In general, say a torus T acts on X; Atiyah-Bott localization relates the equivariant cohomology of X to that of its fixed point set F. In particular, the kernel and cokernel of the map

$$i^* : H^*_T(X) \to H^*_T(F)$$

has torsion kernels and cokernels as $H^*_T(*)$-modules, which in the case of $T = S^1$ is $H^*(\mathbb{CP}^\infty, A) = A[x^2]$. In the case of $T = \mathbb{Z}/2$, this basically means that when we invert/localize by x, we get the described isomorphism.

1.2 Equivariant K-theory

We recall that maps to $BU(n) = \mathrm{Gr}(n, \infty)$ gives vector bundles of rank n, so that $K(X) = [X, \mathbb{Z} \times BU]$. There is also equivariant K-theory, denoted by $K_G(X)$, where we look at equivariant vector bundles instead. This is contrasted with the K-theory of the homotopy quotient $K(X \times_G EG/G)$, but there is a map $K_G(X) \to K(X//G)$. The Atiyah-Segal completion theorem is the corresponding localization theorem in this setting. https://ncatlab.org/nlab/show/Atiyah-Segal+completion+theoremv

Example 3: Let $R = \mathbb{Z}[x]/(x^2 - 1)$. Then $\hat{R} \cong \mathbb{Z} \oplus \mathbb{Z}_2$. We have $K(\mathbb{RP}^\infty) \equiv [\mathbb{RP}^\infty, \mathbb{Z} \times BU] \cong \hat{R}$.

Here, the group is $\mathbb{Z}/2$ again with $B\mathbb{Z}/2 = \mathbb{RP}^\infty$. The equivariant K-theory of a point is the representation ring of the group, which in the case of $G = \mathbb{Z}/2$ is $R = \mathbb{Z}[x]/(x^2 - 1)$. Now the Atiyah-Segal completion theorem says that $K(X//G)$ is the formal completion of $K_G(X)$ at the augmentation ideal of $K_G(*)$. This specializes to the statement above when X is taken to be a point. Moreover, if we complete with respect to $(x + 1)^j$, then this is related to the K-theory of the fixed points.

References

[1] BBD https://publications.ias.edu/sites/default/files/Faisceaux%20pervers.pdf

[2] Weights of exponential sums, intersection cohomology, Newton polyhedra

[3] THE MOMENT MAP AND EQUIVARIANT COHQMOLOGY M. F. ATIYAH and R. BOTT https://www.math.stonybrook.edu/~mmovshev/MAT570Spring2008/BOOKS/atiyahbott_moment.pdf