Midterm Solutions

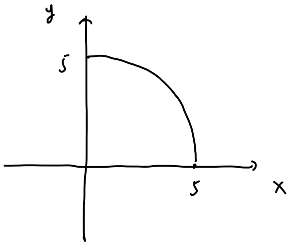
1. $$\iint_R f(x,y)\,dA = \int_0^5 \int_0^\pi x^3 \cos(y)^2\,dy\,dx$$

$$= \int_0^5 x^3\,dx \cdot \int_0^\pi \cos(y)^2\,dy$$

$$= \frac{x^4}{4}\Big|_0^5 \cdot \int_0^\pi \frac{1+\cos(2y)}{2}\,dy$$

$$= \frac{625}{4} \times \frac{\pi}{2} = \frac{625}{8}\pi$$

2. (a)



(b) In polar coordinate, the integral region can be described by:

$$0 \le r \le 5, \quad 0 \le \theta \le \frac{\pi}{2}$$

then

$$\int_0^5 \int_0^{\sqrt{25-x^2}} e^{-x^2-y^2}\,dy\,dx = \int_0^5 \int_0^{\frac{\pi}{2}} e^{-r^2} r\,d\theta\,dr$$

$$= \frac{\pi}{2} \cdot \int_0^5 e^{-r^2} r\,dr$$

$$= \frac{\pi}{4} \cdot -e^{-r^2}\Big|_0^5 = \frac{\pi}{4}\left(1 - e^{-25}\right)$$

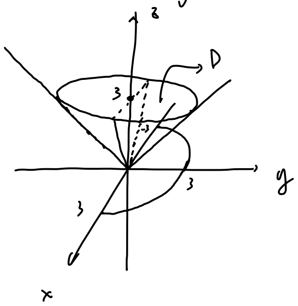
3. $z = \sqrt{x^2+y^2}$, then $\dfrac{\partial z}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}$, $\dfrac{\partial z}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$

then

$$A = \iint_{x^2+y^2\le 1} \sqrt{1 + \frac{x^2+y^2}{x^2+y^2}}\,dA$$

$$= \iint_{x^2+y^2\le 1} \sqrt{2}\,dA = \sqrt{2} \times \pi = \sqrt{2}\,\pi$$

4. Sketch the region



In cylindrical coordinates, it can be described by

$D: \quad 0 \le r \le 3, \quad 0 \le \theta \le \pi, \quad r \le z \le 3$

hence

$$\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{3} yz \, dz \, dy \; x$$

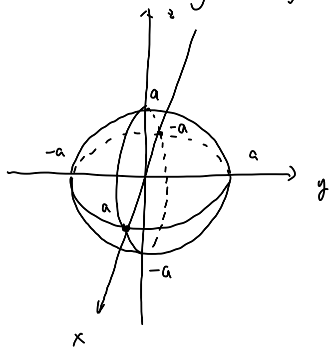
$$= \int_{0}^{\pi} \int_{0}^{3} \int_{r}^{3} r\sin\theta \cdot z \, dz \; r \, dr \, d\theta$$

$$= \int_{0}^{\pi} \sin\theta \, d\theta \cdot \int_{0}^{3} \int_{r}^{3} r^2 z \, dz \, dr$$

$$= 2 \cdot \int_{0}^{3} r^2 \cdot \frac{9-r^2}{2} \, dr$$

$$= \int_{0}^{3} r^2 (9-r^2) \, dr = 81 - \frac{3^5}{5} = \frac{405-243}{5} = \frac{162}{5}$$

5. Sketch the integral region:



In Spherical coordinate, this region is

$$0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi$$

hence

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi \cdot \int_0^{\pi} \sin\phi \, d\phi \cdot \int_0^1 \rho^3 \, d\rho$$

$$= 2\pi \cdot 2 \cdot \frac{1}{4} = \pi$$