Ramification theory

Caleb Ji

1 The Grothendieck-Ogg-Shafarevich formula

1.1 Setup

We begin by recalling some basic definitions from ramification theory. Let \( L/K \) be a finite Galois extension fields complete under a discrete valuation, with Galois group \( G \), degree \( n \), and residue fields of characteristic \( p \). Then we have a chain

\[
K \subset L_1 = L \cap K^{ur} \subset L_2 = L \cap K^{ur} \subset L.
\]

Here \( [L_1 : K] = [l : k] = f_{L/K} \) and \( [L : L_1] = e_{L/K} \). Moreover, if \( \pi_K \) is a uniformizer for \( K \), then \( L_2 = L_1[\pi_K^{1/e}] \) for some \( p \nmid e \). If \( \mathcal{O}_L \) and \( \mathcal{O}_K \) are the valuation rings, then we may write \( \mathcal{O}_L = \mathcal{O}_K[z] \) for some \( x \in \mathcal{O}_L \).

Definition 1.1. The ramification groups \( G_i \subset G \), for \( i \geq 0 \), are defined by

\[
G_i := \left\{ \sigma \in G \, | \, \nu_p(\sigma(z) - z) > i \right\} = \left\{ \sigma \in G \left| \frac{\sigma(z)}{z} \in 1 + m_i^{i+1} \right. \right\}.
\]

In particular, \( G_0 \) is the inertia group.

The setting we will be interested in is when \( Y \) is a curve over an algebraically closed field and \( G \) is a finite group of automorphisms of \( Y \). For every point \( y \in Y \), we use consider the decomposition group \( G_y = \{ \sigma \in G | \sigma(y) = y \} \). We can apply the setup to this group \( G_y \) (that is, what we called \( G \) before is now \( G_y \), not the whole group \( G \)). Indeed, we have \( G_y = \text{Gal}(\hat{L}_y/\hat{K}_x) \), where these fields are the completions of the function fields \( L, K \) with respect to the valuations given by \( y, x \).

1.2 The Artin and Swan characters

We return to the general theory where we call \( G \) the Galois group of \( L/K \) as above. Let \( r \) be the character of the regular representation of \( G \) and let \( u_i \) be the character of the augmentation representation of \( G_i \). We define the Artin representation of \( G \) through its character \( a_G \). Let \( i_G(\sigma) = \nu_p(\sigma(z) - z) \).

Definition 1.2. Define

\[
a_G(\sigma) = \begin{cases} -f \cdot i_G(\sigma) & \sigma \neq \text{id} \\ f \sum_{\theta \neq \text{id}} i_G(\theta) & \sigma = \text{id}. \end{cases}
\]

Example 1.3. Say \( L/K \) is unramified. Then \( G_0 \) is trivial and we have \( i_G(\sigma) = 0 \) for all \( \sigma \in G \), so \( a_G = 0 \).

Theorem 1.4. The character \( a_G \) is the character of a representation of \( G \).
To prove this, one first notes that we have

\[ a_G = \sum_{i=0}^{\infty} \frac{u_i^*}{|G_0 : G_i|} . \]

This implies that \(|G_0|a_G| is the character of some representation. The other ingredients needed are Brauer's theorem, which states that every character of \(G\) can be written as a sum of induced characters of degree 1 of subgroups of \(G\), and the Hasse-Arf theorem.

It is not hard to show that \(a_G\) is the induced character of \(aG_0\).

We can now globalize in two ways. The first is to number theory, in the AKLB setup. For each \(q\mid p\) we have the Artin representation of \(D_q\), which we can extend to the Artin representation of \(G\) by setting \(a_p = \sum_{q\mid p} a_q\). For each character \(\chi\) of \(G\), we define the conductor of \(\chi\) to be

\[ f(\chi) = \prod_p p^{(\chi, a_p)}. \]

Note that if \(p\) is unramified, then \((\chi, a_p) = 0\). Thus conductors provide some measure of ramification. In fact, we have the Führerdiskriminantenproduktformel, which states

\[ D_{L/K} = \prod \chi f(\chi)^{-1}. \]

Alternatively, we can globalize to algebraic curves. This is what we’ll be mainly interested in here. In this case, we have the map \(q : Y \to X = Y/G\), where \(Y\) and \(X\) are curves over an algebraically closed field. That means there is no residue field extension, so \(f = 1\). As before, we have an Artin representation \(a_y\) of \(G_y\) for each \(y \in Y\). We extend \(a_y\) by 0 to all of \(G\) and set \(a_x = \sum_{f(y) = x} a_y\) to obtain a character of a representation of \(G\) associated to a point \(x \in X\).

We will also be interested in the Swan character \(p_y : G_y \to \mathbb{Z}\), which is defined by

\[ p_y = a_y - u_y = a_y - r_y + 1. \]

We claim that the Swan character is 0 if and only if \(q : Y \to X\) is tamely ramified at \(y\). Indeed, the character \(u_y\) satisfies \(u_y(\sigma) = -1\) for \(\sigma \neq \text{id}\) and \(u_y(\text{id}) = |G| - 1\). In the case of tame ramification, we can write \(O_L = O_K[\pi^{1/n}]\) where \(p \nmid n\). Then we can take \(z = \pi^{1/n}\), and \(\sigma(z) = \omega^z\). We conclude that \(i_G(\sigma) = -1\) for \(\sigma \neq \text{id}\), from which we conclude that \(a_y = u_y \Rightarrow p_y = 0\). The other direction is left as an exercise.

1.3 The formula

We recall the Riemann-Hurwitz formula, which says that for \(f : Y \to X\) a finite separable morphism of curves of degree \(n\), we have

\[ 2g_Y - 2 = n(2g_X - 2) + \deg R, \]

where \(R\) is the ramification divisor. \(R\) is also known as the discriminant, and we have \(R = \sum_{x \in X} a_x(1)\). When \(y\) is tamely ramified, we see that \(a_y(1) = c_y - 1\). However, when there is wild ramification, this number is apparently larger (exercise?).

We begin with a simple case of the Grothendieck-Ogg-Shafarevich formula.
Proposition 1.5. Given a finite group $G$ of automorphisms of $Y$, set $X = Y/G$ and consider the action of $G$ on $H^1(Y, \mathbb{Q}_l)$. Then the character $\psi$ of $G$ acting on $H^1(Y, \mathbb{Q}_l)$ is given by the formula

$$\psi(\sigma) = 2 + (2g - 2)r(\sigma) + \sum_{x \in |X|} a_x(\sigma).$$

Proof. For $\sigma = \text{id}$, this is literally Riemann-Hurwitz. For $\sigma \neq \text{id}$, this formula states that

$$2 - \psi(\sigma) = \sum_{x \in |X|} a_x(\sigma).$$

The LHS is just $\sum_{i=0}^2 (-1)^i \text{Tr}(\sigma|H^i(Y, \mathbb{Q}_l))$. Indeed, the for $i = 0$ the trace is 1, and for $i = 2$ it is multiplication by the degree which must also be 1 as $\sigma$ is an automorphism. Thus the Grothendieck-Lefschetz trace formula tells us that the LHS is equal to $-|\Gamma_\sigma \cdot \Delta|$. But $i_y(\sigma)$ is precisely the multiplicity of $(y, y)$ in $\Gamma_\sigma \cdot \Delta$, as desired.

The Grothendieck-Ogg-Shafarevich formula extends this to constructible sheaves $\mathcal{F}$ of $\mathbb{F}_l$ modules. We take $k$ to be algebraically closed of characteristic $p$. Let $P_y$ be the unique projective $\mathbb{Z}_l[G_y]$-module whose character is $p_y$. We define the exponent of the wild conductor of $\mathcal{F}$ at $x$ to be

$$\alpha_x(\mathcal{F}) = \dim_{\mathbb{F}_l} \text{Hom}_{G_y}(P_y, \mathcal{F})$$

for any $y$ mapping to $x$. The exponent of the conductor of $\mathcal{F}$ at $x$ is defined to be

$$c_x(\mathcal{F}) = \dim \mathcal{F}_x - \dim \mathcal{F} + \alpha_x(\mathcal{F}).$$

Theorem 1.6 (Grothendieck-Ogg-Shafarevich). For any constructible sheaf $\mathcal{F}$ of $\mathbb{F}_l$-modules on $X$, we have

$$\chi(X, \mathcal{F}) = (2 - 2g) \dim \mathcal{F} - \sum_{x \in |X|} c_x(\mathcal{F}).$$

Remark. In the tame case, we don’t need the wild conductor term and this was due to Ogg and Shafarevich.

Here is another formulation that works for $U$ a non-empty open subset of $X$.

$$\chi_c(U, \mathcal{F}) = \text{rank}(\mathcal{F}) \cdot \chi_c(U, \mathbb{Q}_l) - \sum_{x \in X \setminus U} \text{Sw}_x(\mathcal{F}).$$

See https://www.math.arizona.edu/~swc/aws/2012/2012MiedaSaitoProjectDescription.pdf.

2 Ideas

1. Formula for character of $G$ acting on $H^1(X, \mathbb{F})$ when it does indeed act on it?

2. About the action of $G$ on $H^1(Y, \mathbb{Q}_l)$, can you get it through the outer action from

$$1 \to \pi_1(Y) \to \pi_1(X) \to G \to 1?$$

3. Can you get a formula describing $\pi_1$, or the (abelianized?) outer action through local terms?

4. GCFT: Given $X \to X_m$, can you describe either formula (character of $G$ acting, Euler characteristic of $\mathcal{F}$)? Can you describe the theory in terms of the Jacobian or get results for ramified covers?

5. What if you have an algebraic group acting on $Y$?
3 Neron-Ogg-Shafarevich

References

http://www.numdam.org/item/10.24033/asens.1257.pdf


http://www.numdam.org/item/10.24033/asens.1257.pdf