Ramification theory

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1 The Grothendieck-Ogg-Shafarevich formula

1.1 Setup

We begin by recalling some basic definitions from ramification theory. Let L/K be a finite Galois extension fields complete under a discrete valuation, with Galois group G, degree n, and residue fields of characteristic p. Then we have a chain

$$K \subset L_1 = L \cap K^{\mathrm{ur}} \subset L_2 = L \cap K^{\mathrm{tr}} \subset L.$$

Here $[L_1 : K] = [l : k] = f_{L/K}$ and $[L : L_1] = e_{L/K}$. Moreover, if π_K is a uniformizer for K, then $L_2 = L_1[\pi_K^{1/e}]$ for some $p \nmid e$. If $\mathcal{O}_L$ and $\mathcal{O}_K$ are the valuation rings, then we may write $\mathcal{O}_L = \mathcal{O}_K[z]$ for some $x \in \mathcal{O}_L$.

Definition 1.1. *The ramification groups $G_i \subset G$, for $i \geq 0$, are defined by*

$$G_i := \{\sigma \in G | \nu_L(\sigma(z) - z) > i\} = \{\sigma \in G | \frac{\sigma(z)}{z} \in 1 + \mathfrak{m}_L^{i+1})\}.$$

In particular, G_0 is the inertia group.

The setting we will be interested in is when Y is a curve over an algebraically closed field and G is a finite group of automorphisms of Y. For every point $y \in Y$, we use consider the decomposition group $G_y = \{\sigma \in G | \sigma(y) = y\}$. We can apply the setup to this group G_y (that is, what we called G before is now G_y, not the whole group G). Indeed, we have $G_y = \mathrm{Gal}(\hat{L}_y/\hat{K}_x)$, where these fields are the completions of the function fields L, K with respect to the valuations given by y, x.

1.2 The Artin and Swan characters

We return to the general theory where we call G the Galois group of L/K as above. Let r be the character of the regular representation of G and let u_i be the character of the augmentation representation of G_i. We define the Artin representation of G through its character a_G. Let $i_G(\sigma) = \nu_{\mathfrak{p}}(\sigma(z) - z)$.

Definition 1.2. *Define*

$$a_G(\sigma) = \begin{cases} -f \cdot i_G(\sigma) & \sigma \neq \mathrm{id} \\ f \sum_{\theta \neq \mathrm{id}} i_G(\theta) & \sigma = \mathrm{id}\,. \end{cases}$$

Example 1.3. Say L/K is unramified. Then G_0 is trivial and we have $i_G(\sigma) = 0$ for all $\sigma \in G$, so $a_G = 0$.

Theorem 1.4. *The character a_G is the character of a representation of G.*

To prove this, one first notes that we have

$$a_G = \sum_{i=0}^{\infty} \frac{u_i^*}{|G_0 : G_i|}.$$

This implies that $|G_0|a_G$ is the character of some representation. The other ingredients needed are Brauer's theorem, which states that every character of G can be written as a sum of induced characters of degree 1 of subgroups of G, and the Hasse-Arf theorem.

It is not hard to show that a_G is the induced character of a_{G_0}.

We can now globalize in two ways. The first is to number theory, in the AKLB setup. For each $\mathfrak{q}|\mathfrak{p}$ we have the Artin representation of $D_\mathfrak{q}$, which we can extend to the Artin representation of G by setting $a_\mathfrak{p} = \sum_{\mathfrak{q}|\mathfrak{p}} a_\mathfrak{q}$. For each character χ of G, we define the conductor of χ to be

$$f(\chi) = \prod_{\mathfrak{p}} \mathfrak{p}^{(\chi, a_\mathfrak{p})}.$$

Note that if $\mathfrak{p}$ is unramified, then $(\chi, a_\mathfrak{p}) = 0$. Thus conductors provide some measure of ramification. In fact, we have the Führerdiskriminantenproduktformel, which states

$$D_{L/K} = \prod_{\chi} f(\chi)^{\chi(1)}.$$

Alternatively, we can globalize to algebraic curves. This is what we'll be mainly interested in here. In this case, we have the map $q : Y \to X = Y/G$, where Y and X are curves over an algebraically closed field. That means there is no residue field extension, so $f = 1$. As before, we have an Artin representation a_y of G_y for each $y \in Y$. We extend a_y by 0 to all of G and set $a_x = \sum_{f(y)=x} a_y$ to obtain a character of a representation of G associated to a point $x \in X$.

We will also be interested in the Swan character $p_y : G_y \to \mathbb{Z}$, which is defined by

$$p_y = a_y - u_y = a_y - r_y + 1.$$

We claim that the Swan character is 0 if and only if $q : Y \to X$ is tamely ramified at y. Indeed, the character u_y satisfies $u_y(\sigma) = -1$ for $\sigma \neq \mathrm{id}$ and $u_y(\mathrm{id}) = |G| - 1$. In the case of tame ramification, we can write $\mathcal{O}_L = \mathcal{O}_K[\pi_K^{1/n}]$ where $p \nmid n$. Then we can take $z = \pi_K^{1/n}$, and $\sigma_i(z) = \omega_j^i z$. We conclude that $i_G(\sigma) = -1$ for $\sigma \neq \mathrm{id}$, from which we conclude that $a_y = u_y \Rightarrow p_y = 0$. The other direction is left as an exercise.

1.3 The formula

We recall the Riemann-Hurwitz formula, which says that for $f : Y \to X$ a finite separable morphism of curves of degree n, we have

$$2g_Y - 2 = n(2g_X - 2) + \deg R,$$

where R is the ramification divisor. R is also known as the discriminant, and we have $R = \sum_{x \in |X|} a_x(1)x$. When y is tamely ramified, we see that $a_y(1) = e_y - 1$. However, when there is wild ramification, this number is apparently larger (exercise?).

We begin with a simple case of the Grothendieck-Ogg-Shafarevich formula.

Proposition 1.5. *Given a finite group G of automorphisms of Y, set $X = Y/G$ and consider the action of G on $H^1(Y, \mathbb{Q}_l)$. Then the character ψ of G acting on $H^1(Y, \mathbb{Q}_l)$ is given by the formula*

$$\psi(\sigma) = 2 + (2g_X - 2)r(\sigma) + \sum_{x \in |X|} a_x(\sigma).$$

Proof. For $\sigma = \mathrm{id}$, this is literally Riemann-Hurwitz. For $\sigma \neq \mathrm{id}$, this formula states that

$$2 - \psi(\sigma) = \sum_{x \in |X|} a_x(\sigma).$$

The LHS is just $\sum_{i=0}^{2} (-1)^i \operatorname{Tr}(\sigma | H^i(Y, \mathbb{Q}_l))$. Indeed, the for $i = 0$ the trace is 1, and for $i = 2$ it is multiplication by the degree which must also be 1 as σ is an automorphism. Thus the Grothendieck-Lefschetz trace formula tells us that the LHS is equal to $-|\Gamma_\sigma \cdot \Delta|$. But $i_y(\sigma)$ is precisely the multiplicity of (y, y) in $\Gamma_\sigma \cdot \Delta$, as desired. $\square$

The Grothendieck-Ogg-Shafarevich formula extends this to constructible sheaves $\mathcal{F}$ of $\mathbb{F}_l$ modules. We take k to be algebraically closed of characteristic p. Let P_y be the unique projective $\mathbb{Z}_l[G_y]$-module whose character is p_y. We define the exponent of the wild conductor of $\mathcal{F}$ at x to be

$$\alpha_x(\mathcal{F}) = \dim_{\mathbb{F}_l}(\operatorname{Hom}_{G_y}(P_y, \mathcal{F}_{\overline{\eta}})$$

for any y mapping to x. The exponent of the conductor of $\mathcal{F}$ at x is defined to be

$$c_x(\mathcal{F}) = \dim \mathcal{F}_{\overline{\eta}} - \dim \mathcal{F}_x + \alpha_x(\mathcal{F}).$$

Theorem 1.6 (Grothendieck-Ogg-Shafarevich)**.** *For any constructible sheaf $\mathcal{F}$ of $\mathbb{F}_l$-modules on X, we have*

$$\chi(X, \mathcal{F}) = (2 - 2g) \dim \mathcal{F}_{\overline{\eta}} - \sum_{x \in |X|} c_x(\mathcal{F}).$$

Remark. In the tame case, we don't need the wild conductor term and this was due to Ogg and Shafarevich.

Here is another formulation that works for U a non-empty open subset of X.

$$\chi_c(U, \mathcal{F}) = \operatorname{rank}(\mathcal{F}) \cdot \chi_c(U, \overline{\mathbb{Q}}_l) - \sum_{x \in X \setminus U} \operatorname{Sw}_x(\mathcal{F}).$$

See https://www.math.arizona.edu/~swc/aws/2012/2012MiedaSaitoProjectDescription.pdf.

2 Ideas

1. Formula for character of G acting on $H^1(X, \mathbb{F})$ when it does indeed act on it?
2. About the action of G on $H^1(Y, \mathbb{Q}_l)$, can you get it through the outer action from
$$1 \to \pi_1(Y) \to \pi_1(X) \to G \to 1?$$
3. Can you get a formula describing π_1, or the (abelianized?) outer action through local terms?
4. GCFT: Given $X \to X_{\mathfrak{m}}$, can you describe either formula (character of G acting, Euler characteristic of $\mathcal{F}$)? Can you describe the theory in terms of the Jacobian or get results for ramified covers?
5. What if you have an algebraic group acting on Y?

3 Neron-Ogg-Shafarevich

References

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