Faltings-Lawrence-Venkatesh

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These are the notes for the seminar on the Lawrence-Venkatesh proof of Faltings's theorem, whose website can be found here: http://www.math.columbia.edu/~calebji/faltings.html. The paper of Lawrence and Venkatesh can be found here: https://arxiv.org/abs/1807.02721. The talks for chapter 3 were given by Ivan Zelich, and the notes for it were taken by Amal Mattoo.

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Chapter 1

Introduction

Let C be a smooth projective curve defined over a number field K of genus g. The nature of the set of rational points $C(K)$ depends heavily on g. As in many other scenarios, we have a trichotomy corresponding to the cases $g = 0, g = 1$, and $g \geqslant 2$. Let us give an overview of what occurs in each case.

1.1 The Hasse principle ($g = 0$)

In the case $g = 0$, the anticanonical bundle has degree 2. Since $2 \geqslant 2g + 1 = 1$, the anticanonical bundle is very ample and gives a closed embedding $C \to \mathbb{P}^2_K$ of degree 2. Thus C is isomorphic to a conic. Now there are two possibilities: either $C(K) = \varnothing$ or there exists some $P \in C(K)$. In the first case there is nothing more to say regarding $C(K)$; in the second we may project from P onto some copy of $\mathbb{P}^1_K$ not going through P. This map gives an isomorphism of C onto $\mathbb{P}^1_K$. Alternatively, the point P defines a line bundle $\mathcal{L}(P)$ of degree 1. By Riemann-Roch, $h^0(C, \mathcal{L}(P)) = 2$ and thus the two sections define a closed embedding of C into $\mathbb{P}^1_K$, which must be an isomorphism.

We conclude that either C has no rational points or has infinitely many. The Hasse principle gives a criterion for determining which of these cases C satisfies. It states that a quadratic form over a number field K has a solution in K if and only if it has a solution over all completions K_v with respect to all places (including the infinite ones). Since C is isomorphic to a conic, we have the following result.

Theorem 1.1.1. *Let $g = 0$. Then if C has a solution over all completions K_v, then C is isomorphic to $\mathbb{P}^1_K$ and has infinitely many rational points. Otherwise, C is isomorphic to a conic in $\mathbb{P}^2_K$ and has no rational points.*

1.2 The Mordell-Weil theorem ($g = 1$)

If $g = 1$, then C is an elliptic curve which we denote as E. The points of E form an abelian group; one way to see this is by viewing its points as a complex torus, another way is through theory of divisors. Furthermore, the sum of two rational points is rational, so $E(K)$ is an abelian group. Using Galois cohomology and some classical algebraic number theory, one proves the weak Mordell-Weil theorem, which states that $E(K)/nE(K)$ is finite for each n. Then by the theory of heights, we arrive at the following result.

Theorem 1.2.1 (Mordell-Weil theorem)**.** *Let E/K be an elliptic curve. Then $E(K)$ is a finitely generated abelian group.*

In fact, this result holds for all abelian varieties (and this is the full statement of the theorem). The Hasse principle does not hold for cubic forms. Though every elliptic curve has a rational point by definition, there may be curves of genus 1 with rational points at every completion K_v but no global rational point. In fact, one may reformulate the Hasse principle in terms of Galois cohomology and show that the obstruction to its truth is described by the Tate-Shafarevich group $Ш(E/K)$. Indeed, we define

$$Ш(E/K) := \ker(H^1(G_K, E) \to H^1(G_{K_v}, E).$$

Here, $H^1(G_K, E)$ classifies torsors over E, which may be interpreted as curves of genus 1 which are isomorphic to E over $\overline{K}$. Having a rational point is equivalent to being 0 in this cohomology class. Interpreting $H^1(G_{K_v}, E)$ similarly, we see that if C represents a nontrivial element in $Ш(E/K)$, then it has rational points in each K_v) but no rational point in K.

We may write $E(K) \cong \mathbb{Z}^r \oplus G$, where G is some finite abelian group. Both the torsion and the rank of $E(K)$ are of enormous interest. They are described by the famous theorem of Mazur and the Birch and Swinnerton-Dyer conjecture.

Theorem 1.2.2 (Mazur, Merel). *Let E/K be an elliptic curve. Then the torsion part of $E(K)$ is $\mathbb{Z}/n\mathbb{Z}$ with $1 \leqslant n \leqslant 10$ or $n = 12$, or it is $\mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with $1 \leqslant n \leqslant 4$.*

Conjecture 1.2.3 (Birch and Swinnerton-Dyer). *Let E/K be an elliptic curve. Then the rank of $E(K)$ is given by the order of the pole of the Hasse-Weil L-function $L(E, s)$ at $s = 1$.*

1.3 Faltings's theorem ($g \geqslant 2$)

The primary goal of this seminar is to understand the Lawrence-Venkatesh proof of the following theorem, previously nown as Mordell's conjecture.

Theorem 1.3.1 (Faltings's theorem). *Let C/K be a smooth projective curve of genus $g \geqslant 2$. Then $C(K)$ is finite.*

We will now sketch Faltings's original proof, which may be found in [1]. We list the main steps. First, let A/K be an abelian variety over a number field, let $G_K = \mathrm{Gal}(\overline{K}/K)$, and let $V_l(A) = T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ be the Tate modules of A for some prime l.

1. (Finiteness I) There are finitely many abelian varieties B which are isogenous to A.
2. (Tate conjecture I) a) The representation of G_K on $V_l(A)$ is semisimple.

   b) The natural map $\mathrm{Hom}_K(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}_l \to \mathrm{Hom}_{G_K}(T_l(A), T_l(B))$ is an isomorphism.
3. (Shafarevich conjecture for AV) Let S be a finite set of places of K and fix a positive integer g. Then there are only finitely many isomorphism classes of abelian varieties A/K of dimension g with good reduction outside S.
4. (Shafarevich conjecture) With the notation above, there are only finitely many isomorphism classes of smooth projective curves C of genus g with good reduction outside S.
5. (Mordell's conjecture) If $g \geqslant 2$, then $C(K)$ is finite.

We will now say something about each step, obviously not trying to give full details.

1. Actually, Faltings originally proved something slightly different from (Finiteness I). At any rate, this is one of the most technically difficult parts of the proof, and involves an intensive analysis of heights. Briefly, we canonically embed the Siegel moduli space of abelian varieties $\mathbf{A}_g$ into $\mathbb{P}^n$; then the height on $\mathbb{P}^n$ gives a height function H defined on $\mathbf{A}_g$. Then Faltings defined a different height function, now known as the Faltings height. First, we take a Neron model $A' \to \operatorname{Spec} \mathcal{O}_K$. Then $\omega_{A'/\mathcal{O}_K}$ is a *metrized line bundle* on $\operatorname{Spec} \mathcal{O}_K$, and we may define

$$h(A) = \frac{1}{[K:\mathbb{Q}]} \deg(\omega_{A'/\mathcal{O}_K}).$$

Now the point of this is to compare these two definitions of height and show that they are not too different. There are serious technical difficulties that arise when analyzing them near the boundary of the moduli space. Then Faltings analyzes the behavior of the Faltings height under isogeny, showing it varies in a controlled way. Then since H and h are not too different and we have finiteness theorems for h, he is able to deduce the finiteness of isogeny classes.

1. $\Rightarrow$ 2. Next, Faltings proved Tate conjecture I using a similar argument to Tate's own proof of this conjecture for abelian varieties over finite fields. Indeed, Tate proved the same statement for finite fields using the fact that there are only finitely many isomorphism classes of abelian varieties of dimension g over $\mathbb{F}_q$. This fact may be replaced by Finiteness I. The injectivity of part b) is not too difficult. Moreover, we may assume $A = B$ by using this statement on $A \times B$. Now both statements are proved together in the following way.

First, one shows that all finite subgroups of $A(\overline{K})$ stable under G_K arise as the kernels of isogenies $A \to B$. Then every G_K-stable $\mathbb{Z}_l$-submodule of $T_l B$ arises as the image of some isogeny $A \to B$. Now consider any G_K-invariant subspace $W \subset V_l(A)$. We claim there is some $u \in \operatorname{End}(A) \otimes \mathbb{Q}_l$ such that $uV_l A = W$. Indeed, we apply the previous correspondence between representations and isogenies to the $\mathbb{Z}_l$-submodules

$$(T_l(A) \cap W) + l^n T_l(A)$$

for each n. Now using Finiteness I along with a compactness argument, we obtain the desired u. This allows us to construct complementary subspaces to prove semisimplicity. Finally, semisimplicity applied to a suitable graph construction yields

$$\operatorname{End}(A) \otimes \mathbb{Q}_l \cong \operatorname{End}_{G_K}(V_l A),$$

which gives b).

2. $\Rightarrow$ 3. By Finiteness I, we only need to show finiteness up to isogeny. Recall that the Néron-Ogg-Shafarevich criterion says that if v is a finite place of K not dividing some prime l, then A has good reduction at v if and only if the representation of G_K on $V_l A$ is unramified at v. This implies that isogenous abelian varieties over K have good reduction at the same finite places.

Now if A has good reduction over v, let $A(v)/K(v)$ be the corresponding abelian variety and let $P_v(A,t) := P(A(v),t)$ be the characteristic polynomial. We claim that if $P_v(A,t) = P_v(B,t)$ for all v in a certain finite set T, then the corresponding G_K-representations $V_l A, V_l B$ are isomorphic. This can be proven using some classical algebraic number theory involving Hermite-Minkowski finiteness and the Chebotarev density theorem.

Now by the Tate conjecture, we have that if $V_l A$ and $V_l B$ are isomorphic, then A and B are isogenous. The final step is thus to show that there are finitely many possible polynomials $P_v(A, t)$ where $v \in T$. But $P_v(A, t)$ is a monic polynomial of degree $2g$ whose roots are the eigenvalues of the Frobenius. These are bounded by the Weil conjectures, so the result follows.

3. ⇒ 4. If we have an abelian variety over a field, there are only finitely many isomorphism classes of principal polarizations on it. Furthermore, if C has good reduction at a prime v, then so does its Jacobian. Then using 3) we may apply the Torelli theorem, which for $g \geqslant 2$ tells us that two curves are isomorphic if their principally polarized Jacobians are.

4. ⇒ 5. (This was proved by Parshin [**?**]) The key is the construction of the so-called **Kodaira-Parshin family**, an abelian scheme over C with good ramification properties. To be precise, if S is a finite set of primes containing the ones dividing 2, we can find a finite extension L/K and curves C_P for each $P \in C(K)$ satisfying the following properties. The genus of C_P is bounded, C_P has good reduction outside the places dividing S, and there are finite maps $\phi_P : C_P \to C$ over L ramified at exactly P. Thus every rational point P gives a pair (C_P, ϕ_P), and by Shafarevich's theorem the C_P fall into finitely many isomorphism classes.

   Next, we use de Franchis's theorem, which states that if C' and C/k are fields and $g_C \geqslant 2$, then there are only finitely many nonconstant maps $C' \to C$. In particular, there are only finitely many possibilities for ϕ_P corresponding to a fixed isomorphism class C_P. The Mordell conjecture follows.

1.4 The Chabauty-Kim approach

Recall that the Mordell-Weil theorem implies that $J(K)$ is finitely generated, where J is the Jacobian of C. Chabauty proved the Mordell conjecture in the case that the rank of $J(K)$ is less than g. The ideas is the following. Take $P_0 \in C(K)$; this gives an embedding

$$\phi : C \to J \qquad \phi(P) = [P - P_0],$$

so $C(K) = C \cap J(K)$. Now embed K into L, some finite extension of $\mathbb{Q}_p$. The logarithm map gives a local isomorphism between $U \subset J(L)$ and $\mathrm{Lie}(J) \cong \mathcal{O}_L^g$. Let $\overline{\Gamma}$ be the closure of $J(K)$ in $J(L)$. Since $J(L)$ is compact, if the intersection is infinite then we get a convergent sequence of points in the intersection to one of them, which we may assume to be 0. Note that $J(K) \cap U$ is free of rank less than g. Changing coordinates, we get a function x_1 on the curve with infinitely many zeroes accumulating at 0; thus $x_1 = 0$ in a neighborhood of 0 on C. But dx_1 has at most $2g - 2$ zeroes, contradiction.

Minhyong Kim generalized this method by using deeper quotients of the fundamental group. Indeed, T_l may be viewed as the first étale cohomology group of C, which is (more or less) the abelianization of the étale fundamental group. Roughly, one finds a middle ground between the étale cohomology and torsor given by the étale fundamental group as described by the section conjecture, which the rational points of C are mapped to. One then analyzes a p-adic period map, which as we will see is also done in the Lawrence-Venkatesh approach. Kim has made significant progress through this approach, though a complete of Faltings's theorem in this way has not yet appeared.

1.5 Outline of the Lawrence-Venkatesh approach

The method we will be studying arose from Brian Lawrence's PhD thesis under Akshay Venkatesh. It gives a full proof of the Mordell conjecture and can also be applied to give results for higher dimensional varieties. As the authors say, it uses the setup of Faltings's proof but is close in spirit to the methods of Chabauty and Kim.

Let Y/K be a curve of genus $g \geqslant 2$. Actually, we will eventually want to run this argument for other Y. The starting point is a smooth projective family $X \to Y$ based on the Kodaira-Parshin family used in the last step of Faltings's proof. This family satisfies certain desired properties we will now describe.

Let $\mathcal{O}$ be the S-integers of K and say $X \to Y$ extends to $\pi : \mathcal{X} \to \mathcal{Y}$ over $\mathcal{O}$. For every p unramified in K and not dividing any prime in S, every $y \in \mathcal{Y}(\mathcal{O})$ gives a Galois representation

$$\rho_y : G_K \circlearrowright H^*_{et}(\overline{X_y}, \mathbb{Q}_p).$$

Now recall that in the step $2 \to 3$ of Faltings's proof, it was (essentially) proven that there are only finitely many possibilities for semisimple representations ρ_y^{ss} that are unramified outside a finite set of primes that are moreover these kinds of Galois representations on étale cohomology. As a reminder, classical algebraic number theory results such as Hermite-Minkowski finiteness and the Chebotarev density theorem are used to show that the representation is determined by its characteristic polynomial for a finite subset of Frobenius elements. By the Weil conjectures, there are only finitely many such polynomials that come from these representations.

Now Faltings worked with abelian varieties where he showed that every ρ_y is semisimple and determines X_y up to isogeny – this is Tate's conjecture. In the approach we are now considering, we take the semisimplification ρ^{ss} and restrict it to G_{K_v} for a suitable place v. Then it is proven that the fibers of this mapping from $Y(K)$ to these p-adic representations are not Zariski dense.

To prove this last statement, we use p-adic Hodge theory. Using this theory, each point $y \in Y(K_v)$ gives a filtered ϕ-module over K_v:

$$y \mapsto (H_{\mathrm{dR}}(X_y/K_v), \mathrm{Fil}^\bullet, \phi).$$

The Gauss-Manin connection allows us identify $H_{\mathrm{dR}}(X_z/K_v) \cong H_{\mathrm{dR}}(X_y/K_v)$ as we vary z in a residue disk in $Y(K_v)$ around y. What changes is the Hodge filtration. This variation is described by the p-adic period map, which sends points in the residue disk to K_v-points of a certain flag variety of subspaces of $H_{\mathrm{dR}}(X_y/K_v)$. The p-adic period map is injective, but there may be different filtrations which give rise to the isomorphic filtered ϕ-modules. Thus, we must also show that the image of the period map has finite intersection with an orbit on the period domain of the centralizer $Z(\phi)$. For example, when Y is a curve we will show that the $Z(\phi)$-orbit of the filtrations is a proper subvariety of the ambient flag variety, and that the image of the p-adic period map is Zariski dense. Then the fiber is given by the intersection points which are the zeroes of a nonvanishing K_v-analytic function in a residue disc, which is finite.

To check Zariski density, one passes to the corresponding complex period map which satisfies the same differential equation coming from the Gauss-Manin connection. Checking the result here is done through analyzing monodromy representations and mapping class groups. Finally, for higher dimensions Bakker and Tsimerman used o-minimality to prove the Ax-Schanuel theorem for period mappings. This gives us better control about the intersection of the $Z(\phi)$-orbit and Y_v.

Part I

Background

Chapter 2

The Gauss-Manin connection and the period map

2.1 Local systems, monodromy, and connections

Our goal is to relate isomorphism classes of complex local systems with representations of the fundamental group of a topological space B, and in the case that B is a complex manifold, to holomorphic vector bundles over B equipped with a flat connection. In this section, we follow [3] and [2].

2.1.1 The monodromy representation

Let B be a topological space. We assume it is nice; e.g. path-connected, locally path-connected and locally simply connected, so it has a nice universal cover.

Definition 2.1.1. *A complex local system on B is a locally free sheaf on B whose fibers are complex vector spaces and transition functions are linear.*

Remark. It is important to clearly distinguish the concepts of being locally constant (local systems) and being locally trivial (vector bundles).

Example 2.1.2. *Consider the sheaf of holomorphic global solutions to a homogeneous system of n linear first order differential equations with holomorphic coefficients on an open subset $U \subset \mathbb{C}$. They form a local system!*

Every (henceforth assumed complex and finite-dimensional) local system H with fiber V on $[0,1]$ is constant. This follows from the fact that $[0,1]$ is compact, so every element of a fiber H_t extends uniquely to a global section by continuing it along the intersections of the trivializations. Furthermore, we see that this construction gives an isomorphism $H_0 \cong H_1$ along any path on a general space B. If two paths γ_1, γ_2 are homotopic, then by covering the corresponding square I^2 with a trivialization, we see that there is a unique way to extend every element of a fiber to a global section of this square. The corresponding linear transformation of fibers at the endpoints coincides with the ones given by both the top and bottom sides of the square. As a consequence, if B is simply connected, then all local systems over B are constant.

Fix a basepoint $b \in B$. If we have a local system H with fiber V, we have constructed a representation, known as the **monodromy representation**,

$$\pi_1(B, b) \to GL(H_b).$$

Theorem 2.1.3. *The functor just defined yields an equivalence between isomorphism classes of local systems with fiber V and isomorphism classes of representations of the fundamental group $\pi_1(B, b)$.*

We sketch the inverse. Let $p : (\tilde{B}, \tilde{b_0}) \to (B, b_0)$ be a universal covering space. Then $\pi_1(B, b_0)$ may be canonically identified with the covering transformations of p. Then we may define a local system H by first defining a constant sheaf $H(\tilde{B})$. Then we take H to be the equivariant sections of $H(\tilde{B})$ under the action of the fundamental group.

2.1.2 Vector bundles with flat connections

Now let us take B to be a complex manifold. Let $E \to B$ be a holomorphic vector bundle.

Definition 2.1.4. *A **connection** ∇ on the vector bundle $E \to B$ is a $\mathbb{C}$-linear map*

$$\nabla : \Gamma(E) \to \Gamma(E) \otimes_{\mathcal{O}_B} \Omega_B$$

that satisfies the Leibniz rule:

$$\nabla(f\sigma) = f\nabla(\sigma) + \sigma \otimes df$$

for $\sigma \in \Gamma(E)$ and $f \in \mathcal{O}_B$.

We can further differentiate $\nabla : \Gamma(E) \otimes \Omega_B \to \Gamma(E) \otimes \bigwedge^2 \Omega_B$ by the rule

$$\nabla(\sigma \otimes \alpha) = \nabla\sigma \wedge \alpha + \sigma \otimes d\alpha.$$

The curvature of a connection is then given by

$$\Theta := \nabla \circ \nabla : \Gamma(E) \to \Gamma(E) \otimes \bigwedge^2 \Omega_B.$$

Finally, we say that a connection is **flat** if it has curvature zero. Another way of expressing this in terms of vector fields is the condition

$$[\nabla_X, \nabla_Y] = \nabla_{[X,Y]}.$$

Our goal is to now associate a local system H over B with a holomorphic vector bundle with a flat connection.

Construction 2.1.5. *Let H be a local system over B and define $\mathcal{H} := H \otimes_{\mathbb{C}} \mathcal{O}_B$. Then $\mathcal{H}$ is a holomorphic vector bundle over B, and define a connection $\nabla : \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{O}_B} \Omega_B$ on it in the following way. For $\sigma \in \mathcal{H}$, write $\sigma = \sum_i \alpha_i \sigma_i$ with σ_i a local trivialization of H. Then set*

$$\nabla\sigma = \sum_i \sigma_i \otimes d\alpha_i.$$

We claim the constructed ∇ on $\mathcal{H}$ is flat. Indeed, we have

$$(\nabla \circ \nabla)\sigma = \sum_i (\nabla\sigma_i \otimes d\alpha_i + \sigma_i \otimes d^2\alpha_i) = 0,$$

since $\nabla\sigma_i = \sigma_i \otimes d(1) = 0$.

Theorem 2.1.6. *The above construction produces a bijection between isomorphism classes of local systems and isomorphism classes of holomorphic vector bundles equipped with a flat connection.*

The inverse map associates to $(\mathcal{H}, \nabla)$ the local system of *flat sections* of $\mathcal{H}$, i.e. those annihilated by ∇. Showing this works boils down to showing that if we pick a point in some fiber $x \in \mathcal{H}_b$, there is a unique way to continue it to a flat section around b. The idea is to use the flatness of the connection to create a suitable *integrable distribution* D, which then gives by the Frobenius theorem a local foliation of $\mathcal{H}$ by submanifolds locally isomorphic to B. Then locally around b, flat sections correspond to the leaves of these local foliations, which are determined by their fiber in $\mathcal{H}_b$.

Here, a distribution E is a subbundle of the tangent bundle of a manifold, and it is integrable if X is covered by open sets with a differentiable map $\phi_U : U \to \mathbb{R}^{n-k}$ such that for all $x \in U$, we have $E_x = \ker d\phi_x$. In other words, we can find a submanifold with E locally giving its tangent spaces. The Frobenius theorem says the following.

Theorem 2.1.7 (Frobenius). *A distribution E is integrable if and only if it is closed under Lie bracket.*

The flatness of the connection in the previous theorem allows us to check that the corresponding distribution is closed under the Lie bracket.

Remark. The Riemann-Hilbert correspondence is a generalization of this correspondence to algebraic varieties. In full generality, it is a very deep theorem.

2.2 The Gauss-Manin connection

Given a suitable fiber bundle, we are interested in the local system determined by de Rham cohomology groups of the fibers. By the theory above, we can study this by looking at the flat sections of the associated connection, known as the *Gauss-Manin connection*. This theory applies both to the manifolds and in the setting of algebraic varieties. We will focus on the case of manifolds here and discuss the algebraic case in the next chapter. Here we follow [2].

2.2.1 Geometric background

Theorem 2.2.1 (Ehresmann's theorem). *Let $\phi : X \to B$ be a proper surjective submersion between two differentiable manifolds, where B is contractible with base point 0. Then there exists a diffeomorphism*

$$T : X \cong X_0 \times B$$

over B.

In particular, if ϕ is a proper surjective submersion of manifolds, then it gives a fiber bundle. The proof of this uses the tubular neighborhood theorem, which says that there is a neighborhood W of X_0 in X with a differentiable retraction $T_0 : W \to X_0$. The map

$$T = (T_0, \phi) : W \to X_0 \times B$$

has invertible differential along X_0, and thus in some open set containing X_0, since X_0 is compact. Then T must be the projection of a direct product in some smaller open set.

In the case of complex manifolds, we will need a slightly stronger result. Namely, that the fibers of $T_0 : W \to X_0$ are complex submanifolds of X.

Next we recall the classical de Rham theorem stating that for a smooth manifold X, we have

$$H^*_{dR}(X) \cong H^*(X, \mathbb{R}).$$

Indeed, we have two flasque resolutions of the constant sheaf $\underline{\mathbb{R}}$ given by

$$0 \to \underline{\mathbb{R}} \to \Omega_X^0 \xrightarrow{d^0} \Omega_X^1 \xrightarrow{d^1} \cdots$$

and

$$0 \to \underline{\mathbb{R}} \to \mathcal{C}^0_{\text{sing}} \xrightarrow{\partial} \mathcal{C}^1_{\text{sing}} \xrightarrow{\partial} \cdots$$

(Note that in the second, we are taking the sheafification, and it takes some work to show that the cohomology agrees with singular cohomology.) Thus they both give the cohomology of the constant sheaf $\underline{\mathbb{R}}$. The same holds for the complex cohomology, where $\underline{\mathbb{R}}$ is replaced with $\underline{\mathbb{C}}$. For complex manifolds there is also a Hodge filtration, which can be calculated with the Dolbeault resolution. We will discuss this in the following section. There is also a resolution of $\underline{\mathbb{C}}$ with the holomorphic de Rham complex; in this case we must take the hypercohomology. When we go to the algebraic setting in the next chapter, we will replace the holomorphic de Rham complex with the algebraic de Rham complex.

Finally, we state the proper base change theorem in topology, which is actually much more general than what we need.

Theorem 2.2.2. *Let $\mathcal{F}$ be a sheaf on X. Then the natural map*

$$g^* R^i f_* \mathcal{F} \to R^i f'_* g'^* \mathcal{F}$$

is an isomorphism of sheaves on A.

$$\begin{tikzcd} X \times_B A \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

2.2.2 Definition of the Gauss-Manin connection

We will put everything together to define the Gauss-Manin connection. Let $\pi : X \to B$ be a surjective proper submersion. Consider the sheaf $R^k\pi_*\underline{\mathbb{C}}$ on B (we use $\mathbb{C}$ for $\underline{\mathbb{C}}$ from now on). We claim it forms a local system. Indeed, fixing a basepoint $0 \in B$, Ehresmann's theorem implies that X is isomorphic to $X_0 \times U$ in a neighborhood $U \subset X_0$. That is, $X \times_B U \cong X_0 \times U$. Then it is evident that $R^k\pi_*\mathbb{C}$ is constant on U.

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \times_B U \arrow[r, "i'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ 0 \arrow[r, hook] & U \arrow[r, hook, "i"'] & B \end{tikzcd}$$

Next, by the proper base change theorem, each fiber $(R^k\pi_*\mathbb{C})_b$ for $b \in U$ is isomorphic to $(R^k\pi'_* i'^*\mathbb{C})_b \cong H^k(X_b, \mathbb{C})$. Thus the cohomology of each fiber X_b is the same. Really though, this is overkill because the fact that we have a fiber bundle $T : X \times_B U \cong X_0 \times U \to X_0$ gives a diffeomorphism between X_b and X_0, and thus we have isomorphisms $H^k(X_0, \mathbb{C}) \cong H^k(X_b, \mathbb{C})$. In fact, since we can make U contractible, we actually obtain canonical isomorphisms.

Now recall that in the previous section, we associated a flat connection to each local system.

Definition 2.2.3 (Gauss-Manin connection). *Let $\mathcal{H}^k = R^k\pi_*\mathbb{C} \otimes_{\mathbb{C}} \mathcal{O}_B$ be the vector bundle associated to the local system $R^k\pi_*\mathbb{C}$. The associated flat connection*

$$\nabla : \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega_B \quad \textit{where} \quad \nabla\Big(\sum_i \alpha_i\sigma_i\Big) = \sum_i \sigma_i \otimes d\alpha_i$$

is called the Gauss-Manin connection.

The sections of $\mathcal{H}^k$ restrict to kth cohomology classes on each fiber. For example, if ω is a complex differential form of degree k on X such that over $b \in U$, we have that ω_b is closed. Then $\omega : b \mapsto [\omega_b]$ defines a section of $\mathcal{H}^k$ over U.

2.2.3 Example: the Picard-Fuchs equation

Let us try to work through a concrete example. Consider the Legendre family

$$E_\lambda : y^2 = x(x-1)(x-\lambda), \quad \text{where} \quad \lambda \in \mathbb{P}^1 - \{0,1,\infty\}.$$

Formally, this is an elliptic surface which looks as follows.

$$\begin{tikzcd}
E_\lambda \arrow[r] \arrow[d] & E_U \arrow[r] \arrow[d] & E \arrow[d] \\
\lambda \arrow[r, hook] & U \arrow[r, hook] & \mathbb{P}^1 - \{0,1,\infty\}
\end{tikzcd}$$

Write $B = \mathbb{P}^1 - \{0,1,\infty\}$. The local system we are working with is $\mathcal{H} = R^1\pi_*\mathbb{C}$ on B. The fiber of this at λ is given by $H^1(E_\lambda, \mathbb{C})$, which has dimension 2. A natural way to construct a section is by taking $\omega = \frac{dx}{y}$ to be the one-form on E that restricts to the holomorphic differential

$$\omega_\lambda = \frac{dx}{x(x-1)(x-\lambda)}.$$

We identify ω_λ with its cohomology class. We may apply the connection along the tangent vector λ to differentiate $\nabla_\lambda(\omega) = (\nabla\omega)d\lambda \in \mathcal{H}^k$. That is, we are using the connection to use tangent vectors to differentiate the sections. Then $\omega, \nabla_\lambda\omega, \nabla_\lambda^2\omega$ are linearly dependent when restricted to any fiber $H^1(E_t, \mathbb{C})$, and thus we obtain a differential equation ...

How to actually find it, how are periods solutions, why are the cycle classes flat sections?

2.3 The period mapping

We follow [2].

2.3.1 Hodge structures

We begin with a rapid overview of some basic Hodge theory. If X is a complex manifold, let $A^{p,q}$ denote the sheaf of (p,q)-forms on X and let A^k be its k-forms. Then we have the Dolbeault resolution

$$0 \to \Omega^p \to A^{p,0} \to A^{p,1} \to \cdots$$

The cohomology of this complex is denoted $H^{p,q}(M) = H^q(M, \Omega^p(M))$.

Now let g be a Riemannian metric on X; this induces an L^2 metric on $A^k(X)$. Then we can define the Hodge star operator, an isomorphism

$$*: A^k(X) \xrightarrow{\cong} A^{n-k}(X)$$

such that

$$(\alpha, \beta)_{L^2} = \int_X \alpha \wedge *\beta.$$

Then define $d^* : A^k(X) \to A^{k-1}(X)$ by

$$d^* = (-1)^k *^{-1} d * .$$

By construction, this operator is the adjoint for d:

$$(\alpha, d^*\beta)_{L^2} = (d\alpha, \beta)_{L^2}.$$

Finally, define the Laplacian $\Delta_d = dd^* + d^*d$. A harmonic form ω is one where $\Delta_d\omega = 0$. Such forms are automatically closed. By analytic results involving elliptic differential operators, one (ideally) proves the following celebrated result, sometimes known as the Hodge theorem.

Theorem 2.3.1 (Hodge). *The natural map from the space of harmonic forms $A^k(X)$ to the complex de Rham cohomology $H_{dR}(X) \otimes_{\mathbb{R}} \mathbb{C}$ is an isomorphism.*

In the case of a Kähler manifold, one can show that $\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$. Here, $\Delta_{\overline{\partial}}$ preserves the type of a differential form, so Δ_d does too. This implies that the harmonic form decomposition descends down to the decomposition of forms, and thus we have the Hodge decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

Leaving the Kähler case, we would still like to define a sort of 'Hodge structure' with a 'Hodge filtration' on an arbitrary complex manifold X. In general, we can define an integral Hodge structure on any free abelian group V by

$$V \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$$

satisfying $V^{p,q} = \overline{V^{q,p}}$. But this doesn't work nicely in general for the complex cohomology of X . Instead we define a Hodge filtration which can be used to define a 'mixed Hodge structure.' We do this is by using the holomorphic de Rham complex. Take the resolution

$$0 \to \mathcal{O}_X \xrightarrow{\partial} \Omega_X \xrightarrow{\partial} \Omega_X^2 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega_X^n \to 0.$$

One shows that the inclusion of the holomorphic de Rham complex into the de Rham complex is a quasi-isomorphism (as complexes of sheaves). Then the de Rham cohomology is given by the hypercohomology of the holomorphic de Rham complex:

$$H^k(X, \mathbb{C}) = \mathbb{H}^k(X, \Omega_X^\bullet).$$

We can now define the Hodge filtration.

Definition 2.3.2. *Let $F^p\Omega_X^\bullet$ be the truncated holomorphic de Rham complex*

$$0 \to \Omega_X^p \to \Omega_X^{p+1} \to \cdots$$

The Hodge filtration is defined by

$$F^pH^k(X, \mathbb{C}) = \mathrm{im}(\mathbb{H}^k(X, F^p\Omega_X^\bullet) \to \mathbb{H}^k(X, \Omega_X^\bullet)).$$

When X is Kähler, this coincides with the filtration coming from the Hodge decomposition where $F^pH = \bigoplus_{i \geqslant p} H^{i,n-i}$.

2.3.2 The period mapping and the period domain

We return to the setting of a fibration $X \to B$; now assume that X is Kähler and shrink B so that X is a product of it and a fiber. As before, we have that $H^k(X_0, \mathbb{C}) \cong H^k(X, \mathbb{C}) \cong H^k(X_b, \mathbb{C})$. Moreover, using the Frölicher spectral sequence, more is true: all the Hodge numbers are constant in B. However, the Hodge filtrations differ. Let $b^{p,k} = h^{p,q}(X_b)$.

Definition 2.3.3. *The period map*

$$\mathcal{P}^{p,k} : B \to \mathrm{Gr}(b^{p,k}, H^k(X_0, \mathbb{C}))$$

is the map which sends $b \in B$ to the subspace

$$F^p H^k(X_b, \mathbb{C}) \subset H^k(X_b, \mathbb{C}) \cong H^k(X_0, \mathbb{C}).$$

A theorem of Griffiths states that the period map is holomorphic for all p, k. One can prove this using some version of a base change theorem.

Taking all p for a fixed k, we see that the period mappings sends b to a complete flag of $H^k(X, \mathbb{C})$. Denote the set of such flags by $\mathrm{Fl}(H^k(X, \mathbb{C}))$. If we restrict to the points $b \in B$ such that X_b is Kähler, then the Hodge filtration must also satisfy the condition

$$F^p H^k(X_b, \mathbb{C}) \oplus \overline{F^{k-p+1} H^k(X_b, \mathbb{C})} = H^k(X_b, \mathbb{C}).$$

The open set $\mathcal{D} \subset \mathrm{Fl}(H^k(X, \mathbb{C}))$ which satisfies this conditions is known as a **period domain**. The polarized period domain, $\mathcal{P}$, is a subset which satisfies additional properties.

Chapter 3

Algebraic de Rham cohomology and crystalline cohomology

3.1 Algebraic de Rham cohomology

3.1.1 Background on differentials

Definition 3.1.1 (derivations). *Let $f : A \to B$ be a ring homomorphism and let B be a B-module. We call $\gamma : B \to M$ be an A-linear derivation if $\gamma(ab) = a\gamma(b)$, $\gamma(b_1 b_2) = b_1\gamma(b_2) = b_2\gamma(b_1)$.*

Definition 3.1.2 (Kähler differentials). *The module of Kähler differentials is a B-module $\Omega^1_{B/A}$ that represents the functor*

$$M \mapsto \mathrm{Der}_A(B, M),$$

so in particular $\mathrm{Der}_A(B, M) = \mathrm{Hom}_B(\Omega^1_{B/A}, M)$ and we have the universal property

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[d, "\gamma"] & \Omega^1_{B/A} \arrow[dl, dashed, "\exists!"] \\ M & \end{tikzcd}$$

We construct $\Omega^1_{B/A}$ in two ways. First, take the free B-module generated by the symbols da, and quotient by relations. Alternatively, define $I := \ker(B \otimes_A B \to B)$. Then define $\Omega^1_{B/A} := I/I^2$. Then we have $d : B \to I/I^2$ by $b \mapsto b \otimes 1 - 1 \otimes b$.

Properties of differentials on rings.

1. If $A \to A'$ is a ring homomorphism, then $\Omega_{B\otimes_A A'/A'} \cong \Omega_{B/A} \otimes_A A'$.
2. $\Omega_{S^{-1}B/A} \cong \Omega_{B/A} \otimes_B S^{-1}B$.
3. Given an exact sequence $A \to B \to C$, we have an exact sequence

   $$C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0.$$

   When $B \to C$ is formally smooth, then this sequence is exact on the left and admits a splitting.
4. Given $B \twoheadrightarrow C$ then

   $$I/I^2 \xrightarrow{\delta} C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to 0.$$

From 1 and 2, we can define a quasi-coherent sheaf on $\operatorname{Spec} B$ by taking $\widetilde{\Omega^1_{B/A}}$. Furthermore, we let $\Omega^n_{B/A} := \wedge^n \Omega^1_{B/A}$. Then given the differential $d : B \to \Omega^1_{B/A}$, this extends to

$$B \to \Omega^1_{B/A} \xrightarrow{d} \Omega^2_{B/A} \xrightarrow{d} \cdots .$$

We now extend the sheaf of differentials to general separated schemes.

Definition 3.1.3 (Sheaf of differentials). *Let $f : X \to S$ be separated. Let I be the ideal sheaf corresponding to the closed immersion $X \xrightarrow{\Delta} X \times_S X$. Then we define the sheaf of differentials*

$$\Omega^1_{X/S} := (\Delta_{X/S})^*(I).$$

The above properties of affine differentials extend to the case of schemes.

3.1.2 Algebraic de Rham cohomology groups

Let $\pi : X \to Y$ be a morphism of schemes. Then the algebraic de Rham cohomology groups are defined as

$$H^q(X, \Omega^\bullet_{X/Y}) = R^q\pi_*(\Omega^\bullet_{X/Y}).$$

We can compute this as the hypercohomology of the complex $R\pi_*(\Omega^\bullet_{X/Y})$. We get the Hodge-de Rham spectral sequence

$$E_1 : H^q(X, \Omega^p_{X/Y}) \Rightarrow R^{p+q}\pi_*(\Omega_{X/Y}) = \mathbb{H}^{p+q}_{dR}(X).$$

Deligne proved that the spectral sequence degenerates at E_1; for $Y = \operatorname{Spec} \mathbb{C}$ this gives the Hodge decomposition.

We can use another filtration. Assume $\pi : X \to Y$ is smooth. Recall there is an exact sequence

$$0 \to \pi^*(\Omega^1_{Y/k}) \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0.$$

We use the fact that the $\Omega_{-/-}$ are locally free. In the Katz-Oda paper, the authors assign a filtration on $\Omega^\bullet_{X/k}$ as follows. Let

$$F^i = \operatorname{im}[\Omega^{\bullet - i}_{X/K} \otimes_{\mathcal{O}_X} \pi^*(\Omega^i_{Y/K}) \to \Omega^\bullet_{X/K}].$$

Then the associated grading is given by

$$\mathrm{Gr}_i = \pi^*(\Omega^i_{Y/K}) \otimes_{\mathcal{O}_X} \Omega^{\bullet - i}_{X/Y}.$$

With respect to this filtration, we compute the spectral sequence for $R\pi_*(\Omega^\bullet_{X/K})$ gives

$$E_1^{p,q} = \Omega^p_{X/k} \otimes_{\mathcal{O}_Y} R^q\pi_*(\Omega^\bullet_{X/Y}).$$

Then we get an exact sequence

$$0 \to H^q_{dR}(X/Y) \xrightarrow{d} \Omega^1_{Y/k} \otimes_{\mathcal{O}_Y} H^q_{dR}(X/Y) \xrightarrow{d} \Omega^2_{Y/k} \otimes_{\mathcal{O}_Y} H^q_{dR}(X/Y) \xrightarrow{d} \cdots$$

This d is the Gauss-Manin connection with $d^2 = 0$. This implies that $H^q_{dR}(X/Y)$ is a crystal on the crystalline site of Y.

3.1.3 The Gauss-Manin connection arising from the spectral sequence for algebraic de Rham cohomology

Recall:

$$\begin{tikzcd} X \arrow[r, "\text{smooth}"] \arrow[dr] & Y \arrow[d] \\ & \text{Spec } k \end{tikzcd}$$

We defined the relative de Rham complex $\Omega^\bullet_{X/Y}$ as an $\mathcal{O}_X$-module and the de Rham cohomology

$$\mathbf{H}_{dR}(X/Y) := R\pi_*(\Omega^\bullet_{X/Y})$$

We showed last time that there is a spectral sequence

$$E_1^{pq} := H^q(X, \Omega^p_{X/Y}) \implies R^{p+q}\pi_*(\Omega^\bullet_{X/Y})$$

Over $\mathbb{C}$, when π is proper in addition to smooth, the spectral sequence terminates at E^1 page, get the Hodge decomposition.

This comes from the stupid filtration. We can use a cleverer one (relying heavily on smoothness).

$$F^i\Omega^\bullet_{X/Y} := \text{Im}[\pi^*(\Omega^i_{Y/k}) \otimes_{\mathcal{O}_X} \Omega^{\bullet,i}_{X/k}[-i] \to \Omega^\bullet_{X/k}]$$

$$0 \to \pi_*(\Omega^1_{Y/k}) \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0$$

The filtration essentially takes the forms in $\Omega_{X/k}$ that have i forms coming from $\Omega^1_{Y/k}$.

$$\text{gr}^i \simeq F^i/F^{i+1} = \pi^*(\Omega^i_{Y/k}) \otimes_{\mathcal{O}_X} \Omega^{\bullet,-i}_{X/Y}$$

(write it out). For a filtration on a complex, we get a spectral sequence.

$$E_0^{p,q} := F^p\Omega^{p+q}_{X/k}/F^{p+1}\Omega^{p+q}_{X/k} = \text{gr}^p\Omega^{p+q}_{X/k}$$

We want to compute $R\pi_*(\Omega^\bullet_{X/k})$.

$$\begin{aligned} E_1^{p,q} &= R^{p+q}\pi_*\text{gr}^p(\Omega^\bullet_{X/k}) \implies R^{p+q}_{\pi_*}(\Omega_{X/k}) \\ &= R^{p+q}\pi_*(\pi^*\Omega^p_{Y/k}) \otimes \Omega^{\bullet,-p}_{X/S} \\ &= R^q\pi_*(\pi^*(\Omega^p_{Y/k}) \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/Y}) \\ &= \Omega^p_{Y/k} \otimes_{\mathcal{O}_Y} R^q\pi_*\Omega^\bullet_{X/Y} \\ &= \Omega^p_{Y/k} \otimes_{\mathcal{O}_Y} H^\bullet_{dR}(X/Y) \end{aligned}$$

We have $F_i \wedge F_j \subset F_{i+j}$ with $R\pi_*$ respecting multiplication

$$\bigwedge : E_r^{p,q} \times E_r^{p',q'} \to E_r^{p+p',q+q'}$$

$$(e, e') \to e \wedge e' = (-1)^{(p+q)(p'+q')}e' \wedge e, \quad d_r(e \wedge e') = dr(e) \wedge e' + (-1)^{p+q}e \wedge dr(e')$$

along $E_1^{\bullet,0} : dr/R \otimes k$ with a section $\Omega^i_{Y/k}$, e a section of $H_q(q, Y)$.

$$d_q^{i,q}(w \wedge e) = d_{Y/K}w \wedge e + (-1)^{i+q}w \wedge d^{1,q}e$$

$$H^q_{dr}(X/Y) \xrightarrow{\nabla} \Omega^1_{Y/k} \otimes H^q_{dR}(X, Y) \xrightarrow{\nabla} ...$$

where $\nabla = d_q^{1,0}$ with $\nabla^2 = 0$. We get an integrable connection: the Gauss-Manin connection.

∇_{GM} gives us a connection on the $\mathcal{O}_X$ module $H^q_{dR}(X/Y)$, this connection means that it will have good properties for the soon to be defined crystalline site. Connection we claim is the same data as infinitesimal firest order descent information.

Descent data: $X \xrightarrow{\pi} Y$, $X \to X \times_Y X$, sheaf of ideals I defining this, $X \to \Delta^1_X$ where Δ^1_X is the closed subscheme associated to I^2. And $X \to \Delta^1_X$ the sheaf of ideals is $\Omega^1_{X/Y}$.

There are two projections $\mathrm{pr}_1, \mathrm{pr}_2 : \Delta^1_X$ induced by the two projections $X \times_Y X \to X$. In general, given an $\mathcal{O}_X$-module M, there is no canonical way to identify $\mathrm{pr}_1^*(M), \mathrm{pr}_2^*(M)$ as Δ^1_X-modules.

Claim: having an Δ^1_X-linear isomorphism $\mathrm{pr}_1^*(M) \xrightarrow{\simeq} \mathrm{pr}_2^*(M)$ is equivalent to defining a connection on M. Stipulating that $X \to X \times_Y X \times_Y X$ satisfying cocycle then in fact is equivalent to $\nabla^2 = 0$.

Definition 3.1.4. $\nabla : M \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} M$*, by* $\nabla(am) = da \otimes m + a\nabla(m)$.

Defining a connection in terms of descent data is more generally a *Grothendieck connection*.

What's the point of all this? This comparison with descent data will give us essentially that having a sheaf that has an integral connection defines a crystal on the crystalline site.

3.2 PD structures remedying the lack of a Poincaré lemma

We have ℓ-adic cohomology which, given a $\mathbf{Z}_p$-variety gives us a good cohomology theory for all $\ell \neq p$. When $\ell = p$ it is not a Weil-cohomology theory. And de Rham cohomology is a good cohomology theory, so you might be tempted to define, say, for an $\mathbf{F}_p$-variety $\overline{X}$ a lift $\tilde{X}$ to $\mathbf{Z}_p$ and take $H_{dR}(\tilde{X}/\mathbf{Z}_p)$. But if we rely on lifts, we don't know if it will be intrinsic or even exist.

So, will the de Rham cohomology have good properties for $\mathbf{Z}_p$ varieties?

$$\mathrm{Spec}(\mathbf{Z}_p[t])$$

$$\mathbf{Z}_p[t] \xrightarrow{d} \mathbf{Z}_p[t]dt, \quad t_p^p \to t^{p-1}dt$$

No Poincare lemma.

Now we talk about PD (divided power structure).

Definition 3.2.1. *Let A be a ring, $I \subset A$ an ideal, $\gamma_{n\geqslant 1} : I \to I$. Then (A, I, γ) is a* PD structure *if for all $x, y \in I$*

1. $\gamma_0(x) = 1$, $\gamma_1(x) = x$, $\gamma_i(x) \in I$
2. $\gamma_n(x + y) = \sum \gamma_i(x)\gamma_{n-i}(y)$
3. *For $a \in A$ we have* $\gamma_n(ax) = a^n\gamma_n(x)$
4. $\gamma_n(x)\gamma_m(x) = \binom{n+m}{m}\gamma_{n+m}(x)$
5. $\gamma_p(\gamma_1(x)) = C_{p,q}\gamma_{pq}(x)$ *with* $C_{p,q} = \frac{(pq)!}{p!q^p}$.

Example 3.2.2. *For a* $\mathbf{Q}$*-algebra,* $\gamma_n(x) = \frac{x^n}{n!}$ *(missed a bit).*

Proposition 3.2.3. *Given PD structure (A, I, γ) and $f : A \to B$, we get an extension of the PD structure to (B, IB) in the following cases:*

- $IB = 0$
- $I \otimes_A B \cong IB$ *(if f is flat)*

Theorem 3.2.4. *If (A, I, γ) is a PD-algebra and B is a finite A-algebra, $J \subseteq B$, and $f : A \to B$. Then there exists a ring $(D, \overline{J}, \overline{\gamma})$ such that*

$$\begin{tikzcd} & (D, \overline{J}, \overline{\gamma}) \arrow[dd, dashed, "\exists!"] & \\ (B, J) \arrow[ur] \arrow[dr] & & (A, \gamma, I) \arrow[ul] \arrow[dl] \\ & (C, \tilde{J}, \tilde{Y}) & \end{tikzcd}$$

Proof sketch. Consider the graded ring Γ defined by

$$\Gamma_0 := B$$

$$\Gamma_1 = J + IB$$

$\Gamma_n :=$ generated by symbols $[x]_n, x \in J + IB$ modulo the PD-relations $\gamma_n(x) = [x]_n, x = [x]_1$

□

Proposition 3.2.5. *Let (A, I, γ) be a PD, B an A-algebra, $J \subseteq B$, then*

- *B' is a flat B-algebra, $D_{B',\gamma}(JB') = D_{B,\gamma}(J) \otimes_B B'$*
- *$(A, I, \gamma) \to (A', I', \gamma')$, $D_{B \otimes_A A', \gamma'}(J \otimes_A A) = D_{B,\gamma}(J) \otimes_A A'$*

3.3 Crystalline cohomology

Recall: we showed that $\mathcal{O}_X$-module M has infinitesimal descent data if it admits an integrable connection. That is, a map

$$M \xrightarrow{\nabla} M \otimes \Omega^1_{X/S} \to$$

such that the extension $\nabla^2 = 0$.

$H^q_{dR}(X/Y)$ as an $\mathcal{O}_Y$-module admitted a Gauss-Manin connection, so as an $\mathcal{O}_Y$ module we have infinitesimal descent.

PD0structure: for A, $I \subseteq A$, maps $\gamma_n I \to I$ satisfying $\gamma_0(x) = 1$, $\gamma_1(x) = x$ and

- $\gamma_n(x + y) = \sum_{i=0}^n \gamma_i(x)\gamma_{n-i}(y)$
- $\gamma_n(ax) = a^n \gamma_n(x)$
- $\gamma_n(x)\gamma_m(x) = \binom{n+m}{n}\gamma_{n+m}(x)$
- These imply $n!\gamma_n(x) = x^n$

Example 3.3.1.
- *For $\mathbf{Q}$-algebras, there exists a unique PD-structure $\gamma_n(x) = \frac{x^n}{n!}$.*
- *If $mA = 0$, in I, $x^m = 0 \; \forall \, x \in I$ we have $m!\gamma_n(x) = x^m$.*
- *$\mathbf{Z}/2\mathbf{Z}$ has a PD-structure*

we also defined a universal PD-algebra over a PD-algebra (A, I, γ_0) and B, $J \subseteq B$

$$\begin{tikzcd}
& D_{B,\gamma}(J) \arrow[dd] & \\
(B,J) \arrow[ur] \arrow[dr] & & (A,I,\gamma) \arrow[ul] \arrow[ll] \arrow[dl] \\
& (C,I',\gamma) &
\end{tikzcd}$$

This construction is compatible with flat base change. This means that given

$$\begin{tikzcd}
& X_Z \arrow[dl] \arrow[dr] & \\
X \arrow[rr] \arrow[dr] & & Z \arrow[dl] \\
& (S,X) &
\end{tikzcd}$$

where $X \to Z$ is a closed immersion $A \to A/I$, then X_Z will be $D_{A,\gamma}(I)$.

Example 3.3.2. *Given $A[t]$, we can put a PD-structure on (t) by $A[\gamma_n(t)] = A\langle t\rangle$ the free PD-polynomial algebra over A.*

The PD structure will be the thing that allows you to let the Poincaré lemma work.

$$\mathbf{Z}_p\langle t\rangle \to \mathbf{Z}_p\langle t\rangle dt, \quad d(\gamma_n(t)) \to \gamma_{n-1}(t)dt$$

taking $\frac{x^n}{n!} \to \frac{x^{n-1}}{(n-1)!}$.

The idea of Crystalline cohomology: if we have an $\mathbf{F}_p$ variety X, the l-adic cohomology groups for $l \neq p$ form a good cohomology theory, but for $l = p$ we do not have that.

One possible remedy is to consider a lift to $\mathbf{Z}_p$, $\tilde{X}$ over X, smooth over $\mathbf{Z}_p$, then define

$$H^i_{\text{crys}}(X, \mathbf{F}_p) = H^i_{dR}(\tilde{X}/\mathbf{Z}_p)$$

Two problems: there may be no such lift, and it depends on the left if it exists.

But you can lift locally on affine stuff (that's deformation theory). Grothendieck's solution was to consider all possible lifts of affine objects (U, T) where U is open in X and $U \hookrightarrow T$ is a nil-immersion.

Now, we have to introduce notions of a site on a fibered category.

Definition 3.3.3. *Let C be a fibered category. A* Grothendieck Top *consists of covering families $(U_i \to U)$ such that*

- *$id : U \to U$ is in the covering family*
- *Given a morphism $V \to U$ and covering family $\{U_i \to U\}$, then $\{V \times_U U_i \to V\}$ is also a covering family*
- *If $\{U_i \to U\}$ and $\{V_{ij} \to U_j\}$ are covering families, then $\{U_{ij} \to U\}$ is a covering family.*

This is sort of an axiomatization of open sets of a topological space.

Now, Crystalline Site. Let $W = \text{Spec}(\mathbf{Z}_p)$ and X finite type over W when p is locally nilpotent. X will be a finite type $\mathbf{Z}_p/p^N\mathbf{Z}_p$-algebra.

We say a closed immersion $X \to T$ over W is a *PD-thickening* if (T, I, γ) is compatible with $(\mathbf{Z}_p, (p), \gamma)$.

Claim: $X \to T$ is a nil-immersion. $\gamma_n : I \to I$ with $p^N A = 0$, $n!\gamma_n(x) = x^n$ for $n = pN$.

Definition 3.3.4. *Let $k = \mathbf{F}_p$ so X is over $W(k)$. The* crystalline site *$X/W(k)_{crys}$ consists of all PD-thickenings (nil-immersions) (U, T), with $U \subset X$ open. And $\{(U_i, T_i) \to (U, T)\}$ is a covering if $T_i \to T$ is a Zariski covering.*

We also have truncated sites: $X/W_{n,\text{crys}}$ will consist of (U, T) PD-thickenings of X for T a W_n-scheme.

Given a sheaf F on a crystalline site and (U, T, γ), we have a Zariski sheaf $F_T(W) := F(U \cap W, W, \gamma|_W)$ for each open $W \subset T$.

Given a map $f : T \to T'$ we have a comparison between (U, T, δ) and (U', T', δ') for $f(U) \subset U'$

$$C_f : f^{-1}F_{T'} \to F_T$$

given $W' \subset T'$, f induces a morphism

$$f|_{f^{-1}W} : (U \cap f^{-1}(W'), f^{-1}W', \delta|_{f^{-1}W'}) \to (U' \cap W', W', \delta|_W)$$

Key point: given Zariski sheaves F_T for each (U, T, δ) and comparison maps $(f, f : T \to T', W')$ which is an isomorphism for open immersions and satisfies transitivity, then get a sheaf on (X/W_{crys}).

Definition 3.3.5. *A quasi-coherent sheaf on X/W_{crys} is called a* crystal *if all comparison maps are isomorphisms.*

Two things

- We're asking for a lifting problem

$$\begin{array}{ccc} ? & & \overline{B} \\ & & \uparrow \\ A & \longrightarrow & A/I \end{array}$$

- Another thing we can see is the following. Given $X \to X \times_S X$ with $X \to S$ smooth, there is a sheaf of ideals I defining the closed immersion. But we can also define the infinitesimal neighborhood $\Delta_1(X)$ by $X \to \Delta_1(X) \to I^2 \subset X \times_S X$. For $X \to \Delta_1(X)$ a closed immersion, we can assign a PD-structure and $(X, \Delta_1 X)$ will be in X/W_{crys}. Recall we have two projections $\text{pr}_1, \text{pr}_2 : \Delta_1(X) \to X$. The isomorphism condition on comparison maps implies $\text{pr}_1^{-1}(X) \cong \text{pr}_2^{-1}(X)$ and the data that gives this is a connection. The point is that given a quasi-coherent sheaf F on X/W_{crys} (U, T, δ), $\tilde{\Omega}_{T/S}$ it will be $\Omega_{T/S}$ quotient by $d\gamma_n(x) = \gamma_{n-1}(x)dx$. It's saying that F is a crystal iff there exists an integral connection of F on X/W_{crys},
$$F_T \to F \otimes_{\mathcal{O}_T} \tilde{\Omega}_{T/S}$$

There is a structure sheaf on X/W_{crys} which assigns to every (U, T, δ) as $\Gamma(T, \mathcal{O}_X)$. Check that this forms a crystal.

For (U, T, δ), assign $H^i_{dR}(T/S)$; this is a crystal. But $\tilde{\Omega}^i(T/S)$ is not, as it doesn't a priori have descent data.

Theorem 3.3.6. *If X admits a smooth $\mathbf{Z}_p$ lifting $\tilde{X}$, then*

$$H^i_{crys}(X/W_{crys}, \mathcal{O}_{X/crys}) = H^i_{dR}(\tilde{X}/W)$$

So this says that $H^i_{dR}(\tilde{X}/W)$ only depends on the special fiber. If we have two liftings $\tilde{X}'$ and $\tilde{X}$, then

$$\begin{tikzcd} H^i_{dR}(\tilde{X}/W) \arrow[rr, "{\simeq, \nabla\mathrm{GM}}"] \arrow[dr] & & H^i_{dR}(\tilde{X}'/W) \arrow[dl] \\ & H^i_{\text{crys}}(X/\text{crys}, D_{X/W}) & \end{tikzcd}$$

they are isomorphic by the Gauss Manin connection.

Chapter 4

p-adic Hodge theory

Three rings for the Elven-kings under the sky,

$\mathbf{B}_{\mathrm{cris}}, \mathbf{B}_{\mathrm{st}}, \mathbf{B}_{\mathrm{dR}}$.

Seven for the Dwarf-lords in their halls of stone,

$\mathbf{E}_{\mathbb{Q}_p}, \mathbf{A}_{\mathbb{Q}_p}, \mathbf{B}_{\mathbb{Q}_p}, \mathbf{E}, \mathbf{A}, \mathbf{B}, \widetilde{\mathbf{A}}$.

Nine for mortal Men doomed to die,

$\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{F}_p, \overline{\mathbb{Q}_p}, \overline{\mathbb{F}_p}, \mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p}, \mathbb{Q}_p^{\mathrm{ur}}, \mathbf{B}_{\mathrm{HT}}$

One ring to rule them all,

$\mathbf{A}_{\mathrm{inf}}$.

Fontaine

Notation:

- K is a finite extension of $\mathbb{Q}_p$ (in fact, the results hold when K is a complete discrete valued field of characteristic 0 with perfect residue field k of characteristic p).
- $\mathcal{O}_K$ is the ring of integers of K, with maximal ideal $\mathfrak{m}_K$ and uniformizer $\pi = \pi_K$.
- $k = \mathcal{O}_K/\mathfrak{m}_K$ is the residue field of K. In our case, k is a finite extension of $\mathbb{F}_p$.
- $\mathbb{C}_p$ is the completion of $\overline{K} = \overline{\mathbb{Q}}_p$ with respect to the p-adic metric.
- $G_K = \mathrm{Gal}(\overline{K}/K)$.
- K^{ur} is the maximal algebraic unramified extension of K.
- K^{tr} is the maximal tame extension of K.
- $I_K = \mathrm{Gal}(K/K^{\mathrm{ur}}) \subset G_K$ is the inertia group of K.
- $P_K = \mathrm{Gal}(K/K^{\mathrm{tr}}) \subset I_K$ is the wild inertia subgroup.
- μ_n is the group of nth roots of unity.

4.1 l-adic and p-adic Galois representations

4.1.1 Preliminaries on local fields

Studying representations of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is a big deal. There is an important local variant: the representations of $G_{\mathbb{Q}_p}$. These are related in the following way.

Proposition 4.1.1. *There is an injection $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$ identifying $G_{\mathbb{Q}_p}$ with the decomposition group of the place (p) in $G_{\mathbb{Q}}$.*

Proof. First we exhibit an injection $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$. Indeed, fixing inclusions $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$, we obtain a map $G_{\mathbb{Q}_p} \to G_{\mathbb{Q}}$. To show this is an injection, it suffices to show two things: first, $G_{\mathbb{Q}_p}$ acts continuously on $\overline{\mathbb{Q}_p}$, and second, $\overline{\mathbb{Q}}$ is dense in $\overline{\mathbb{Q}_p}$.

For the first, note that $G_{\mathbb{Q}_p}$ sends $\mathbb{Z}_p$ to $\mathbb{Z}_p$, because it preserves the ring of integers. It follows easily that $G_{\mathbb{Q}_p}$ also preserves $p\mathbb{Z}_p$ and acts by an isometry with respect to the p-adic metric. The second follows from Krasner's lemma, which in fact gives that $\overline{\mathbb{Q}}$ is dense in $\mathbb{C}_p$. Now we see that $G_{\mathbb{Q}_p}$ is identified with the stabilizer of the place (p) of $\overline{\mathbb{Q}}$, so it is identified with the decomposition group of (p) in $G_{\mathbb{Q}}$ as desired. $\square$

Given this, $G_{\mathbb{Q}_p}$ fits into an exact sequence involving the inertia group and the Galois group of the residue field. More generally, this holds for K as well. Recall that there is a unique way to extend the valuation on $\mathbb{Q}_p$ to K, by setting

$$\nu_p(\alpha) = \frac{1}{[K:\mathbb{Q}_p]}\nu_p(N_{K/\mathbb{Q}_p}).$$

Then $\mathcal{O}_K$ is also a complete PID with some uniformizer π. The same proof as above shows that G_K preserves $\mathcal{O}_K$ and $\mathfrak{m}_K$, so we have an exact sequence

$$1 \to I_K \to G_K \to G_k \cong \widehat{\mathbb{Z}} \to 1.$$

Evidently, I_K consists of the elements $\sigma \in G_K$ such that $\sigma(x)-x \in \mathfrak{m}_K$. It corresponds to the Galois group $\mathrm{Gal}(\overline{K}/K^{\mathrm{ur}})$, while G_k is given by $\mathrm{Gal}(K^{\mathrm{ur}}/K)$. Indeed, the finite extensions of the residue field k correspond bijectively to the finite unramified extensions of K. The inertia group I_K is also known as G_0, with the higher ramification groups G_i being defined as the subgroups of G_K with $\sigma(x) - x \in \mathfrak{m}_K^{i+1}$. We are particularly interested in $G_1 = P_K$, which fits into another exact sequence

$$1 \to P_K \to I_K \to \prod_{l \neq p} \mathbb{Z}_l(1) \to 1.$$

This deserves some explanation. First, a tamely ramified extension L/K is one for which $p \nmid e_{L/K}$. Thus K^{tr} is the union of all extensions of K^{ur} with degree relatively prime to p. Then P_K is the wild inertia group, corresponding to $\mathrm{Gal}(\overline{K}/K^{\mathrm{tr}})$. Thus this exact sequence is saying that the tamely ramified portion, $\mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{ur}})$, is well-understood as just $\prod_{l\neq p}\mathbb{Z}_l(1)$. To show this, we use the following fact.

Proposition 4.1.2. *If L/K^{ur} is a totally tamely ramified extension of degree e (so $p \nmid e$), then $L = K[\pi_{K^{\mathrm{ur}}}^{1/e}]$.*

Proof. Being totally ramified means that $\mathcal{O}_{K^{\mathrm{ur}}}/(\pi_{K^{\mathrm{ur}}}) \cong \mathcal{O}_L/(\pi_L)$ and $\pi_{K^{\mathrm{ur}}} = u(\pi_L)^e$ for some $u \in \mathcal{O}_L^\times$. Furthermore, it also means that $L = K^{\mathrm{ur}}[\pi_K]$[1]. Thus it suffices to choose a different uniformizer π_L' which is an eth root of $\pi_{K^{\mathrm{ur}}}$. By an appropriate choice of $\pi_{K^{\mathrm{ur}}}$, we may assume that $u \equiv 1 \pmod{\pi_L}$. Then by Hensel's lemma, we have a solution to $x^e = u$ in $\mathcal{O}_L$ (this uses the fact that $p \nmid e$). We conclude by setting $\pi_L' = \pi_L/x$. $\square$

[1] In fact, $\mathcal{O}_L = \mathcal{O}_{K^{\mathrm{ur}}}[\pi_L]$ and π_L is the root of an Eisenstein polynomial.

From here, it follows that $\mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{ur}}) \cong \varprojlim_{p\nmid n} \mathbb{Z}/n\mathbb{Z} \cong \prod_{l\neq p} \mathbb{Z}_l$. Moreover, since K^{tr} can be viewed as a cyclotomic extension of K^{ur}, it naturally carries the action of the cyclotomic character, which is why we write $\prod_{l\neq p} \mathbb{Z}_l(1)$. The '$(1)$' is known as a Tate twist.

The moral is the following: we have a filtration

$$1 \subset P_K \subset I_K \subset G_K \quad \text{corresponding to} \quad \overline{K} \supset K^{\mathrm{tr}} \supset K^{\mathrm{ur}} \supset K.$$

The parts $G_K/I_K = \mathrm{Gal}(K^{\mathrm{ur}}/K) \cong \hat{Z}$ and $I_K/P_K = \mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{ur}}) \cong \prod_{l\neq p} \mathbb{Z}_l(1)$ are well-understood. The wild ramification group, $P_K = \mathrm{Gal}(\overline{K}/K^{\mathrm{tr}})$, is where the difficulty lies. But one can show, essentially due to a 'clash of topologies,' that the image of P_K under any (continuous, as always) representation is finite. This is already pretty good. To take this the full way, we have Grothendieck's monodromy theorem. We call an l-adic representation ρ potentially semi-stable if there is an open subgroup on which ρ acts unipotently. Equivalnetly, there is a finite extension of K such that ρ is semi-stable.

Theorem 4.1.3 (Grothendieck's monodromy theorem). *Every l-adic representation of G_K is potentially semistable.*

4.1.2 Good reduction

Recall that a variety X over K is said to have good reduction if it admits a smooth model over $\mathcal{O}_K$, namely a smooth variety $\mathfrak{X}/\mathcal{O}_K$ with the following Cartesian diagram.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} \mathcal{O}_K \end{tikzcd}$$

Now consider the action of G_K on $H^1_{et}(X, \mathbb{Z}_l)$. The Néron-Ogg-Shafarevich theorem says that for elliptic curves, X has good reduction if and only if this representation is unramified; i.e. the action of I_K is trivial. Using Néron models, Serre and Tate extended this to abelian varieties.

Theorem 4.1.4 (Serre-Tate). *Let X/K be an abelian variety and let $l \neq p$ be a prime. Then X has good reduction if and only if the representation $H^1_{et}(X, \mathbb{Z}_l)$ is unramified.*

In their original paper, Serre-Tate used $T_l(A)$, which gives essentially the same thing. Let us briefly describe why for elliptic curves, good reduction implies the Galois representation is unramified. If E/K has good reduction, then we may pass to its reduction $\mathcal{E}/\mathbb{F}_p$. The point is that as G_K-representations, we have

$$T_l(E) \cong T_l(\mathcal{E}),$$

where the representation on $T_l(\mathcal{E})$ is obtained through the surjection $G_K \to G_k$. This implies that the action of G_K factors through G_k, so $T_l(E)$ is unramified.

This never works if $l = p$, because in this case $T_p(\mathcal{E}_{\mathbb{F}_p})$ has dimension either 0 or 1, not 2 – we can't simply divide by p in characteristic p. Thus being unramified is not the correct criterion for having good reduction when $l = p$. Grothendieck found the correct answer through Barsotti-Tate groups, which turned out to give one of the first incarnations of p-adic Hodge theory. In more modern terms, X has good reduction when its etale cohomology groups are **crystalline representations**, a term which we will explain in the following section.

4.2 Cohomological comparison theorems

Remark. Fontaine was the primary architect of these period rings. The proofs of the hardest theorems are initially due to Fontaine-Messing-Kato-Tsuji. Faltings also proved them around the same time using an independent method: his theory of almost mathematics. Niziol and Beilinson have also proved them with different methods. We refer the reader to https://arxiv.org/pdf/2005.07919.pdf for an excellent survey.

4.2.1 The Hodge-Tate decomposition

Let us recall the classical theorems in the complex case.

Theorem 4.2.1 (Hodge decomposition)**.** *Let X be a compact Kähler manifold. Then there is a canonical isomorphism*

$$H^k_{\text{sing}}(X^{\text{an}}, \mathbb{R}) \otimes \mathbb{C} \cong \bigoplus_{p+q=k} H^p(X, \Omega^q_{X/\mathbb{C}}).$$

Using étale cohomology rather than singular cohomology, p-adic Hodge theory will not only give analogues of this decomposition, but also do so in a way that respects filtrations and the Galois and Frobenius actions that arise when considering étale cohomology. Rather than tensoring with $\mathbb{C}$, which can be thought of as a ring of periods, we will tensor with p-adic analogues. The most obvious idea is to tensor with $\mathbb{C}_K = \mathbb{C}_p$, and this does indeed give the desired 'Hodge-Tate' decomposition.

Theorem 4.2.2 (Hodge-Tate decomposition)**.** *Let X be a proper smooth variety over K. Then there is a canonical isomorphism*

$$H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong \bigoplus_{i+j=n} H^j(X, \Omega^i_{X/K}) \otimes_K \mathbb{C}_K(-i).$$

compatible with G_K-actions.

Note that here, G_K acts on every term **except** for $H^j(X, \Omega^i_{X/K})$. This means that the information we get from the right hand side is essentially limited to the 'Hodge-Tate weights' appearing in $\mathbb{C}_K(-i)$. For instance, we could write the right hand side as $\bigoplus_{i+j=n} \mathbb{C}_K(-i)^{h^{i,j}}$. This is great, but note that the left hand side doesn't have a grading, while the right hand side does. In particular, we don't have a formula for the Hodge numbers of X using the étale cohomology. To remedy this, we define

$$\mathbf{B}_{\text{HT}} := \bigoplus_{n \in \mathbb{Z}} \mathbb{C}_K(n).$$

With this new period ring, the Hodge-Tate decomposition becomes

$$H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{HT}} \cong \left(\bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}) \right) \otimes_K \mathbf{B}_{\text{HT}},$$

and this is a graded isomorphism. Here, the grading on the left is given by the natural grading on $\mathbf{B}_{\text{HT}}$, and the grading on the right is given by the sum of i for $H^j(X, \Omega^i_{X/K})$ and the grading on $\mathbf{B}_{\text{HT}}$. Now we use the following important theorem.

Theorem 4.2.3 (Ax-Sen-Tate)**.** *We have $\mathbf{B}_{\text{HT}}^{G_K} = K$.*

In particular, this copy of K is coming from the weight 0 portion $\mathbb{C}_K^{G_K} = K$. We have

$$(H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{HT}})^{G_K} \cong \bigoplus_{i+j=n} H^j(X, \Omega^i_{X/K}).$$

The ith graded piece of the left hand side therefore recovers the Hodge number $h^{i,j}$.

4.2.2 Étale to de Rham

Recall that there is a Hodge-de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_{X/K}) \Rightarrow H^{p+q}_{\mathrm{dR}}(X/K).$$

In fact, this degenerates at the E_1 page, so we know that the de Rham cohomology is abstractly (as $\mathbb{C}$-vector spaces) isomorphic to $\bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K})$. Although the Hodge-Tate decomposition respects grading, it doesn't respect the filtration of $H^k_{\mathrm{dR}}(X/K)$. For this we will need another ring, $\mathbf{B}_{\mathrm{dR}}$, which has a fairly complicated construction.

This ring $\mathbf{B}_{\mathrm{dR}}$ itself has a filtration such that its associated graded ring is $\mathbf{B}_{\mathrm{HT}}$. Furthermore, it also carries an action of G_K such that $\mathbf{B}_{\mathrm{dR}}^{G_K} = K$. Then we have the following comparison theorem

Theorem 4.2.4 (C_{dR} theorem). *Let X be a proper smooth variety over K. Then there is a canonical isomorphism*

$$H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}} \cong H^n_{\mathrm{dR}}(X/K) \otimes_K \mathbf{B}_{\mathrm{dR}}.$$

compatible with G_K-actions and filtrations.

The filtration on the left hand side comes from the one on $\mathbf{B}_{\mathrm{dR}}$, while the one on the right hand side is the convolution of the Hodge filtration and the one on $\mathbf{B}_{\mathrm{dR}}$. Furthermore, by taking G_K-invariants, we obtain

$$\left(H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}}\right)^{G_K} \cong H^n_{\mathrm{dR}}(X/K).$$

4.2.3 Étale to crystalline: the mysterious functor

There is even more structure involved once we reflect on the fact that if X admits a smooth $\mathcal{O}_K$ lifting $\mathfrak{X}$ (i.e., has good reduction), then the de Rham cohomology is given by the crystalline cohomology of the special fiber of $\mathfrak{X}$. Moreover, the crystalline cohomology carries a Frobenius action, something not seen by the étale cohomology alone because we are in the situation $l = p$.

Grothendieck conjectured the existence of a "mysterious functor" which would allow one to go from the étale cohomology to the crystalline cohomology. Fontaine reformulated this conjecture in terms of a comparison theorem involving another period ring $\mathbf{B}_{\mathrm{cris}}$ equipped with an action of G_K and a Frobenius-semilinear endomorphism with the following property.

Theorem 4.2.5 (C_{cris} theorem). *Suppose X/K has good reduction with a proper smooth $\mathfrak{X}/\mathcal{O}_K$. Then there is a canonical isomorphism*

$$H^n_{et}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{cris}} \cong H^n_{\mathrm{cris}}(\mathfrak{X}_k/W(k))[1/p] \otimes_{K_0} \mathbf{B}_{\mathrm{cris}}$$

compatible with G_K-actions, filtrations, and Frobenius actions.

The mysterious functor is none other than the fully faithful functor (requires extra work)

$$D_{\mathbf{B}_{\mathrm{cris}}} : \{\text{crystalline representations}\} \to \{\text{Filtered } K_0 \text{ vector spaces } + \text{ Frobenius.}\}$$

Finally, if we are interested in varieties with semistable reduction[2], there is yet another period ring $\mathbf{B}_{\mathrm{st}}$, satisfying $\mathbf{B}_{\mathrm{cris}} \subset \mathbf{B}_{\mathrm{st}} \subset \mathbf{B}_{\mathrm{dR}}$, that satisfies a suitable semistable comparison theorem. These have additional structure, namely a monodromy action.

[2] proper and flat model that is regular, generically smooth, with special fiber a reduced divisor with normal crossings

4.2.4 Recap: connection to representations

Let us explain the meaning of crystalline representations and put all these statements in the context of p-adic representations. We saw earlier that l-adic representations are reasonably well-understood. For p-adic representations $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, we classify them based off of whether they satisfy an appropriate comparison theorem when tensored with various period rings B. To be precise, define

$$D_B(V) := (V \otimes_{\mathbb{Q}_p} B)^{G_K}.$$

Definition 4.2.6 (B-admissibility). *We say that V is B-admissible if the natural morphism*

$$\alpha_V : D_B(V) \otimes_{B^{G_K}} B \to V \otimes_{\mathbb{Q}_p} B$$

is an isomorphism.

We can now call a representation Hodge-Tate if it is $\mathbf{B}_{\mathrm{HT}}$-admissible, etc. We have that the inclusions

$$\{\text{crystalline representations}\} \subset \{\text{de Rham representations}\} \subset \{\text{Hodge-Tate representations}\}.$$

The various comparison theorems say that various representations coming from geometry are crystalline, etc. In particular, if X has good reduction, then its associated representation is crystalline. Moreover, D_B respects the structures of the ring B. For example, when we apply $D_{\mathbf{B}_{\mathrm{cris}}}$, the filtration and Frobenius action on $\mathbf{B}_{\mathrm{cris}}$ are reflected by that on what we get, which is crystalline cohomology.

The semistable representations fit in too, along with potentially semistable representations, which are those that become semistable after a finite extension of the base field. We have

$$\{\text{crystalline}\} \subset \{\text{semistable}\} \subset \{\text{potentially semistable}\} = \{\text{de Rham}\} \subset \{\text{Hodge-Tate}\}.$$

The equality, saying that all de Rham representations (and thus all those coming from geometry) are potentially semistable, is a recently obtained deep theorem known as the p-adic monodromy theorem. It is an analogue of Grothendieck's monodromy theorem.

4.3 More on the period rings

4.3.1 $\mathbb{C}_p$ and $\mathbf{B}_{\mathrm{HT}}$

First, we revisit the definition of B-admissibility. A period ring $\mathbf{B}$ is a K-algebra that satisfies various properties – all the period rings mentioned so far will work. Taking $V \in \mathrm{Rep}_K(G_K)$ finite-dimensional and setting $W = B \otimes_K V$, we say that V is B-admissible if any of the equivalent conditions hold.

1. W is trivial; i.e. isomorphic to B^d as a G_K-representation.
2. The natural morphism $\alpha_W : B \otimes_{B^{G_K}} W^{G_K} \to W$ is an isomorphism.
3. $\dim_{B^{G_K}} W^{G_K} = \dim_K V$.

Then we have the following characterization of $\mathbb{C}_p$-admissible representations.

Theorem 4.3.1. *V is $\mathbb{C}_p$-admissible if and only if I_K acts on V through a finite quotient.*

Thus, $\mathbb{C}_p$-admissibility is equivalent to being potentially unramified. This theorem is not easy to prove, and in particular requires the Ax-Sen-Tate theorem, which we recall below.

Theorem 4.3.2 (Ax-Sen-Tate)**.** *We have* $\mathbb{C}_p^{G_K} = K$ *and* $\mathbb{C}_p(n)^{G_K} = 0$ *for* $n \neq 0$.

Actually, it uses the fact that $\mathbb{C}_p^{G_K} = K$, and can be used to prove that $\mathbb{C}_p(n)^{G_K} = 0$ for $n \neq 0$. Indeed, the cyclotomic character is infinitely ramified, as it includes the action of $K(\mu_{p^r})/K$ for all r, which comes from the inertia group.

Using this, we can also prove that Hodge-Tate weights are well-defined. Indeed, it suffices to show that

$$\dim \operatorname{Hom}_{G_K}(\mathbb{C}_p(m), \mathbb{C}_p(n)) = \delta_{mn}.$$

But $\operatorname{Hom}_{G_K}(\mathbb{C}_p(m), \mathbb{C}_p(n)) = \mathbb{C}_p(n-m)^{G_K}$, so we are done by the Ax-Sen-Tate theorem.

4.4 $\mathbf{B}_{\mathrm{dR}}, \mathbf{B}_{\mathrm{cris}}, \mathbf{B}_{\mathrm{st}}$

The construction is very complicated. See, we aren't messing around:

$$\begin{tikzcd}
 & & & \mathbf{B}_{\mu,K} \arrow[rr, hook] & & \mathbf{B}_{\mathrm{cris},K} \arrow[rr, hook] & & \mathbf{B}_{\mu,K} \arrow[rr, hook] & & \mathbf{B}_{\max,K} \arrow[r, hook] & \mathbf{B}_{\mathrm{dR}} \\
 & & \mathbf{B}_{\mu} \arrow[ur, hook] \arrow[rr, hook] & & \mathbf{B}_{\mathrm{cris}} \arrow[ur, hook] \arrow[rr, hook] & & \mathbf{B}_{\mu} \arrow[ur, hook] \arrow[rr, hook] & & \mathbf{B}_{\max} \arrow[ur, hook] & & \\
 & \mathbf{B}^+_{\mathrm{inf},K} & & \mathbf{B}^+_{\mu,K} \arrow[uu] \arrow[rr, hook] & & \mathbf{B}^+_{\mathrm{cris},K} \arrow[uu] \arrow[rr, hook] & & \mathbf{B}^+_{\mu,K} \arrow[uu] \arrow[rr, hook] & & \mathbf{B}^+_{\max,K} \arrow[uu] \arrow[r, hook] & \mathbf{B}^+_{\mathrm{dR}} \arrow[uu, hook] \\
\mathbf{B}^+_{\mathrm{inf}} \arrow[ur, hook] \arrow[rr, hook] & & \mathbf{B}^+_{\mu} \arrow[uu] \arrow[ur, hook] \arrow[rr, hook] & & \mathbf{B}^+_{\mathrm{cris}} \arrow[uu] \arrow[ur, hook] \arrow[rr, hook] & & \mathbf{B}^+_{\mu} \arrow[uu] \arrow[ur, hook] \arrow[rr, hook] & & \mathbf{B}^+_{\max} \arrow[uu] \arrow[ur, hook] & &
\end{tikzcd}$$

$(\mu > p-1)$ $(1 \leqslant \mu \leqslant p-1)$

(from Caruso's article *An Introduction to p-adic period rings*)

We will only describe $\mathbf{B}^+_{\mathrm{inf}}$, at the base of all these, which itself is built up from $\mathbf{A}_{\mathrm{inf}}$[3] via $\mathbf{B}^+_{\mathrm{inf}} = \mathbf{A}_{\mathrm{inf}}[1/p]$.

Note that the Frobenius is a ring homomorphism on $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p} \cong \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$. Then we let $\mathcal{R}$ be the projective limit of the system

$$\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p} \xrightarrow{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p} \xrightarrow{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p} \xrightarrow{x\mapsto x^p} \cdots$$

In other words, $\mathcal{R}$ is the **perfection** of $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$. THen we define $\mathbf{A}_{\mathrm{inf}}$ to be the Witt vectors $W(\mathcal{R})$.

[3] the one ring to rule them all

Part II

Ingredients of the proof

Chapter 5

A prototype result

In this chapter we will explain the proof of a prototype result, Proposition 5.1.1. This statement gives a general bound on the number of S-integer points of Y given various conditions. We do not make many assumptions about Y here; in particular we do not take it to be the Kodaira-Parshin family used for the proof of Faltings's theorem. However, the proof strategy and ingredients in this chapter will be present in the further applications of the Lawrence-Venkatesh approach.

Notation:

- K is a number field
- $G_K = \mathrm{Gal}(\overline{K}/K)$
- S is a finite set of places of K including the archimedean places
- $\mathcal{O} = \mathcal{O}_S$ is the ring of S-integers of K
- p a prime that does not divide any place of S
- K_w the completion of K at some $w \in \mathrm{Spec}\,\mathcal{O}$
- v some finite place of K such that K_v is unramified over $\mathbb{Q}_p$ and the rational prime p below v is not 2 and does not lie in S.

5.1 Basic strategy

Let $\pi : X \to Y$ be a proper smooth morphism over K. Suppose π extends to a proper smooth morphism $\pi : \mathcal{X} \to \mathcal{Y}$ of smooth $\mathcal{O}$-schemes. Then we have a natural inclusion $\mathcal{Y}(\mathcal{O}) \hookrightarrow Y(K)$. Given $y_0 \in \mathcal{Y}(\mathcal{O})$, we will use the Galois representation ρ on the p-adic étale cohomology of X_{y_0} to bound $|\mathcal{Y}(\mathcal{O})|$. Because there is a smooth model of Y, by definition Y has good reduction. By p-adic Hodge theory, this implies that this representation is *crystalline*. Then we can use p-adic Hodge theory again to translate crystalline representations into certain filtered vector spaces, which we can then analyze through the period map and the Gauss-Manin connection. We describe this in more detail in the following steps.

1. Prove that there are only finitely many isomorphism classes of semisimple representations $\rho : G_K \to \mathrm{GL}_d(\mathbb{Q}_p)$ unramified outside S that come from geometry (i.e. satisfy Weil conjectures). This was proven by Faltings as part of the original proof of the Mordell conjecture.

2. It now suffices to show that the map

$$y \mapsto \rho_{y,v} : G_{K_v} \hookrightarrow G_K \to \mathrm{Aut}\, H^q_{et}((X_y)_{\overline{K}}, \mathbb{Q}_p)$$

has finite fibers, where we chose an appropriate place v. Moreover, fixing some $y_0 \in \mathcal{Y}(\mathcal{O})$, we may restrict our consideration to the y satisfying $y \equiv y_0 \pmod v$.

3. The existence of a smooth proper model $\mathcal{Y}$ implies that implies that all the representations are crystalline, so by p-adic Hodge theory each y corresponds to a triple

$$(H^q_{\mathrm{dR}}(X_y/K_v), \phi_v, \Psi_v(y)).$$

Here ϕ_v is a Frobenius-semilinear automorphism acting on the de Rham cohomology and the third entry is the Hodge filtration.

4. Use the Gauss-Manin connection to conclude an isomorphism

$$H^q_{\mathrm{dR}}(\mathcal{X}_{y_0}/K_v) \cong H^q_{\mathrm{dR}}(\mathcal{X}_y/K_v)$$

for all $y \equiv y_0 \pmod v$. This isomorphism respects the associated Frobenius actions, but not the filtrations. A similar isomorphism holds in the complex case.

5. The filtration is determined by the period map. Show that the image of the period map is large; it's closure has dimension at least that of the monodromy $\Gamma \cdot h_0^\iota$.

6. Show that the set of possible filtrations is determined by the size of the centralizer of the Frobenius. If this is smaller than monodromy, then the set of possible filtrations is a proper Zariski closed subset of the period map.

Then we have the following general bound, which we will explain in more detail in the following sections.

Proposition 5.1.1. *With the notation above, suppose that*

$$\dim_{K_v}\left(Z(\phi_v^{[K_v:\mathbb{Q}_p]})\right) < \dim_{\mathbb{C}} \Gamma \cdot h_0^\iota.$$

Then the set

$$\{y \in Y(\mathcal{O}) | y \equiv y_0 \pmod v, \rho_y \textit{ semisimple}\}$$

is contained in a proper K_v-analytic subvariety of the residue disk of $Y(K_v)$ at y_0.

5.2 Finiteness of Galois representations

This theorem was proven by Faltings. As of now, a proof is not included here. I will probably add it when I revisit this.

5.3 The complex and p-adic Gauss-Manin connections

Recall that we have extended $\pi : X \to Y$ to a proper smooth morphism $\pi : \mathcal{X} \to \mathcal{Y}$. Recall that the relative de Rham cohomology sheaf is defined by $\mathcal{H}^q = R^q\pi_*\Omega^\bullet_{\mathcal{X}/\mathcal{Y}}$. By a result of Deligne (in his degeneration of the Leray spetral sequence paper), these are coherent and locally free over the generic point of $\mathcal{O}$. Enlarging S, we can assume they are locally free $\mathcal{O}_Y$-modules. Then by Katz-Oda, we obtain a Gauss-Manin connection

$$\nabla : \mathcal{H}^q \to \mathcal{H}^q \otimes_{\mathcal{O}_Y} \Omega^1_{\mathcal{Y}/\mathcal{O}}.$$

Fixing a basis $\{v_1, \ldots, v_r\}$ for $\mathcal{H}^q$ in a neighborhood of some $y_0 \in \mathcal{Y}(\mathcal{O})$, we may write $\nabla v_i = \sum_{j=1}^n A_{ij} v_j$ where A_{ij} are sections of $\Omega^1_{\mathcal{Y}}$. Recalling that $\nabla(f_i v_i) = d(f_i)v_i + f_i \nabla v_i$, we see that a local section $\sum f_i v_i$ is flat precisely when it satisfies the equation

$$d(f_i) = -\sum_j A_{ji} f_j.$$

We may write down a formal solution to this equation in power series, but we want it to converge. That is why we introduce a finite place v of K. We pick it so that if it lies over $p \in \operatorname{Spec} \mathbb{Z}$, then $p > 2$ and does not lie below anything in S, and $K_v/\mathbb{Q}_p$ is unramified. Denote the residue field of K_v by $\mathbb{F}_v$. Now consider the set

$$U := \{y \in \mathcal{Y}(\mathcal{O}) | y \equiv y_0 \pmod v\}.$$

One checks that the solutions to the Gauss-Manin equation are convergent in this case, and in fact we have an isomorphism

$$GM : H^q_{\mathrm{dR}}(\mathcal{X}_{y_0}/K_v) \cong H^q_{\mathrm{dR}}(\mathcal{X}_y/K_v)$$

for all $y \in \mathcal{Y}(\mathcal{O}_v)$ with $y \equiv y_0 \pmod v$ (call this set Ω_v). Similarly, there is an isomorphism in the complex case for $y \in Y(\mathbb{C})$ sufficiently close to y_0.

Finally, by results on crystalline cohomology we also have an isomorphism to the crystalline cohomology, which comes with a semilinear Frobenius action. The Gauss-Manin connections is compatible with the Frobenius action.

5.4 Bounding period mappings with monodromy

Recall that the goal was to show that the $y \in U$ that give rise to isomorphic representations, or filtered vector spaces with Frobenius, is small. Through the Gauss-Manin connection, we know they all have isomorphic de Rham cohomologies with Frobenius, so it remains to analyze their Hodge filtrations. The period mapping sends each Hodge filtration to a flag:

$$\Phi_{\mathbb{C}} : \Omega_{\mathbb{C}} \to \mathcal{H}_{\mathbb{C}}(\mathbb{C}) \quad \text{and} \quad \Phi_v : \Omega_v \to \mathcal{H}(K_v).$$

We are interested in analyzing the image of the p-adic period map. In particular, we are interested in first bounding it below by the image of some monodromy representation. The filtrations equivalent to that of y_0 come from the centralizer of the Frobenius ϕ_v. Thus if the dimension of this centralizer is less than that of monodromy, it cannot be Zariski dense in the entire image of the period map.

We begin by bounding the image of the period map below by monodromy. In the complex case, the period map extends to a map

$$\Phi_{\mathbb{C}} : \widetilde{Y_{\mathbb{C}}} \to \mathcal{H}_{\mathbb{C}}(\mathbb{C})$$

that is equivariant for the monodromy action of $\pi_1(Y_{\mathbb{C}})$ on $H_{\mathbb{C}}(\mathbb{C})$. In some sense it 'contains' the monodromy action, since monodromy is essentially just restriction to the points of $\widetilde{Y_{\mathbb{C}}}$ that project back down to y_0. Let Γ be the Zariski closure of the image of the monodromy representation. Fix the initial flag $h_0 = \Phi_{\mathbb{C}}(y_0)$. Then we have that

$$\Gamma \cdot h_0 \subset \overline{\Phi_{\mathbb{C}}(\Omega_{\mathbb{C}})}$$

where on the right we are taking the Zariski closure inside $\mathcal{H}_{\mathbb{C}}$.

The key to relating the p-adic version to this comes from the fact that the coefficients of the Gauss-Manin equation are defined over K, and the following lemma.

Lemma 5.4.1. *Suppose $B_0, \dots B_N \in K[[z_1, \dots, z_m]]$ are absolutely convergent power series with on common zero in both v-adic and complex discs U_v and $U_{\mathbb{C}}$. If $Z \subset \mathbb{P}^N_K$ is the subscheme cut out by all polynomials killing $(B_0, \dots, B_n)$, then the base-extension of Z to K_v and $\mathbb{C}$ gives the Zariski closures of $B(U_v)$ and $B(U_{\mathbb{C}})$ respectively.*

Apply this to the power series of the Gauss-Manin connection. This implies that the dimension of the Zariski closure of $\Phi_v(\Omega_v)$ in $\mathcal{H}_{K_v}$ is at least the complex dimension of $\Gamma \cdot h_0$. In particular, if we have a subset $\mathcal{H}_v^{\mathrm{bad}} \subset \mathcal{H}_v$ of dimension less than that of monodromy, then $\Phi_v^{-1}(\mathcal{H}_v^{\mathrm{bad}})$ is contained in a proper K_v-analytic subset of Ω_v.

We now recall the desired statement.

Proposition 5.4.2. *With the notation above, suppose that*

$$\dim_{K_v}\left(Z(\phi_v^{[K_v:\mathbb{Q}_p]})\right) < \dim_{\mathbb{C}} \Gamma \cdot h_0^{\iota}.$$

Then the set

$$\{y \in Y(\mathcal{O}) | y \equiv y_0 \pmod{v}, \rho_y \textit{ semisimple}\}$$

is contained in a proper K_v-analytic subvariety of the residue disk of $Y(K_v)$ at y_0.

Proof. In light of the discussion above, the set of interest can be identified with triples $(V_v, \phi_v, \Phi_v(y))$, which are all isomorphic to one of a finite number of representatives (V_v, ϕ_v, h_i). This means that each such y must satisfy

$$\Phi_v(y) \in \bigcup_i Z(\phi_v) \cdot h_i.$$

But we have $Z(\phi_v) \subset Z(\phi_v^{[K_v:\mathbb{Q}_p]}) \subset \mathrm{Aut}_{K_v}(V_v)$. Since $\phi_v^{[K_v:\mathbb{Q}_p]}$ is a Zariski closed subset, the earlier results apply to the given condition and give that the preimage lies in a proper K_v analytic subvariety, as desired. $\square$

Chapter 6

The S-unit equation

In this chapter we use our tools to prove a non-trivial theorem: the S-unit equation has finitely many solutions. The proof is more involved than the prototype result of the previous chapter, but still significantly simpler than the proof of Mordell conjecture. It can be thought of as a proof-of-concept for the Lawrence-Venkatesh method.

6.1 Statement and initial reductions

We will explain a proof of the following theorem.

Theorem 6.1.1. *The set*

$$U = \{t \in \mathcal{O}_S^* | 1 - t \in \mathcal{O}_S^*\}$$

is finite.

We begin by enlarging S and K so that S contains the primes above 2 and K contains μ_8. It suffices to show the finiteness of U_1, where $U_1 \subset U$ consists of non-squares. Let m be the largest power of 2 dividing the order of the roots of unity in K. It suffices to check finiteness for t not a square. Now there are only finitely many choices for $K(t^{1/m})$ (??? Kummer + Hermite-Minkwoski?), so in fact we can fix some cyclic degree m extension L and restrict our attention to proving the finiteness of

$$U_{1,L} = \{t \in U_1, K(t^{1/m}) \cong L\}.$$

After choosing an appropriate prime $v \in \operatorname{Spec} \mathcal{O}_K$, we restrict even more to the $t \equiv t_0 \pmod{v}$.

6.2 The chosen family, a variant of Legendre

The family we use is the composite

$$\mathcal{X} \to \mathcal{Y}' \xrightarrow{\pi} \mathcal{Y}$$

where $\mathcal{Y}' = \mathbb{P}^1_{\mathcal{O}} - \{0, \mu_m, \infty\}$, $\mathcal{Y} = \mathbb{P}^1_{\mathcal{O}} - \{0, 1, \infty\}, \pi$ is the map $u \mapsto u^m$, and $\mathcal{X} \to \mathcal{Y}'$ is the Legendre family. In particular, the geometric fiber X_t of $t \in Y(K)$ is the disjoint union of the curves $y^2 = x(x-1)(x - t^{1/m})$.

A point that we glossed over (??? fix this up!) is that when we pass to local Galois representations, we need to make sure ρ_y is semisimple first before going to $\rho_{y,v}$. In this case, we can show that there are only finitely many solutions $(t, 1-t)$ to the S-unit equation where the corresponding Tate module fails to be simple. This is a separate argument from the rest of the proof, and we do not give it here (for now ???).

Some comments on why we chose this family.

6.3 Reduction to big monodromy

We now run our machine: p-adic Hodge theory to send the Galois representations associated to points to filtered vector spaces with Frobenius, Gauss-Manin to identify fibers, and the period mapping to analyze when two fibers are isomorphic.

Fix $t_0 \in U_{1,L}$. We want to show the finiteness of $t \in U_{1,L}$ with $t \equiv t_0 \pmod v$. The corresponding Galois representation ρ_t on $H^1_{\mathrm{dR}}(X_{t,K_v}/K_v)$ is a 2-dimensional vector space over $K_v(t^{1/m})$ and a $2d$-dimensional vector space over K_v. The Gauss-Manin connection for $X \to Y$ gives

$$H^1_{\mathrm{dR}}(X_{t,K_v}/K_v) \cong H^1_{\mathrm{dR}}(X_{t_0,K_v}/K_v)$$

respecting the Frobenius action, as usual. The period mapping sends t to a $K_v(t^{1/m})$-line, or an m-dimensional K_v-subspace in $H^1_{\mathrm{dR}}(X_{t,K_v}/K_v)$. We are interested in the dimension of the orbit $Z(\phi_v) \cdot h_i$. As before, this is contained in $Z(\phi_v^{[K_v:\mathbb{Q}_p]})$. By some general theory (Lemma 2.1 in LV), we have

$$\dim_{K_v} Z \leqslant (\dim_{K_v(t_0^{1/m})} H^1_{\mathrm{dR}})^2 = 4.$$

In particular, the set of $t \in U_{1,L}$ congruent to $t_0 \pmod v$ is contained in $\Phi^{-1}(Z)$, where $Z \subset \mathrm{Gr}_{K_v}(2m, m)$ has dimension at most 4. Then by the results of the previous chapter, it suffices to show that the dimension of the orbit of the complex monodromy is greater than 4.

6.4 Big monodromy

The complex monodromy action is the following. The fiber over $t \in \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$ is the union of the elliptic curves E_z, with $z^m = t$. Then the monodromy representation is a map

$$\pi_1(\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}, t_0) \to \mathrm{Aut}\left(\bigoplus_{z^m = t_0} H^1_B(E_z, \mathbb{Q})\right).$$

Lemma 6.4.1. *The Zariski closure of the image of monodromy Γ contains $\prod_z SL(H^1_B(E_z, \mathbb{Q}))$.*

Proof. □

The corresponding orbit then contains all of $\prod_{i=1}^m \mathbb{P}V_i$, which has dimension $m \geqslant 8 > 4$ as desired.

Chapter 7

Proof of Faltings's theorem modulo facts about the Kodaira-Parshin family

7.1 Paradigm of the proof

We use a similar paradigm as in the previous chapter, but the details are much more complicated. The families we will consider $X \to Y$ will be abelian-by-finite families, defined below.

Definition 7.1.1. *An abelian-by-finite family $X \to Y$ is one that factors as*

$$X \to Y' \xrightarrow{\pi} Y,$$

where π is finite étale and $X \to Y'$ is a polarized abelian scheme.

We would also like there to be a smooth model over some S-integers $\mathcal{O} \subset K$, given by $\mathcal{X} \to \mathcal{Y}' \to \mathcal{Y}$. For each point $y \in Y_K$, we consider cohomology of the fiber $H^i_{et}((X_y)_{\overline{K}}, \mathbb{Q}_p)$ along with its action of G_K. The existence of the smooth model implies that such representations are crystalline, so they are associated to triples $(H^q_{\mathrm{dR}}(X_y/K_v), \phi_v, \Psi_v(y))$ where ϕ_v is a semilinear Frobenius action on $H^q_{\mathrm{dR}}(X_y/K_v)$ and $\Psi_v(y)$ is the Hodge filtration. If $y \equiv y_0 \pmod{v}$, then the first entries for y and y_0 are identified via the Gauss-Manin connection. If two such triples are isomorphic, they must differ by some element in the centralizer of ϕ. We would like to bound this from above so that we have finite fibers from points to isomorphism classes of Galois representations, the latter of which has finite cardinality by a result of Faltings.

If $k(y) = E$, then the Frobenius acts on E/K_v, and we have $\dim_{K_v}(Z(\phi)) = \dim_E Z(\phi^e) \leqslant \dim_E(H^q_{\mathrm{dR}}(X_y/K_v))^2$. The upshot is that we want the residue fields of the y to be large. This is the purpose of the abelian-by-finite families: π ensures that the Galois orbits on its fibers are large. This can be quantified in the following way.

Definition 7.1.2. *Let G_K act on E and let v be a place of K at which the action is unramified. Let*

$$\mathrm{size}_v(E) = \frac{\textit{number of elements of } E \textit{ that belong to } \mathrm{Frob}_v\textit{-orbits of size } < 8}{\textit{number of elements of } E}.$$

If $E \to E'$ is a morphism of G_K sets with all fibers having the same cardinality, then $\mathrm{size}_v(E) \leqslant \mathrm{size}_v(E')$.

7.2 Monodromy of abelian-by-finite families

Generally speaking, we want to show that

$$\text{centralizer} < \text{monodromy} \leqslant \text{image of period map}$$

We can then conclude that bad points, corresponding to the centralizer, are contained in a lower dimensional subset of Y. The second inequality has already been shown, so what we need to do is show that the centralizer is small and monodromy is large. We make the second statement precise in this section.

Begin with an abelian-by-finite family $X \to Y' \to Y$ and take a complex point $y_0 \in Y(\mathbb{C})$ and consider the action of $\pi_1(Y(\mathbb{C}), y_0)$ on

$$H^1_B(X_{y_0}, \mathbb{Q}) \cong \bigoplus_{\pi(\tilde{y})=y_0} H^1_B(X_{\tilde{y}}, \mathbb{Q}).$$

Definition 7.2.1. *We say that the family has* ***full monodromy*** *if the Zariski closure of the image of $\pi_1(Y, y_0)$ under this representation contains the following product of symmetric groups:*

$$\overline{(\textit{image of } \pi_1(Y(\mathbb{C}), y_0))} \supset \prod_{\pi(\tilde{y})=y_0} \mathrm{Sp}(H^1_B(X_{\tilde{y}}, \mathbb{Q}), \omega)$$

where the symplectic group is with reference to the form ω defined by the polarization.

Let us briefly explain this last part. We recall some definitions, taken from Milne's notes on abelian varieties.

The dual abelian variety, also known as the Picard variety, is an abelian variety $A^\vee$ that parametrizes the elements of $\mathrm{Pic}^0(A)$. Let us give a proper definition.

Definition 7.2.2. *Let $(A^\vee, \mathcal{P})$ be a pair where $\mathcal{P}$ is an invertible sheaf on $A \times A^\vee$. Assume that $\mathcal{P}|_{A\times\{b\}} \in \mathrm{Pic}^0(A_b)$ and $\mathcal{P}|_{\{0\}\times A^\vee}$ is trivial. Then $A^\vee$ is the* ***dual abelian variety*** *of A and $\mathcal{P}$ is the Poincaré sheaf if $(A^\vee, \mathcal{P})$ satisfies the following universal property. For every other such pair $(T, \mathcal{L})$, there is a unique regular map $\alpha : T \to A$ such that $(1 \times \alpha)^*\mathcal{P} \cong \mathcal{L}$.*

In more conceptual terms, $(A^\vee, \mathcal{P})$ represents the functor sending a variety T to the set of line bundles on A parameterized by T.

The construction of the dual abelian variety is a special case of the construction of the Picard scheme, which was famously done by Grothendieck. However, even this special case is rather involved.

Definition 7.2.3. *A* ***polarization*** *λ of an abelian variety is an isogeny $A \to A^\vee$ such that, over $\overline{k}$, we have that λ becomes of the form $\lambda_{\mathcal{L}}$ for some ample sheaf $\mathcal{L}$ on $A_{\overline{k}}$. If the degree of a polarization is 1, then λ is called a* ***principal polarization.***

Recall that $\lambda_{\mathcal{L}} : A(k) \to \mathrm{Pic}(A)$ is defined by $\lambda_{\mathcal{L}}(a) = t_a^*\mathcal{L} \otimes \mathcal{L}^{-1}$. In fact, $\mathrm{Pic}^0(A)$ may be defined as those $\mathcal{L}$ for which $\lambda_{\mathcal{L}} = 0$.

But this may not be too helpful for our purposes. Over $\mathbb{C}$, we can describe polarizations as follows. Let $L \subset C^n$ be a lattice. Then the torus $\mathbb{C}^n/L$ is an abelian variety if it has a Riemann form H, which is a positive definite Hermitian form on V such that $E = \mathrm{im}\, H$ is integer valued on L. In such a case, we have

$$H(u, v) = E(iu, v) + iE(u, v)$$

where E is a symplectic form.

Remark. A principal polarization is when E can be represented as the standard symplectic matrix with I and $-I$.

Remark. One can see where this form comes from by looking at Jacobians of curves over $\mathbb{C}$. One has the intersection pairing on the first homology, and one obtains the dual pairing on the first cohomology by the cup product. The Hermitian form and polarization then follow from integrating. ???

In our setting, we have a symplectic form ω defined by the polarization of the abelian variety $X_{\tilde{y}}$. Then after fixing a basis, $\mathrm{Sp}(V, \omega)$ is given by the matrices M acting on V with $V^\intercal \omega M = \omega$. We will explain why the image of monodromy lies in this symplectic group in a future section. ???

7.3 Properties of the Kodaira-Parshin family

In future sections, we will construct a specific abelian-by-finite family $X_q \to Y_q' \xrightarrow{\pi} Y$ for each prime $q \geqslant 3$. This is called the Kodaira-Parshin family for the group $\mathrm{Aff}(q)$. Let us list the properties we need.

1. It has full monodromy.
2. The relative dimension d_q of $X_q \to Y_q'$ is given by $d_q = (q-1)(g - \frac{1}{2})$.
3. For each $y \in Y(K)$ there is a G_K-equivariant identification of $\pi^{-1}(y_0)$ with the conjugacy classes of surjections $\pi_1^{et}(Y_{\mathbb{C}} - y, *) \twoheadrightarrow \mathrm{Aff}(q)$ that are nontrivial on a loop around y.

These will be proven in future sections, but let us explain what the third means.

Definition 7.3.1. *Let* $\mathrm{Aff}(q) \subseteq \mathrm{Sym}(\mathbb{F}_q)$ *be the subgroup of linear permutations of* $\mathbb{F}_q$*, defined by* $x \mapsto ax + b$ *where* $a \in \mathbb{F}_q^*, b \in \mathbb{F}_q$.

Then we have an exact sequence

$$0 \to \mathbb{F}_q^+ \to \mathrm{Aff}(q) \to \mathbb{F}_q^* \to 0,$$

and we have $\mathrm{Aff}(q) \cong \mathbb{F}_q^+ \rtimes \mathbb{F}_q^*$.

Now we can view $\pi_1^{et}(Y_{\mathbb{C}} - y_0, *)$ as the profinite completion of the free group on $x_1, x_1', \ldots, x_g, x_g'$ with a loop around y_0 corresponding to the conjugacy class of $[x_1, x_1'][x_2, x_2'] \cdots [x_g, x_g']$. Therefore, the set of surjections $\pi_1^{et}(Y_{\mathbb{C}} - y, *) \twoheadrightarrow \mathrm{Aff}(q)$ are identified with the following set.

Let $f : \mathrm{Aff}(q)^{2g} \to \mathbb{F}_q^+$ be given by

$$f(y) = f(y_1, y_1', \cdots, y_g, y_g') = [y_1, y_1'] \cdots [y_g, y_g'].$$

Now consider the set

$$T = \{y | f(y) \neq 0, y \text{ generates } \mathrm{Aff}(q)\}.$$

There is a map $g : T \to [\mathbb{F}_q^*]^{2s}$ given by sending each coordinate $ax + b \mapsto a$. The image of g consists of those $(2g)$-tuples of $\mathbb{F}_q^*$ which generate $\mathbb{F}_q^*$, and it is easily shown that any point in the image has fiber of size $q^{2g-1}(q-1)$.

7.4 Proof of Faltings's theorem

Assuming the results stated in the previous section, we can now prove Faltings's theorem.

Theorem 7.4.1. *Let Y be a curve over the number field K with genus $g \geqslant 2$. Then $Y(K)$ is finite.*

We outline the proof in the following steps.

1. Reduce to showing that all y have fibers of size bounded above by $\frac{1}{d_q+1}$. That is, prove that the cardinality of such points is finite.
2. Use algebraic number theory to choose appropriate q and v. Form the associated Kodaira-Parshin family.
3. Obtain an appropriate map of G_K-sets

$$\pi^{-1}(y) \to H^1_{et}(Y_{\overline{K}}, \mathbb{Z}/(q-1)) = M.$$

   It suffices to show that size_v of the image I is less than $\frac{1}{d_q+1}$.
4. Use the perfect Weil pairing on M and the Frobenius at v to bound the number of elements of M belonging to Frobenius orbits of size less than 8.
5. Use basic inequalities to bound $\text{size}_v(I)$ given the previous result.

As usual, let Y be a curve over K of genus $g \geqslant 2$. Let $X \to Y' \xrightarrow{\pi} Y$ be an abelian-by-finite family over Y, with full monodromy. Let d be the relative dimension of $X \to Y'$. Suppose that $X \to Y' \xrightarrow{\pi} Y$ admits a good model over the ring $\mathcal{O}$ of S-integers of K. Let $v \notin S$ be a friendly place of K.

Step 1. This is the following proposition.

Proposition 7.4.2 (Proposition 5.3). *The set*

$$Y(K)^* := \{y \in Y(K) | \, \text{size}_v(\pi^{-1}(y)) < \frac{1}{d+1}\}$$

is finite.

This proof is all of Section 6 in the Lawrence-Venkatesh paper. It is a more difficult version of the prototype result proved earlier. We will not prove it here. ??? Assuming this, we need to chose q and v in such a way that v is friendly and

$$\text{size}_v(\pi^{-1}(y)) < \frac{1}{d+1}$$

for all $y \in Y(K)$.

Step 2. Let us state the precise conditions we want.

1. $q-1$ is not divisible by 4 or by any or primes less than $8[K:\mathbb{Q}]$.
2. The Galois closure k' of K is linearly disjoint from $\mathbb{Q}(\zeta_{q-1})$ over $\mathbb{Q}$.
3. $\frac{8 \cdot 2^{g+1}}{(q-1)^g} < \frac{1}{(g-1/2)(q-1)+1}$.

and

1. v is friendly.
2. $(q_v, q-1) = 1$ (recall that q_v is the cardinality of the residue field at v).

3. For any odd prime factor r of $q-1$, the class of q_v in $(\mathbb{Z}/r)^*$ has order at least 8.

For q, note that for the second condition, if there is some common subfield greater than $\mathbb{Q}$, then a prime that ramifies in it must divide both $q-1$ and the discriminant. Thus the first and second conditions can be satisfied by some admissible congruence condition. Condition 3 is satisfied when q is sufficiently large. Thus Dirichlet's theorem suffices. For v, it is a bit harder but the Chebotarev density theorem suffices.

Step 3. The desired map is simply the one we described at the end of the previous section. Indeed, for $y \in Y(K)$, property 3 of the Kodaira-Parshin family identifies $\pi^{-1}(y)$ with the set

$$T = \{y | f(y) \neq 0, y \text{ generates } \mathrm{Aff}(q)\}.$$

Moreover, the surjection $\mathrm{Aff}(q) \to \mathbb{F}_q^*$ gives rise to the map $g : T \to (\mathbb{F}_q^*)^{2s} \cong M := H^1_{et}(Y_{\overline{K}}, \mathbb{Z}/(q-1))$.
This can also be written as the composite

$$T : \pi^{-1}(y) \to H^1_{et}(Y_{\overline{K}}, \mathrm{Aff}(q)) \to H^1_{et}(Y_{\overline{K}}, \mathbb{Z}/(q-1)),$$

and the key is that this is a G_K-equivariant map. Its image I also has fibers of the same size, so as noted in the first section of this chapter, we have $\mathrm{size}_v(\pi^{-1}(y)) \leqslant \mathrm{size}_v(I)$, so we just need to show that $\mathrm{size}_v(I) \leqslant \frac{1}{d_q+1}$.

Step 4. To show that $\mathrm{size}_v(I) \leqslant \frac{1}{d_q+1}$, we will first show that the Galois orbits on M are large. In the following step we will pass to I. We use the fact that M is equipped with a Galois-equivariant Weil pairing that is perfect.

$$\langle -, - \rangle : M \times M \to \mu_{q-1}^\vee := \mathrm{Hom}(\mu_{q-1}, \mathbb{Z}/(q-1)\mathbb{Z}).$$

Recall that for elliptic curves, we can define this Weil pairing as follows. ???
Since this pairing is Galois-equivariant, if we take $T : M \to M$ to be the map on M coming from the Frobenius at v, we have

$$\langle Tv_1, Tv_2 \rangle = q_v^{-1} \langle v_1, v_2 \rangle$$

since the Frobenius acts on μ_{q-1} by raising to the q_v power.

We would like to bound the size of $\bigcup_{i=1}^8 \ker(T^i - 1)$. Note that if $m_1, m_2 \in \ker(T^i - 1)$, then $(q_v^{-i} - 1)\langle m_1, m_2 \rangle = 0 \Rightarrow 2\langle m_1, m_2 \rangle = 0$ by our choices of q and v. However, in general if we have a nondegenerate pairing $A \times A \to \mathbb{Q}/\mathbb{Z}$ with a subgroup $B \subset A$ satisfying $\langle B, B \rangle = 0$, then $|B| \leqslant \sqrt{|A|}$. Applying this to $A = 2M$, we get

$$|2\ker(T^i - 1)| \leqslant \left(\frac{q-1}{2}\right)^g \Rightarrow |\bigcup_{i=1}^8 \ker(T^i - 1)| \leqslant 8 \cdot 2^g (q-1)^g.$$

This is the desired bound for the number of elements of M with Frobenius orbits of size at most 8.

Step 5. The quantity $\mathrm{size}_v(I)$ can now be bounded using the above bound and the fact that I consists of most of M. Recall that I is the image of $g(T)$ in $(\mathbb{F}_q^*)^{2s}$, which consists of all generating $(2g)$-tuples in $\mathbb{Z}/(q-1)$. This number is just

$$(q-1)^{2g} \prod_{p|N} (1 - p^{-2g}) \geqslant \frac{1}{2}(q-1)^{2g}.$$

In total, we have seen there are at most $8 \cdot 2^g (q-1)^g$ elements with Frobenius orbits of size at most 8. Thus, we finally have

$$\mathrm{size}_v(\pi^{-1}(y)) \leqslant \mathrm{size}_v(I) \leqslant \frac{8 \cdot 2^g (q-1)^g}{\frac{1}{2}(q-1)^{2g}} < \frac{1}{(g-1/2)(q-1)+1} = \frac{1}{(g-1/2)(q-1)+1},$$

as desired. Note that the last inequality not only uses the fact that we choose q large (property 3), but also that $g > 1$.

Part III

The Kodaira-Parshin family

Chapter 8

Construction of the Kodaira-Parshin family

The Kodaira-Parshin family was originally used to reduce the Mordell conjecture to the Shafarevich conjecture. For each point $P \in Y(K)$, it gives a finite morphism of curves $Y_P \to Y$ ramified only at P with appropriate good reduction properties. We will construct a variant of this family. As described in the previous section, we want to construct an abelian-by-finite family

$$X_q \to Y_q' \to Y,$$

where as usual Y/K is a curve of genus at least 2 and q is some prime. Briefly, the morphism $Y_q' \to Y_q$ will be a Hurwitz space with group $\mathrm{Aff}(q)$ for Y, and $X_q \to Y_q'$ will be the reduced relative Prym of the associated universal $\mathrm{Aff}(q)$-cover $Z \to Y_q' \times Y$.

8.1 Branched covers of $\mathbb{P}^1$

There are two classical Hurwitz functors:

$$\mathcal{H}_{n,r}(S) = \{f : X \to \mathbb{P}^1_S | \deg f = n;\ f \text{ simple with } r \text{ branch points}\}$$

and

$$\mathcal{H}_{r,G}(S) = \{f : X \to \mathbb{P}^1_S | \ f \text{ Galois with group } G \text{ and has } r \text{ branch points}\}$$

equipped with an isomorphism $\tau : \mathrm{Aut}_{\mathbb{P}^1_{\mathbb{C}}}(X) \cong G$.

There are many possible variants, and it seems that not too much is known. (!) In fact, for the Kodaira-Parshin family be interested in the case not of the projective line, but in the case of our curve Y. Still, for curiosity's sake, we include here a digression on branched covers of $\mathbb{P}^1$.

8.1.1 Dessins d'enfants

Let us explain what dessins d'enfants are and how they relate to algebraic curves.

Definition 8.1.1. *A **dessin d'enfant** is a triple $X_0 \subset X_1 \subset X_2$ where X_2 is some real surface, X_0 is a finite set of vertices, $X_1 \backslash X_0$ is a finite set of disjoint segments, and $X_2 \backslash X_1$ is a finite set of disjoint open cells. Furthermore, as a graph the dessin must be bipartite.*

*A **clean dessin** is one where we may alternately label the vertices with 0 and 1 such that each vertex labeled 1 has degree 2.*

The connection to algebraic curves defined over $\overline{\mathbb{Q}}$ is given by Belyi's theorem.

Theorem 8.1.2 (Belyi). *A complex algebraic curve may be defined over $\overline{\mathbb{Q}}$ if and only if it admits a finite morphism to $\mathbb{P}^1_{\mathbb{C}}$ unramified outside $\{0, 1, \infty\}$.*

Remark. Technically, Belyi's theorem is the 'only if' direction. The 'if' direction actually requires more work to prove, but it was known before the 'only if' direction.

So, let us begin with an algebraic curve X with some $f : X \to \mathbb{P}^1_{\mathbb{C}}$ unramified outside $\{0, 1, \infty\}$. We know by Belyi's theorem that it may be defined over $\overline{\mathbb{Q}}$, and thus there is an interesting action of $G_{\mathbb{Q}}$ on it. This is a crucial point used in Grothendieck's theory of dessins[1], but here we will not go into it and instead focus on more basic combinatorial aspects. Consider drawing the preimage of the line segment $[0, 1]$ on X. Then we get a dessin d'enfant, with the preimages of ∞ corresponding to the faces which X is subdivided into.

Now let's say we want to go in the other direction: from the dessin to the algebraic curve. Note that we can, for example, find the degree of f by the Riemann-Hurwitz formula. To get the curve though, we first pick an edge $E \in X_1$ and consider the monodromy action on it. Indeed, viewing $X \backslash X_0$ as a covering space of $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$, fix a basis for $\pi_1(\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\})$ using a loop l_0 around 0 and a loop l_1 around 1. In fact, we want to use a 'tangential base point' from 0 to 1, which works since the interval is contractible. Then l_0 corresponds to rotation of 0 counter-clockwise around 1 and l_1 corresponds to rotation of 1 counter-clockwise around 0. Let $N \leqslant G$ be the stabilizer of E. This is well-defined up to conjugacy, and by the correspondence between covering spaces and conjugacy classes of subgroups of the fundamental group, we obtain the covering $X \backslash X_0 \to \mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$. Thus we obtain a bijection between isomorphism classes of finite coverings branched only at $\{0, 1, \infty\}$ and isomorphism classes of dessins.

We are not too interested in what structure map $f : X \to \mathbb{P}^1_{\mathbb{C}}$ is used. Therefore we may compose f with a function, say $z \mapsto 4z(1 - z)$, to ensure that the ramification indices over 1 are all 2, which corresponds to the vertices labeled 1 all having degree 2. These are the clean dessins. The purpose of studying clean dessins is to relate them to polyhedral combinatorics. Indeed, if you draw any dessin on a surface forgetting the bipartite condition, we can label all the vertices 0, and then bisect the edges and label them 1. This gives a clean dessin! With the same process described above, we obtain a bijection between isomorphism classes of clean dessins and isomorphism classes of clean Belyi pairs (i.e. where the covering has ramification indices 2 over the point 1), with the knowledge that any algebraic curve over $\overline{\mathbb{Q}}$ is part of a clean Belyi pair.

8.1.2 Polyhedral combinatorics

Take some graph G embedded on some real surface X. We call the resulting configuration (X, G) a **map**. A map has faces, edges, and vertices. Define a **flag** of (X, G) to be a choice of a face, an edge of that face, and a vertex of that edge. Now whatever (X, G) we take, there is

[1]In fact, the birth of anabelian geometry can be traced back to this!

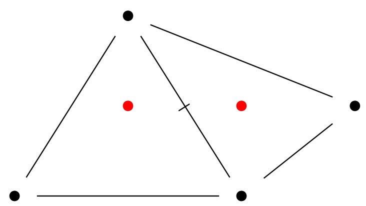
going to be a transitive action on the set of flags by the **cartographic group**[2]

$$\underline{C}_2 = \langle \sigma_0, \sigma_1, \sigma_2 | \sigma_0^2 = \sigma_1^2 = \sigma_2^2 = (\sigma_0\sigma_2)^2 = 1 \rangle.$$

These operations $\sigma_0, \sigma_1, \sigma_2$ correspond to the reflection of the chosen vertex, edge, and face, respectively. There is also an action of the **oriented cartographic group**

$$\underline{C}_2^+ = \langle \rho_v, \rho_f, \rho_e | \rho_2\rho_0 = \rho_1, \rho_1^2 = 1 \rangle$$

where $\rho_0 = \sigma_1 \circ \sigma_2, \rho_1 = \sigma_0 \circ \sigma_2, \rho_2 = \sigma_0 \circ \sigma_1$. Thus, ρ_0, ρ_1, ρ_2 correspond to the rotation of the flag around the vertex, edge, and face, respectively. Draw a picture!



An alternate way of presenting this group is by the relations $\rho_0\rho_1\rho_2 = 1, \rho_1^2 = 1$. We see that this is precisely the fundamental group of $\mathbb{P}^1$ minus 3 points quotiented by $l_1^2 = 1$, so these polyhedra correspond precisely to he algebraic curves described earlier.

8.1.3 Regular polyhedra

Now when do we get a regular polyhedron? Precisely when its automorphism group acts transitively on its flags. Note that this is equivalent to our structure map $f : X \to \mathbb{P}^1_{\mathbb{C}}$ being a Galois covering! We see that every pair of integers $p, q \geqslant 1$ gives rise to a unique connected map by imposing the additional relations

$$\rho_0^p = \rho_2^q = 1$$

on its automorphism group. We see that, after pinning down a flag, this automorphism group determines the polyhedron. In particular, p is the number of faces to a vertex and q is the number of edges to a face. Immediately we see hat not all the regular polyhedra we get in this way are Platonic solids. Rather, only the compact ones are; i.e. those realizable on a sphere. These are the ones with finite automorphism group. The others give regular tilings of either the Euclidean plane or the hyperbolic plane. In fact, this approach leads to an easy classification of Platonic solids! Indeed, one just needs the sum of the angles

$$\frac{\pi(p-2)q}{p} < 2\pi$$

to get a Platonic solid. If it is equal to 2π then we get a tiling of the Euclidean plane and if it is greater than 2π we get a tiling of the hyperbolic plane.

Alternatively, we could have calculated this with the Riemann-Hurwitz formula. If there are b branch points, then we have

$$2d - 2 = \sum (e_y - 1) = bd - |\text{ramification points}| = bd - \sum \frac{d}{e_y}.$$

We thus easily obtain the ramification indices and the degree, but not the explicit realizations from the previous approach. Note that another way is to find the finite subgroups of $PGL(2, \mathbb{C})$,

[2]Beware, my conventions differ slightly from Grothendieck's in the *Esquisse*; here I consider the elements as operators so I multiply in the opposite direction.

which can be identified with the finite subgroups of $SO(3, \mathbb{R})$.

As noted by Grothendieck, it is important not to confine ourselves to the cases where p and q are finite. Indeed, by pondering the universal formulas that are used to define the reflections, we not only get new regular polyhedra but also obtain a method for specializing to characteristic p in a meaningful way! Or, even considering regular polyhedra over any base ring. Indeed, **the formulas for the fundamental reflections σ_i can be written in terms of universal formulae in terms of the cosines of the angles of the polyhedron!**

Fixing a flag v_0, v_1, v_2, we have

$$\begin{aligned}
\sigma_0(v_0) &= 2v_1 - v_0, \\
\sigma_1(v_1) &= (1 - \cos\theta)v_0 - v_1 + (1 + \cos\theta)v_2, \\
\sigma_2(v_2) &= (1 - \cos\gamma)v_1 - v_2.
\end{aligned}$$

The data of the polyhedron is completely contained in these values of $\cos\theta$ and $\cos\gamma$. Thus, taking any base field, we may substitute any pair of values for them and obtain a *regular polyhedron*! Note that there will be many of these which all correspond to $p = q = \infty$. In particular, we may *specialize* from the field $\mathbb{R}$ to finite fields! For instance, in the case of the octahedron, we have $\cos\theta = 1/2, \cos\gamma = -1/3$. For $6 \nmid q$, we see that we can specialize these values to $\mathbb{F}_q$, and therefore obtain an octahedron over $\mathbb{F}_q$! This has the same automorphism group as the ordinary octahedron[3].

However, as Grothendieck writes, the situation is entirely different if we start with an infinite (i.e. Euclidean/hyperbolic tilings) regular polyhedron! Then when we specialize it to $\mathbb{F}_q$, the fact that polyhedra over finite fields must necessarily be finite implies that we get an infinite number of finite regular polyhedra as q varies, whose automorphism group varies arithmetically with q! One of the questions Grothendieck mentions in connection with this is: which algebraic curves come from regular polyhedra over finite fields?

8.1.4 Moduli

At last we come to the question: is there a meaningful way of interpreting these phenomenon through moduli? Indeed, consider the functor

$$F_g(S) = \{X \xrightarrow{f} \mathbb{P}^1_S \mid g_X = g, f \text{ Galois}, \deg(D/S) = 3\}.$$

8.2 Hurwitz spaces

8.2.1 The usual Hurwitz functor

Recall the Hurwitz functor:

$$\mathcal{H}_{r,G}(S) = \{(f, \tau) | f : X \to \mathbb{P}^1_S | \, f \text{ Galois with group } G \text{ and has } r \text{ branch points}, \tau : \mathrm{Aut}_{\mathbb{P}^1_{\mathbb{C}}}(X) \cong G\}.$$

Say we have constructed a coarse moduli space for it. Note that automorphisms of the pair (f, τ) are given by the center of G. Thus if the center of G is just the identity, then there are no automorphisms. Apparently this means we have a fine moduli space???

[3]Grothendieck seems to claim this; I haven't checked it.

Theorem 8.2.1. *There exists a scheme $\mathcal{H}$, smooth and of finite type over $\mathbb{Z}$, that is a coarse moduli space for the functor $\mathcal{H}_{r,G}$. It is a fine moduli space when G is center-free.*

Let us briefly describe how this can be proven. We begin with constructing it analytically over $\mathbb{C}$, which is not so bad. This amounts to putting an appropriate manifold structure on $\mathcal{H}_{\mathbb{C}}$, which as a set is the collection of all Galois covers with r branch points and with Galois group G. The hard part is descending it onto $\mathbb{Z}$. Anyways, the construction follows the same model as the construction of the universal covering space.

Note that as a set, $\mathcal{H}_{\mathbb{C}}$ corresponds to pairs of r points and surjective homomorphisms $\pi_1(\mathbb{P}^1_{\mathbb{C}} - r \text{ points}) \twoheadrightarrow G$ up to inner automorphism of G. Fix a cover f with branch locus $D = \{t_1, \ldots, t_r\}$ and surjective homomorphism $\theta : \pi_1(\mathbb{P}^1_{\mathbb{C}} - r \text{ points}) \twoheadrightarrow G$. Let $\{C_i\}$ be disks around the points of the branch locus that do not intersect each other. Picking another branch locus D' where we just pick different points in the disks C_i, we get a natural isomorphism

$$\pi_1(\mathbb{P}^1_{\mathbb{C}} - D) \cong \pi_1(\mathbb{P}^1_{\mathbb{C}} - D').$$

Then define the topology on $\mathcal{H}_{\mathbb{C}}$ to be generated by $\mathcal{H}(f, C_i)$ which correspond to elements with branch locus in the C_i and whose homomorphisms agree after composition up to inner automorphism of G. Using the Riemann existence theorem, one shows that this works.

8.2.2 Higher genus case

Lawrence and Venkatesh construct a Hurwitz space for Y/K where the genus of Y is at least 2. They do not explicitly state that this is a coarse/fine moduli space...

Proposition 8.2.2. *Let Y/K have genus at least 2 and let G be a center-free finite group. Then there is a curve Y'/K equipped with an étale map $\pi : Y' \to Y$ and a relative curve $Z \to Y'$ with the following properties.*

1. *Y' parameterizes G-covers of Y branched at a single point.*
2. *Z gives the universal G-cover of Y branched at a single point.*

Let us explain these a bit more. The idea is that Y' represents the functor

$$F(S) = \{X \to Y \times_K S | \text{branched at } 1 \text{ point, Galois with group } G\}.$$

There is also an implicit isomorphism to G. Note that this is essentially the Hurwitz functor $\mathcal{H}_{1,G}$, except we have replaced $\mathbb{P}^1$ with Y. Note that since there is only one branch point, there is an identification between the space of points on Y with Y, which is where the map $\pi : Y' \to Y$ comes from. The set $\pi^{-1}(y)$ gives the set of G-conjugacy classes of surjections $\pi_1(Y - y, *) \twoheadrightarrow G$ nontrivial on a loop around y.

The scheme Z comes from taking the universal family. This occurs when we look at $\mathcal{H}_{1,G}(Y')$ and take the identity, which corresponds to a G-cover $Z \to Y' \times_K Y$. It is a covering space away from the graph of π, and each fiber $Z_{y'} \to Y$ is ramified exactly at $\pi(y')$.

The analytic construction follows the same pattern as with the previous Hurwitz functor. One takes Y' as a set to be the union of all $S(y)$, for $y \in Y(\mathbb{C})$, of G-covers of Y ramified only at y. This is equipped with a covering map $e : Y' \to Y$ which gives Y' the structure of a Riemann surface. Then Z is taken to be the union of the corresponding curves $X_{y'}$ for $y' \in Y'$, and we obtain a map $f : Z \to Y' \times Y$. This last map, on $f : X_{y'} \to Y$, is given by (y', f). This can all be made algebraic through GAGA theorems, and as before the difficulty lies in the descent to K.

8.3 Prym varieties

8.3.1 General theory

Definition 8.3.1. *Given a morphism $f : C_1 \to C_2$ of curves over an algebraically closed field, the associated **Prym variety** is the cokernel of the induced map*

$$\mathrm{Pic}^0(C_2) \to \mathrm{Pic}^0(C_1)$$

on Jacobians.

In our setting, if we have an $\mathrm{Aff}(q)$-covering $C_1 \to C_2$, we can take the subgroup $\mathbb{F}_q^* \subset \mathrm{Aff}(q)$ to factor it into a degree $q-1$ covering followed by a degree q covering $C_1 \to C_1' \to C_2$. We can then take the Prym variety of this second covering

$$\mathrm{coker}(\mathrm{Pic}^0(C_2) \to \mathrm{Pic}^0(C_1)).$$

Looking at their images in $\mathrm{Pic}^0(C_1)$, we see that these are the connected components of the identities of $\mathrm{Pic}^0(C_1)^{\mathrm{Aff}(q)}$ and $\mathrm{Pic}^0(C_1)^{\mathbb{F}_q^*}$. This is ??? an abelian variety of dimension $(2g-1) \cdot \frac{q-1}{2}$.

8.3.2 The Kodaira-Parshin family for $\mathrm{Aff}(q)$

For a prime q, we recall we want to construct the Kodaira-Parshin family for $\mathrm{Aff}(q)$ as an abelian-by-finite family $X_q \to Y_q' \xrightarrow{\pi} Y$. The finite étale map $\pi : Y_q' \to Y$ is given by the Hurwitz spaces for the center-free group $\mathrm{Aff}(q)$, as described in the previous section. The previous section also gives a map

$$Z_q \to Y_q' \times Y$$

from the universal G-cover of Y. Recall that the fiber over $y' \in Y$ gives the associated $\mathrm{Aff}(q)$-cover $(Z_q)_{y'} \to Y$ branched at y. Thus $Z_q \to Y_q'$ is a relative curve.

The desired sequence of morphisms giving the Kodaira-Parshin family for $\mathrm{Aff}(q)$

$$X_q \to Y_q' \to Y$$

comes from the reduced relative Prym of $Z_q \to Y_q' \times Y$. Basically, this means that for every $y' \in Y_q'$, the fiber of $X_q \to Y_q'$ is the Prym of $(Z_q)_{y'} \to (Y_q' \times Y)_{y'} = \{y'\} \times Y$. It is possible to make these fit together into an abelian scheme $X_q \to Y_q'$ with a symmetric and fiberwise ample line bundle through a some further construction given by Lawrence-Venkatesh.

Chapter 9

Monodromy of the Kodaira-Parshin family

9.1 Background on mapping class groups

See Benson/Farb. S surface with b boundaries and n punctures. Then $\chi(S) = 2 - 2g - b - n$. If $\chi = 0$ then S admits a Euclidean metric, and if $\chi(S) < 0$ then S admits a hyperbolic metric. In the hyperbolic case, closed curves are homotopic to geodesics.

Simple closed curves. Prop. 1.4: primitive elements of $\pi_1(S)$. 1.2.5: homotopy vs isotopy. Essential (nontrivial in π_1 curves: isotopic if homotopic. 1.2.6: isotopy of surfaces.

2.1: definition of $\mathrm{MCG}(S)$. Lemma 2.1: $\mathrm{MCG}(D^2), \mathrm{MCG}(S_{0,1}), \mathrm{MCG}(S^2)$ trivial. 2.2.2-2.2.4: $\mathrm{MCG}(S_{0,3}) \cong S_3, \mathrm{MCG}(S_{0,2}) \cong S_2, \mathrm{MCG}(A) \cong \mathbb{Z}, \mathrm{MCG}(T^2) \cong \mathrm{SL}(2,\mathbb{Z})$. 2.3: The Alexander method.

3: Dehn twists definition. T^2 example. Nontriviality, infinite order, braid relation.

4: Dehn-Lickorish theorem. Humphries generators.

9.2 A topological reformulation of having full monodromy

Setup

We recall that we constructed the Kodaira-Parshin family as an abelian-by-finite family

$$X_q \to Y'_q \xrightarrow{\pi} Y,$$

and used it to prove Faltings's theorem under the condition that it possesses full monodromy. Briefly, recall that $\pi : Y'_q \to Y$ is a finite étale map whose fiber over a point y classifies Galois $\mathrm{Aff}(q)$-covers branched only at y. If Z is the universal $\mathrm{Aff}(q)$-cover of Y, then the fiber over some point $y' \mapsto y$ gives an $\mathrm{Aff}(q)$-covering $Z_{q,y'} \to Y$ ramified precisely at y. The subgroup $\{ax\} \subset \mathrm{Aff}(q)$ gives a degree q covering $C_{y'} \to Y$, whose Prym is the abelian variety $X_{q,y'}$. These fit together to form the abelian scheme $X_q \to Y'_q$. In the language to come, $Z_{q,y'} \to Y$ is a **singly branched** $\mathrm{Aff}(q)$-cover, while $C_{y'} \to Y$ is a **singly ramified** $\mathrm{Aff}(q)$-cover.

We recall what having full monodromy means. Begin with an abelian-by-finite family $X \to Y' \to Y$ and take a complex point $y_0 \in Y(\mathbb{C})$ and consider the action of $\pi_1(Y(\mathbb{C}), y_0)$ on

$$H^1_B(X_{y_0}, \mathbb{Q}) \cong \bigoplus_{\pi(\tilde{y}) = y_0} H^1_B(X_{\tilde{y}}, \mathbb{Q}).$$

Definition 9.2.1. *We say that the family has* ***full monodromy*** *if the Zariski closure of the image of* $\pi_1(Y, y_0)$ *under this representation contains the following product of symmetric groups:*

$$\overline{(\textit{image of } \pi_1(Y(\mathbb{C}), y_0))} \supset \prod_{\pi(\tilde{y})=y_0} \mathrm{Sp}(H^1_B(X_{\tilde{y}}, \mathbb{Q}), \omega)$$

where the symplectic group is with reference to the form ω *defined by the polarization.*

The goal will be to prove the following theorem.

Theorem 9.2.2. *Let* $Z_1, \ldots, Z_N$ *be the (isomorphism classes of) singly ramified* $\mathrm{Aff}(q)$*-covers of* Y*. Then the map*

$$\mathrm{Mon} : \pi_1(Y, y)_0 \to \prod_{i=1}^{N} \mathrm{Sp}(H_1^{\mathrm{Pr}}(Z_i, Y))$$

has Zariski-dense image.

$\mathrm{Aff}(q)$-covers – not to be confused with (Galois) $\mathrm{Aff}(q)$-covers

Now let us define the terms and see why this implies that the Kodaira-Parshin family has full monodromy.

Definition 9.2.3 ($\mathrm{Aff}(q)$-covers)**.** *Let* Y *be a surface. Then an* $\mathrm{Aff}(q)$*-cover of* Y *is defined to be a connected surface* Z *with a degree* q *covering map* $\pi : Z \to Y$ *whose monodromy representation has image* $\mathrm{Aff}(q) \subset S_q$*.*

Let $y \in Y$ *be a point. Then a singly ramified* $\mathrm{Aff}(q)$*-cover of* Y *is the compactification of an* $\mathrm{Aff}(q)$*-cover of* $Y - \{y\}$ *whose monodromy around* y *is nontrivial, and hence a* q*-cycle.*

By the Riemann-Hurwitz formula, a singly ramified $\mathrm{Aff}(q)$-cover of Y has genus $gq - \frac{q-1}{2}$. Recall that covering spaces are determined by their monodromy representations. Precisely, an $\mathrm{Aff}(q)$-cover of Y is determined by an $\mathrm{Aff}(q)$-conjugacy class of maps $\pi_1(Y, y_0) \twoheadrightarrow \mathrm{Aff}(q)$. Similarly, the singly ramified $\mathrm{Aff}(q)$-covers of Y are given by representatives $\mathrm{Cov}_i : \pi_1(Y - \{y\}, y_0) \twoheadrightarrow \mathrm{Aff}(q)$ for $1 \leqslant i \leqslant N$. Let $Z_1, \ldots, Z_N$ be these singly ramified $\mathrm{Aff}(q)$-covers of Y (up to isomorphism).

9.3 The reformulation explained

The representation on primitive homology

We still have not explained what the map Mon is, or even what $\pi_1(Y, y_0)$ is. We do so now. Note that $\mathrm{MCG}(Y)$ acts on the set of isomorphism classes of $\mathrm{Aff}(q)$-covers of Y. This action can be algebraically viewed as the outer action $\mathrm{MCG}(Y) \to \mathrm{Out}(\pi_1(Y, y_0))$[1].

For every $\mathrm{Aff}(q)$-cover of Z, define $\mathrm{MCG}(Y)_Z$ to be the stabilizer of (Z, π) under this action. Because $\mathrm{Aff}(q)$ has trivial centralizer in S_q, we obtain a well-defined homomorphism

$$\mathrm{MCG}(Y)_Z \to \mathrm{MCG}(Z),$$

where we have the obvious commutative diagram with π. Next, we note that $\mathrm{MCG}(Z)$ acts on $H_1(Z)$ in a way that preserves its symplectic intersection form. Moreover, since $\pi : Z \to Y$ is a covering, we have a decomposition

$$H_1(Z, \mathbb{Q}) = \pi^* H_1(Y, \mathbb{Q}) \oplus \ker(\pi_* : H_1(Z, \mathbb{Q}) \to H_1(Y, \mathbb{Q})).$$

[1] A variant of this observation is crucial in anabelian geometry

We define

$$H_1^{\mathrm{Pr}}(Z,Y;\mathbb{Q}) := \ker(\pi_* : H_1(Z,\mathbb{Q}) \to H_1(Y,\mathbb{Q})).$$

As we will see, Pr can stand for both 'primitive' and 'Prym.' We assume $\mathbb{Q}$ coefficients henceforth. One sees that the mapping class group preserves $H_1^{\mathrm{Pr}}(Z,Y)$, and thus we obtain a monodromy map

$$\mathrm{Mon} : MCG(Y)_Z \to \mathrm{Sp}(H_1^{\mathrm{Pr}}(Z,Y)).$$

In our context, we are actually taking Y right above to be the punctured surface $Y - \{y\}$, and the Z_i are the compactified singly ramified $\mathrm{Aff}(q)$-covers. Note these have the same homology without compactification, so we can invoke Mon as before.

Finally, we have an embedding $\pi_1(Y,y) \hookrightarrow \mathrm{MCG}(Y - \{y\})$. We define $\pi_1(Y,y)_0$ to be the intersection of the image of $\pi_1(Y,y)$ with the intersection $\mathrm{MCG}(Y - \{y\})_0 = \bigcap_{i=1}^N \mathrm{MCG}(Y)_{Z_i}$. Restricting gives the monodromy map

$$\mathrm{Mon} : \pi_1(Y,y)_0 \to \mathrm{Sp}(H_1^{\mathrm{Pr}}(Z,Y)).$$

Back to the full monodromy theorem

We again recall the Kodaira-Parshin family. We have a family $Z_q \to Y_q' \to Y$, and for $y' \in Y_q'$ above $y \in Y$, we take the fiber and obtain a singly branched $\mathrm{Aff}(q)$-cover

$$Z_{q,y'} \to Y.$$

This is a Galois, degree $q(q-1)$ cover. Taking the associated degree q cover, we obtain a singly ramified $\mathrm{Aff}(q)$-cover

$$C_{y'} \to Y.$$

We have that $C_{y'}$ is isomorphic to one of the covers Z_i. Now for the actual Kodaira-Parshin family $X_q \to Y_q' \to Y$, we have an isogeny

$$X_{q,y'} \to \mathrm{Prym}(Z_i \to Y).$$

Looking at rational homology, this precisely gives

$$H_1^{\mathrm{Pr}}(Z_i,Y;\mathbb{Q}) \cong H^1(X_{q,y'},\mathbb{Q}).$$

Moreover, the monodromy representation coincides with that coming from $\pi_1(Y,y)_0$ in the second definition of Mon. Thus it suffices to prove Theorem 9.2.2 to show that the Kodaira-Parshin family has full monodromy.

Reduction from the fundamental group to a mapping class group

Recall that we first defined

$$\mathrm{Mon} : MCG(Y - \{y\})_Z \to \mathrm{Sp}(H_1^{\mathrm{Pr}}(Z,Y)).$$

By taking their intersections we have a map

$$\mathrm{Mon} : MCG(Y - \{y\})_0 \to \mathrm{Sp}(H_1^{\mathrm{Pr}}(Z,Y)).$$

and then restricted it to $\pi_1(Y,y)_0$. We claim that we can just work with $\mathrm{Mon} : MCG(Y - \{y\})_0$; that is, to show that the image of $\pi_1(Y,y)_0$ is Zariski-dense, we just need to show that the image of $MCG(Y - \{y\})_0$ is Zariski-dense.

Theorem 9.3.1 (Birman exact sequence). *Let Y be a surface, possibly with punctures and/or boundary, with $\chi(Y) < 0$. Then there is an exact sequence*

$$1 \to \pi_1(Y, y) \to \mathrm{MCG}(Y, y) \to \mathrm{MCG}(Y) \to 1.$$

Then one shows that the image of $\pi_1(Y, y)_0 \to \mathrm{Sp}(H_1^{\mathrm{Pr}}(Z_i, Y))$ on any of the N factors is not contained in the center of the image. Then since $\pi_1(Y, y)_0$ is a normal subgroup of $\mathrm{MCG}(Y - \{y\})_0$ and the symplectic groups are almost simple, we conclude that it suffices to show that the monodromy map

$$\mathrm{Mon} : MCG(Y - \{y\})_0 \to \mathrm{Sp}(H_1^{\mathrm{Pr}}(Z, Y))$$

has Zariski-dense image.

9.4 A normal form for singly ramified $\mathrm{Aff}(q)$-covers

We would like to put the singly ramified $\mathrm{Aff}(q)$ covers into a nice normal form where they are trivial over S_{g-1} and nontrivial over T, where $Y = S_{g-1} \# T$. Precisely, write $S_{g-1}^\circ = S_{g-1} - D$ and $T^\circ = T - D'$ and identify the boundaries of the open disks D, D' to form Y. This can be done so that the ramifiction point y is in the interior of T°, the cover $Z \to Y$ splits over S_{g-1}°, and when it is restricted to T° extends over T (i.e. has trivial monodromy around the boundary of T). Furthermore, letting β_1, β_2 be a standard basis for $(\pi_1(T - \{y\}, *)$, monodromy sends β_1 to a generator of $\mathbb{F}_q^* \subset \mathrm{Aff}(q)$ and β_2 to a generator of $\mathbb{F}_q^+ \subset \mathrm{Aff}(q)$.

We only give the general idea. A singly ramified $\mathrm{Aff}(q)$-cover $Z \to Y$ is associated to a surjection $\pi_1(Y - \{y\}, y_0) \twoheadrightarrow \mathrm{Aff}(q)$. The map on abelianizations $H_1(Y, \mathbb{Z}) \to \mathbb{F}_q^*$ can be induced by intersecting with a simple closed curve α_1. Cut Y along two such curves α_1^+, α_1^- on either side of y. The result is a surface Y^1 with boundary, Use Poincare duality

$$H_1(Y^1, \partial Y^1; \mathbb{Z}) \times H_1(Y^1; \mathbb{Z}) \to \mathbb{Z}$$

to construct a 'unit' element $\alpha_2 \in H_1(Y^1, \partial Y^1; \mathbb{Z})$. Cutting Y^1 along α_2, we obtain the desired S_{g-1}.

9.5 Dehn twists and completion of the proof

We sketch the completion of the proof. We use the following fact, which can be viewed as an algebraic version of Goursat's lemma.

Proposition 9.5.1. *Suppose G is an algebraic subgroup of $\mathrm{Sp}(V)^N$ such that*

- *Each projection $\pi_i : G \to \mathrm{Sp}(V)$ is surjective.*
- *For $1 \leqslant i < j \leqslant N$, there exists $g \in G$ such that $\pi_i(g)$ and $\pi_j(g)$ are unipotent with fixed spaces of different dimensions.*

For the second, one can show that for two non-isomorphic covers Z_i, Z_j, then there is some simple closed curve η in Y such that the cycle decompositions of monodromy around η in Z_1 and Z_2 are different.

For the first, we reduce the problem even further to just showing that each map

$$\mathrm{Mon} : \mathrm{MCG}(Y)_{Z_i} \to \mathrm{Sp}(H_1^{\mathrm{Pr}}(Z_i, Y))$$

has Zariski-dense image. Indeed, we are just taking a finite-index subgroup of what we started with, which will have the same Zariski closure in the image because the image is connected. We prove Zariski-density by showing that there are enough curves on Y whose corresponding Dehn twists generate what we need. For this we need the normal form of Z_i explained in the previous section.

Part IV

Higher dimensions

Chapter 10

O-minimality and applications to transcendence

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