Generalized Jacobians I

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We explain some of the statements of ‘classical’ geometric class field theory, à la Rosenlicht, Lang, and Serre. Proofs are generally not given; they seem hard for various reasons. I would be very pleased if someone more competent could give a more comprehensive account, with proofs, of this material in the language of schemes (note that [6] is written in Weil’s language).

1 Background from algebraic geometry

The exposition in this section is based on [3].

1.1 General results on the Picard scheme

Let $f : X \to S$ be a separated morphism of finite type. We write $X_T = X \times_S T$ and have a projection $q : X_T \to T$.

Definition 1.1. *The Picard functor* $\mathrm{Pic}_{X/S}$ *is defined by*

$$\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X_T)/q^* \mathrm{Pic}(T).$$

In nice cases this is representable, in which case it is a sheaf in the Zariski, étale, and fppf topologies. Note that if we did not quotient out by $q^* \mathrm{Pic}(T)$, it would not be a separated presheaf because under some covering a nontrivial element would become 0; thus it would not be representable. In any case, we have by taking its sheafification in various topologies we get a sequence of functors

$$\mathrm{Pic}_{X/S}(-) \to \mathrm{Pic}_{X/S\ (\mathrm{Zar})}(-) \to \mathrm{Pic}_{X/S\ (\text{ét})}(-) \to \mathrm{Pic}_{X/S\ (\mathrm{fppf})}(-).$$

If any of these are representable, then we call the representing scheme $\mathbf{Pic}_{X/S}$. There can only be one such scheme; not that if $\mathrm{Pic}_{X/S}$ is representable then they are all the same, while say if $\mathrm{Pic}_{X/S\ (\text{ét})}(-)$ is, $\mathrm{Pic}_{X/S}(-)$ may not be. It turns out that the sheafified versions, e.g. $\mathrm{Pic}_{X/S\ (\text{ét})}(-)$, are the real functors we should try to represent. However, we have the following result.

Theorem 1.2 ([3], 9.2.5)**.** *Assume $\mathcal{O}_S \cong f_*\mathcal{O}_X$ universally. Then all three maps above are injections, and if f has a section, they are all isomorphisms.*

In particular, if $S = \operatorname{Spec} k$ and X has a rational point, then if $\mathbf{Pic}_{X/S}$ exists, it represents $\mathrm{Pic}_{X/S}$. However, for the needs of arithmetic geometry it is important to not assume that this is the case. The next theorem is a key result of Grothendieck.

Theorem 1.3 ([3], 9.4.8)**.** *Assume $f : X \to S$ is projective Zariski locally over S, and is flat with integral geometric fibers. Then $\mathbf{Pic}_{X/S}$ exists, is separated and locally of finite type over S, and represents $\mathrm{Pic}_{X/S\ (\text{ét})}$.*

Let us now restrict to the scenario of the above theorem. In the next section we will sketch (sketchily) the construction in the case of a relative curve. We will also need to discuss the components of $\mathbf{Pic}_{X/S}$. Let $S = \operatorname{Spec} k$ and let $\mathbf{Pic}^0_{X/k}$ be the connected component of the identity.

Theorem 1.4 ([3], 9.5.4). *Let X be projective and geometrically integral. Then $\textbf{Pic}^0_{X/k}$ exists and is quasi-projective. Of X is geometrically normal, then $\textbf{Pic}^0_{X/k}$ is projective.*

We note that this theorem uses Chevalley's structure theorem, which we will soon discuss.

Recall that the Hilbert polynomial ϕ of a coherent sheaf $\mathcal{F}$ on X/S projective, flat with integral geometric fibers is given by $\phi(n) = \chi(\mathcal{F}(n))$.

Theorem 1.5 ([3], 9.6.20). *With the conditions listed above, the $\textbf{Pic}^{\phi}_{X/S}$ are open and closed subschemes of finite type whose formation commute with changing S. They are disjoint and they cover $\textbf{Pic}_{X/S}$.*

In the case of a curve X/k, we have that $\textbf{Pic}_{X/k}/\textbf{Pic}^0_{X/k} \cong \mathbb{Z}$, given by the degree of the line bundle. We use $\textbf{Pic}^r_{X/k}$ to denote the component corresponding to line bundles of degree r. Recall that the degree of the line bundle is given by the formula $\chi(X, \mathcal{L}^n) = n\deg(\mathcal{L}) + 1 - g$. Note that if $\mathcal{L}$ is very ample, then the Hilbert polynomial of $\mathcal{L}$ is thus given by $(\deg L)n + \deg L + 1 - g$. Thus in previous notation we would be using $\textbf{Pic}^{rx+r+1-g}_{X/k}$. (I think!!) Furthermore, each $\textbf{Pic}^r_{Y/k}$ is a principal homogeneous space for $\textbf{Pic}^0_{X/k}$. We will be interested in the case $r = 1$.

In what follows, we will focus on the original $\mathrm{Pic}_{X/S}(-)$ functor and also use $\mathrm{Pic}_{X/S}$ for the representing scheme, rather than $\textbf{Pic}_{X/S}$.

1.2 Chevalley's structure theorem

Theorem 1.6 (Chevalley). *Let G be an algebraic group over a perfect field k (connected smooth group scheme). Then there is a unique short exact sequence of algebraic groups*

$$1 \to H \to G \to A \to 1$$

with H linear algebraic and A an abelian variety. Moreover, the formation of H commutes with base change to an arbitrary perfect field extension over k.

There are expositions of the proof by Conrad [1] and Milne [4].

Moreover, we can decompose in the linear algebraic group H by taking a maximal torus T; then H/T is unipotent. If the unipotent part is 0, then we say G is a semi-abelian variety. If we are thinking about Néron models, then having no affine part corresponds to having good reduction while having no unipotent part corresponds to having semi-stable reduction. This paradigm will be used when we compare the Jacobian of a singular (a generalized Jacobian) to that of its normalization (which is an abelian variety).

2 Construction of generalized Jacobians

2.1 Generalized Jacobians

We follow Serre's book [6].

Let X be a smooth projective curve over a field k. Let S be a set of closed points of X and let $\mathfrak{m} = \sum_P n_P$ be a modulus supported on S (i.e. a finite sum of points in S) with $\deg \mathfrak{m} > 1$. The Picard group of X may be seen as the divisor group of X modulo $\operatorname{div} g$ for $g \in K(X)$. Now consider the construction of the singular curve $X_{\mathfrak{m}}$. Its closed points are given by $X - S \cup \{Q\}$ and the structure sheaf is given by setting the stalk at Q to be

$$\mathcal{O}_{X_{\mathfrak{m}},Q} = k + \{f \equiv 0 \pmod{\mathfrak{m}}\}.$$

Here, $f \equiv 0 \pmod{\mathfrak{m}}$ means $\nu_P(f) \geq n_P$. Then $X_{\mathfrak{m}}$ is constructed so that it has $X \to X_{\mathfrak{m}}$ as its normalization and its Picard group consists of divisors of X up to $\mathfrak{m}$-equivalence; i.e. $D \sim_{\mathfrak{m}} D'$ if $D - D' = \operatorname{div} g$ with $g \equiv 1 \pmod{\mathfrak{m}}$.

Let us now give a few examples. If $\mathfrak{m} = P + Q$ for distinct $P, Q \in X(k)$, then $X_{\mathfrak{m}}$ has a k-rational nodal singularity. If $\mathfrak{m} = P$ with $\deg_k(P) = 2$, then $X_{\mathfrak{m}}$ still has nodal singularity, but it is essentially non-split multiplicative reduction. If $\mathfrak{m} = 2P$ for $P \in X(k)$, then $X_{\mathfrak{m}}$ has a cusp. Working over an algebraically closed field, the nodes give the tori in the Jacobians, while the more complicated singularities give the unipotent part.

By Hartshorne Ex. II.6.9,, if $\tilde{X} \to X$ is a normalization of curves, then there is an exact sequence

$$0 \to \bigoplus_{P \in X} \widetilde{\mathcal{O}_P}^* / \mathcal{O}_P^* \to \operatorname{Pic} X \to \operatorname{Pic} \tilde{X} \to 0.$$

Then for instance, if we take the nodal cubic $\mathbb{P}^1 \to y^2 z = x^3 + x^2 z$, we see that we obtain $\mathbb{G}_m$, whereas for the cuspidal cubic $\mathbb{P}^1 \to y^2 z = x^3$, we obtain $\mathbb{G}_a$.

2.2 Picard of a relative curve

Here we follow Milne [5].

Let $f : X \to S$ be a projective flat morphism whose fibers are integral curves. We are interested in the representability of the following functor.

$$P^r_{\mathcal{C}}(T) = \{\mathcal{L} \in \operatorname{Pic}(\mathcal{C} \times_S T) | \deg(\mathcal{L}_t) = r\} / f_T^* \operatorname{Pic}(T)$$

We will in fact show that for $r > 2g - 2$, this is representable. This will also represent the sheafified version, which will also give the $r = 0$ version if X has a rational point. The construction proceeds in several steps.

1. If $\mathcal{E}$ is a finite vector bundle on S, then let $\operatorname{Grass}^{\mathcal{E}}_n(T) = \{(V, h)\}/\sim$ where V is a rank n vector bundle on T with an epimorphism $h : \mathcal{O}_T \otimes_k \mathcal{E} \twoheadrightarrow V$. Then $\operatorname{Grass}^{\mathcal{E}}_n(-)$ is representable by a projective variety $\operatorname{Gr}^{\mathcal{E}}_n$ over S.

2. Let $\operatorname{Div}^r_{X/S}(T)$ be the set of relative effective Cartier divisors on $C \times T/T$ of degree r (so they are flat over T). In the case of a smooth curve over a field, it is represented by the symmetric power of the curve. Assume there is a section $s : S \to X$ and use it to identify Div^r_X with a closed subscheme of $\operatorname{Gr}^{q_* \mathcal{O}_X(m)}_r$.

3. The generalized Abel-Jacobi map $\operatorname{Div}^r_{X/S} \to P^r_{X/S}$ is defined by sending a divisor to its corresponding line bundle. The fibers of these are projective space bundles, and one show that the quotient by these is again representable.

2.3 Birational group approach

This is the approach found in Serre's book [6]. It is hard to read because it is written in Weil's language. We will just say the following. In the smooth case, one shows that the Jacobian J is the unique abelian variety birationally equivalent to $X^{(g)}$ (the symmetric power). Using Riemann-Roch, one creates a rational group law on $X^{(g)}$, and from this there is a unique birational homomorphism $X^{(g)} \to J$ to some group variety J which is the Jacobian.

For generalized Jacobians, we use the arithmetic genus $\pi = g + \deg \mathfrak{m} - 1$ to construct the generalized Jacobian $J_\mathfrak{m}$ of $X_\mathfrak{m}$. We endow $X^{(\pi)}$ with a rational group law in a similar way and proceed as in the smooth case.

3 Properties of generalized Jacobians

The statements of theorems in this section (without proof), may be found in [2].

3.1 Albanese property

Let $x_0 \in X(k)$ be a closed rational point of a smooth projective variety X/k. Recall that the Albanese variety of (X, x_0) is a pair consisting of an abelian variety over k and a morphism $\alpha_X : X \to \mathrm{Alb}(X)$ satisfying the following universal property. Fix a closed point $x_0 \in X$. Then $\alpha_X(x_0) = 0$ and every morphism $f : X \to T$ to an abelian variety T satisfying $f(x_0) = 0$ factors uniquely through α_X and a morphism of abelian varieties $g : \mathrm{Alb}(X) \to T$.

In general, the Albanese variety is the dual of the 0-component of the Picard variety $\mathrm{Pic}^0(X)$. In the case X is a curve, we have $\mathrm{Pic}^0(X) = J(X)$. But $J(X)$ also turns out to be the Albanese variety; in other words, $J(X)$ is self-dual. One can prove this using the characterization of J as being birational to $X^{(g)}$. From this we get a desired rational map from $J(X)$ to T, but since rational maps from smooth varieties to abelian varieties may be extended to regular ones, we are done.

Let us now explain the corresponding statement for generalized Jacobians. Instead of just abelian varieties, we will consider maps to commutative group schemes. Let $f : X - S \to G$ be a morphism. We say that f admits a modulus $\mathfrak{m}$ if $\mathfrak{m}$ has support S and for all $g \equiv 1 \pmod{\mathfrak{m}}$, we have $f(\div(g)) = 0$.

Proposition 3.1. *Every such f admits some modulus $\mathfrak{m}$.*

In the case of k being algebraically closed, we can use $J_\mathfrak{m}$ directly.

Theorem 3.2. *Assume k is algebraically closed (so there is a rational point.) Then there is a morphism $f_\mathfrak{m} : X - S \to J_\mathfrak{m}$ such that for each morphism to a commutative algebraic group $f : X - S \to G$ admitting the modulus $\mathfrak{m}$, then there is a unique morphism θ such that $f = \theta \circ f_\mathfrak{m}$.*

$$\begin{tikzcd} X - S \arrow[r, "f_\mathfrak{m}"] \arrow[dr, "f"'] & J_\mathfrak{m} \arrow[d, "\theta"] \\ & G \end{tikzcd}$$

As alluded to already, the correct thing to work with is $J^1_\mathfrak{m}$. When X has a rational point, we can use it to translate $J_\mathfrak{m}$ into $J^1_\mathfrak{m}$. Note that we have a canonical map

$$f_\mathfrak{m} : X - S \to J^1_\mathfrak{m}$$

defined by sending $x \mapsto \mathcal{O}(x)$ for each S-point x. Here, $\mathcal{O}(x)$ is the inverse ideal sheaf of the section given by x of $X_S \to S$.

Theorem 3.3. *Let X be a smooth projective and geometrically connected curve over a perfect field k, and let $S \subseteq X$ be a finite set of closed points. Let G be a smooth connected commutative k-group, and let $f : X - S \to G^1$ be a map to a principal homogeneous space for G.*

Then there exists a modulus $\mathfrak{m}$ with support in S and a k-morphism $\theta^1 : J^1_{\mathfrak{m}} \to G^1$ equivariant to a k-group map $\theta : J_{\mathfrak{m}} \to G$ such that $f = \theta^1 \circ f_{\mathfrak{m}}$.

$$\begin{tikzcd} X - S \arrow[r, "f_{\mathfrak{m}}"] \arrow[dr, "f"'] & J_{\mathfrak{m}} \arrow[d, "\theta^1"] \\ & G^1 \end{tikzcd}$$

3.2 Abelian coverings

Let us recall how it works for the unramified case. For now say there is a k-rational point $x \in X(k)$, which gives a map $f_x : X \to J$. Say we are interested in finite abelian Galois covers of X. We can obtain them through taking coverings of J – which are all abelian – and pulling them back to X. If M is a finite abelian group, then M-coverings of X are given by surjective morphisms $\mathrm{Hom}(\pi_1(X, 0), M)$. Thus to show we get all abelian coverings of C this way, we need to show that

$$f^x_* : \mathrm{Hom}(\pi_1(J, 0), M) \to \mathrm{Hom}(\pi_1(C, x), M)$$

is an isomorphism for all finite abelian groups M. Since we can factor coverings by subgroups, we just need to show the result for $M = \mathbb{Z}/n\mathbb{Z}$. We divide into two cases: $(n, \mathrm{char}\, k) = 1$ and $n = p^i$ where $\mathrm{char}\, k = p$.

In the first case, we use the Kummer sequence

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{\bullet^n} \mathbb{G}_m \to 1.$$

The long exact sequence gives an exact sequence

$$1 \to \mathcal{O}_X(X)^*/\mathcal{O}_X(X)^{*n} \to H^1_{\text{ét}}(X, \mu_n) \to \mathrm{Pic}(X)_n \to 1$$

where we use the fact that $H^1(X, \mathbb{G}_m) = \mathrm{Pic}(X)$ and $H^2(X, \mu_n) = 1$. If X is projective over $k = k^{sep}$, then $\mathcal{O}_X(X)^*/\mathcal{O}_X(X)^{*n} = k^*/k^{*n} = 1$ and we have an isomorphism $H^1_{\text{ét}}(X, \mu_n) \to \mathrm{Pic}(X)_n$.

we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(\pi_1(J), \mathbb{Z}/n\mathbb{Z}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}(\pi_1(C, x), \mathbb{Z}/n\mathbb{Z}) \arrow[d, "\cong"] \\ \mathrm{Pic}(J)_n \arrow[r] & \mathrm{Pic}(C)_n \end{tikzcd}$$

Here the horizontal maps are both obtained functorially from the map $f^x : C \to J$. As J is an abelian variety, it is its own Albanese. J is also the Albanese of C. We thus conclude that the bottom arrow is an isomorphism, as desired. For $n = p^k$ we may use the Artin-Schreier sequence and Witt vectors.

Let us now give the statement that works for ramified covers as well.

Theorem 3.4. *Let X be a smooth projective and geometrically connected curve over a perfect field k. Let $\pi : X' \to X$ be a finite covering map of smooth connected curves with ramification locus $S \subseteq X$. Then there exists a modulus $\mathfrak{m}$ with support S and a connected finite abelian covering $G^1 \to J^1_{\mathfrak{m}}$ with the following Cartesian diagram.*

$$\begin{tikzcd} X' - \pi^{-1}(S) \arrow[r] \arrow[d, "\pi"'] & G^1 \arrow[d] \\ X - S \arrow[r] & J^1_{\mathfrak{m}} \end{tikzcd}$$

In Serre's book [6], it seems that the way this is proven is as follows. First, one shows a finite Galois covering with group N can be realized as the pullback of a certain isogeny $G(N) \to G(N)/N$. These $G(N)$, apparently the "bilinear groups" of Elie Cartan, are constructed through Kummer theory and Artin-Schreier theory in the cyclic case. Then one uses the Albanese property of the generalized Jacobians $J_{\mathfrak{m}}$ to construct the desired isogeny. Moreover, the isogeny can be shown to be unique by studying the group $\mathrm{Ext}^1(J_{\mathfrak{m}}, N)$.

We note that in Deligne's Arcata lectures, he claims that Serre's book [6] gives an isomorphism $\mathrm{Hom}((J_{\mathfrak{m}})_n, \mathbb{Z}/n) \to H^1_{\text{ét}}(X, \mathbb{Z}/n)$. This is consistent with our calculation in the unramified case, and would seem useful to showing we get the coverings we desire. However, I could not find this statement in [6]. It seems that one can at least get something close to this from a ramified version of the Kummer sequence.

4 Structure of generalized Jacobians

(Incomplete)

We can compare $J_{\mathfrak{m}}$ to $J_{\mathfrak{m}'}$ where $\mathfrak{m} \geq \mathfrak{m}'$, and in doing so apparently get a projective limit that might be called the generalized Jacobian (I will try to think about this more.) But just in the basic case for $\mathfrak{m}' = 0$, let us consider the structure of the affine part H of the generalized Jacobian, given by

$$1 \to H \to J_{\mathfrak{m}} \to J \to 1.$$

This kernel H consists of those divisors linearly equivalent to 0, so they must be of the form $\mathrm{div}(h)$ with $h \equiv 1 \pmod{\mathfrak{m}}$. This should look something like $R_{\mathfrak{m}}/\mathbb{G}_m$, where $\mathbb{G}_m$ consists of nonzero constants which give rise to the trivial divisor. The group $R_{\mathfrak{m}}$ should look like

$$R_{\mathfrak{m}} = \prod_{P \in S} U_P/U_P^{n_P}.$$

Each of these components should be a (semidirect?) product of $\mathbb{G}_m$ and some unipotent part; the unipotent part admits a composition sequence of factors isomorphic to $\mathbb{G}_a$.

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