Arithmetic Siegel-Weil formula on $\mathcal{X}_0(N)$

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Contents

- Geometric Siegel-Weil formula on $X_0(1)$.
- Integral model $\mathcal{X}_0(N)$.
  - Arithmetic Sigel-Weil formula.
  - Associated RZ spaces.
- Difference formulas.
  - Geometric side
  - Analytic side

Special cycles on the modular curve $X_0(1)$

- Let $Y_0(1) := \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$ be the modular curve, it parameterize elliptic curves over $\mathbb{C}$ by

$$\tau \in \mathcal{H} \longmapsto E_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau.$$

  Let $X_0(1)$ be its compactification.
- Given an integer $m > 0$, we consider the following moduli problem: for a $\mathbb{C}$-scheme S,

$$Z(m)(S) = \{(E, \alpha) : E/S \text{ is an elliptic curve, } \alpha \in \mathrm{End}_S(E) \text{ satisfying } \alpha^2 = -m.\}.$$

Counting points on the modular curve $X_0(1)$

- The moduli problem $Z(m)$ parameterise elliptic curves with complex multiplication by the order $\mathcal{O}_m = \mathbb{Z} + \mathbb{Z} \cdot \sqrt{-m}$.
- It can be shown the set $Z(m)(\mathbb{C})$ consists of finitely many points.

A natural question: what's $\#Z(m)(\mathbb{C})$?

Counting points on the modular curve $X_0(1)$

$$\#Z(m)(\mathbb{C}) = \sum_{E} \#\{\alpha \in \mathrm{End}_{\mathbb{C}}(E) : \alpha^2 = -m.\}/\mathrm{aut}(E).$$

Counting points on the modular curve $X_0(1)$

$$\#Z(m)(\mathbb{C}) = \sum_E \#\{\alpha \in \mathrm{End}_{\mathbb{C}}(E) : \alpha^2 = -m.\}/\mathrm{aut}(E).$$

If $E = E_\tau$ appears on the right hand side, then τ satisfies a quadratic equation

$$a\tau^2 + b\tau + c = 0.$$

where $a, b, c \in \mathbb{Z}$ and $\gcd(a, b, c) = 1$, the discriminant of this equation is $b^2 - 4ac$.

Recall that the discriminant of $\mathcal{O}_m$ is $-4m$, then there exists an integer $k > 0$ such that

$$-4m = k^2(b^2 - 4ac).$$

Counting points on the modular curve $X_0(1)$

Then by the theory of complex multiplication, we have

$$\#Z(m)(\mathbb{C}) = \sum_E \#\{\alpha \in \mathrm{End}_{\mathbb{C}}(E) : \alpha^2 = -m.\}/\mathrm{aut}(E).$$
$$= \sum_{k>0:k^2|4m} h(\frac{4m}{k^2}) = H(4m).$$

Recall that for a positive integer N,

$$H(N) = \#\mathrm{SL}_2(\mathbb{Z})\text{-equivalence classes of positive definite binary quadratic form of disc } -N.$$
$$h(N) = \#\mathrm{SL}_2(\mathbb{Z})\text{-equivalence classes of primitive positive definite binary quadratic form of disc } -N.$$

Geometric Sigel-Weil formula on $X_0(1)$

- The modular curve $X_0(1)$ is the (compactified) GSpin Shimura variety attached to the rank 3 quadratic lattice $V = \mathrm{M}_2(\mathbb{Z})^{\mathrm{tr}=0}$, because

$$\mathrm{GSpin}(V_{\mathbb{Q}}) \simeq \mathrm{GL}_2, \;\; \mathrm{GSpin}(V) \simeq \mathrm{GL}_2(\mathbb{Z})$$

- There is an Eisenstein series $E(z, s, 1_{V \otimes \hat{\mathbb{Z}}})$ associated to the lattice V via Weil representation.

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Theorem (Geometric Sigel-Weil formula on $Y_0(1)$)

Let $m > 0$ be an integer, then

$$\#Z(m)(\mathbb{C}) \cdot q^m = \frac{1}{12} \cdot E_m(z, \frac{1}{2}, 1_{V\otimes\hat{\mathbb{Z}}}).$$

Computing the Eisenstein series

We can also compute the Fourier coefficients $E_m(z, \frac{1}{2}, 1_{V\otimes\hat{\mathbb{Z}}})$ in the following way,

$$E_m(z, \frac{1}{2}, 1_{V\otimes\hat{\mathbb{Z}}}) = 4\pi(1+i)\sqrt{m}\cdot q^m \prod_p W_{m,p}(1, \frac{1}{2}, 1_{V\otimes\mathbb{Z}_p}).$$

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Assume $-m < -4$ is a fundamental discriminant, by the works of Kudla, Rapoport and Yang

$$E_m(z, \frac{1}{2}, 1_{V\otimes\hat{\mathbb{Z}}}) = (3-\chi_m(2))\cdot\frac{\sqrt{m}}{\pi}L(1,\chi_m)\cdot q^m.$$

Geometric Siegel-Weil v.s. Class number formula

On the other hand,

$$H(4m) = h(m) + h(4m) = (3 - \chi_m(2)) \cdot h(m).$$

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The geometric Siegel-Weil formula on $Y_0(1)$ implies

Theorem (Class number formula)

Let $-m < -4$ be a fundamental discriminant, then

$$h(m) = \frac{\sqrt{m}}{\pi} L(1, \chi_m).$$

The modular curve $Y_0(N)$

- Let $Y_0(N) = \Gamma_0(N)\backslash\mathcal{H}$, and $X_0(N) = Y_0(N) \cup \{\text{cusps}\}$. The modular curve $Y_0(N)$ parameterize cyclic isogenies between elliptic curves over $\mathbb{C}$ by the following

$$\tau \in \mathcal{H} \longmapsto (E_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau \to E_{\frac{\tau}{N}}).$$

- Given an integer $m > 0$, we consider the following moduli problem: for a $\mathbb{C}$-scheme S,

$$Z(m)(S) = \{(E \xrightarrow{\pi} E', \alpha) : E \xrightarrow{\pi} E' \text{ is a cyclic isogeny between elliptic curves, } \alpha \in \mathrm{Hom}_S(E, E') \text{ satisfies } \alpha^\vee \circ \pi + \pi^\vee \circ \alpha = 0 \text{ and } \alpha^\vee \circ \alpha = m.\}.$$

Geometric Siegel-Weil formula on $X_0(N)$

Let $N > 0$ be an integer, and $\Delta(N)$ be the following rank 3 quadratic lattice over $\mathbb{Z}$,

$$\Delta(N) = \left\{ x = \begin{pmatrix} -Na & b \\ c & a \end{pmatrix} : \ a, b, c \in \mathbb{Z} \right\}.$$

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Then the geometric Siegel-Weil formula on $X_0(N)$ is proved by Tuoping and Tonghai [DY19],

Theorem (Geometric Siegel-Weil formula on $X_0(N)$)

For an integer $m > 0$, we have

$$\#Z(m)(\mathbb{C}) \cdot q^m = \frac{\psi(N)}{12} E(z, \frac{1}{2}, 1_{\Delta(N)(\hat{\mathbb{Z}})}).$$

here $\psi(N) = N \prod_{p|N} (1 + p^{-1})$.

The stack $\mathcal{X}_0(N)$

Let $\mathcal{Y}_0(N)$ be the stack of $\Gamma_0(N)$-level structures on elliptic curves defined by Katz and Mazur in [KM85]: for a scheme S,

$$\mathcal{Y}_0(N)(S) = \{E \xrightarrow{\pi} E' : \pi \text{ is a cyclic isogeny and } \pi^\vee \circ \pi = N\}.$$

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here π is cyclic means that the order N group scheme $G := \ker(\pi)$ is a cyclic group scheme in the sense that there exists a section $P \in G(S)$ such that for any $f \in \mathcal{O}_G$,

$$\det(T - f) = \prod_{a=1}^{N} (T - f(aP)).$$

Let $\mathcal{X}_0(N)$ be its compactification.

Special cycles on $\mathcal{X}_0(N)$

- Given an integer $m > 0$, we consider the following moduli problem: for a scheme S,

$$Z(m)(S) = \{(E \xrightarrow{\pi} E', \alpha) : \pi \text{ is a cyclic } N\text{-isogeny, } \alpha \text{ is an isogeny from } E \text{ to } E' \text{ satisfying } \alpha^\vee \circ \alpha = m \text{ and } \alpha^\vee \circ \pi + \pi^\vee \circ \alpha = 0.\}.$$

  It is a generalized Cartier divisor on $\mathcal{X}_0(N)$ and has no intersections with cusps.

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  It is a generalized Cartier divisor on $\mathcal{X}_0(N)$ and has no intersections with cusps.

- Given a 2×2 positive definite symmetric matrix T, we define the moduli problem $\mathcal{Z}(T)$ as follows: for a scheme S,

$$\mathcal{Z}(T)(S) = \{(E \xrightarrow{\pi} E', \alpha_1, \alpha_2) : \pi \text{ is a cyclic } N\text{-isogeny, } \alpha_i \text{ are isogenies from } E \text{ to } E' \text{ satisfying } \frac{1}{2}(\alpha_i, \alpha_j) = T \text{ and } \alpha_i^\vee \circ \pi + \pi^\vee \circ \alpha_i = 0.\}.$$

The special cycle $\mathcal{Z}(T)$

Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a nonsingular matrix, we define the following difference set

$$\begin{aligned}\mathrm{Diff}(T, \Delta(N)) = \{l \text{ is a finite prime} : T \text{ is not represented} \\ \text{by } \Delta(N) \otimes \mathbb{Q}_l.\}\end{aligned}$$

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Lemma

Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a nonsingular matrix. If $\mathcal{Z}(T)(\overline{\mathbb{F}}_p) \neq \varnothing$ for some prime p, then T is positive definite, and

$$\mathrm{Diff}(T, \Delta(N)) = \{p\}.$$

Moreover, in this case, the special cycle $\mathcal{Z}(T)$ is supported in the supersingular locus of the special fiber $\mathcal{X}_0(N)_{\mathbb{F}_p}$.

Arithmetic Siegel-Weil formula on $\mathcal{X}_0(N)$

Let T be a 2×2 positive definite symmetric matrix with diagonal elements m_1, m_2, then we define

$$\deg(\mathcal{Z}(T)) = \chi(\mathcal{Z}(T), \mathcal{O}_{\mathcal{Z}(m_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(m_2)}) \cdot \log p.$$

where $p \in \mathrm{Diff}(T, \Delta(N))$.

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where $p \in \mathrm{Diff}(T, \Delta(N))$.

Theorem (Arithmetic Siegel-Weil formula on $\mathcal{X}_0(N)$)

Let $T \in \mathrm{Sym}_2(\mathbb{Q})$ be a positive definite symmetric matrix, then

$$\deg(\mathcal{Z}(T))q^T = \frac{\psi(N)}{24} \cdot E'(z, 0, 1_{(\Delta(N)\otimes\hat{\mathbb{Z}})^2}).$$

where $z = x + iy \in \mathcal{H}_2$ and $q^T = e^{2\pi i \mathrm{tr}(Tz)}$.

Key ingredients

- Formal uniformization of the supersingular locus of $\mathcal{X}_0(N)$.

  It connects intersection numbers on $\mathcal{X}_0(N)$ with local arithmetic intersection numbers on the RZ space associated to $\mathcal{X}_0(N)$.

Key ingredients

- Formal uniformization of the supersingular locus of $\mathcal{X}_0(N)$.

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- Kudla-Rapoport conjecture for the RZ space associated to $\mathcal{X}_0(N)$.

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  It connects local arithmetic intersection numbers on RZ space with Whittaker functions.

Both are proved by embedding trick!

RZ space associated to $\mathcal{X}_0(1) \times \mathcal{X}_0(1)$

Let $\mathbb{X}$ be a p-divisible group of dim 1, height 2. Consider the following functor: for every $S \in \mathrm{Nilp}_W$, the set $\mathcal{N}(S)$ consists of $((X, X'), (\rho, \rho'), (\lambda, \lambda'))$, where
(1) X and X' are two p-divisible group over S, ρ and ρ' are two height 0 quasi-isogenies between p-divisible groups $\rho : \mathbb{X} \times_{\mathbb{F}} \overline{S} \to X \times_S \overline{S}$, $\rho' : \mathbb{X} \times_{\mathbb{F}} \overline{S} \to X' \times_S \overline{S}$.
(2) $\lambda : X \to X^\vee$, $\lambda' : X' \to X'^\vee$ are two principal polarizations, such that Zariski locally on $\overline{S}$, we have

$$\rho^\vee \circ \lambda \circ \rho = c(\rho) \cdot \lambda_0, \quad \rho'^\vee \circ \lambda \circ \rho' = c(\rho') \cdot \lambda_0.$$

for some $c(\rho) = c(\rho') \in \mathbb{Z}_p^\times$.

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for some $c(\rho) = c(\rho') \in \mathbb{Z}_p^\times$.

- $\mathcal{N} \simeq \mathrm{Spf}\, W[[t_1, t_2]]$.

Special cycles on $\mathcal{N}$

Let $\mathbb{B}$ the unique division quaternion algebra over $\mathbb{Q}_p$, it is isometric to $\mathrm{End}^0(\mathbb{X})$.

Definition

For any subset $L \subset \mathbb{B}$, define the special cycle $\mathcal{Z}^\sharp(L) \subset \mathcal{N}$ to be the closed formal subscheme cut out by the condition,

$$\rho'^{\mathrm{univ}} \circ x \circ (\rho^{\mathrm{univ}})^{-1} \in \mathrm{Hom}(X^{\mathrm{univ}}, X'^{\mathrm{univ}}).$$

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for all $x \in L$.

Let L be a rank 3 lattice with basis x_1, x_2 and x_3. Define the local arithmetic intersection number on $\mathcal{N}$ to be

$$\mathrm{Int}^\sharp(L) = \chi(\mathcal{N}, \mathcal{O}_{\mathcal{Z}^\sharp(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_2)} \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{N}} \mathcal{O}_{\mathcal{Z}^\sharp(x_3)}).$$

Difference divisor on $\mathcal{N}$

- For any $x \in \mathbb{B}$, the special cycle $\mathcal{Z}^\sharp(x)$ is cut out by a single equation $f_x \in W[[t_1, t_2]]$, define $d_x = f_x / f_{p^{-1}x} \in W[[t_1, t_2]]$ and the difference divisor $\mathcal{D}(x) = \operatorname{Spf} W[[t_1, t_2]]/(d_x)$

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Theorem

The difference divisor $\mathcal{D}(x)$ is regular.

Recently we have proved that difference divisors on GSpin RZ spaces with hyperspecial level structure are regular, the formal scheme $\mathcal{N}$ is a special example of such RZ spaces.

RZ space associated to $\mathcal{X}_0(N)$

Fix a N-isogeny $x_0 : \mathbb{X} \to \mathbb{X}$. Consider the following functor: for every $S \in \mathrm{Nilp}_W$, the set $\mathcal{N}_0(N)(S)$ consists of $(X \xrightarrow{x} X', (\rho, \rho'), (\lambda, \lambda'))$, where
(1) X and X' ...
(2) $\lambda : X \to X^\vee$, $\lambda' : X' \to X'^\vee$...
(3) $x : X \to X'$ is a cyclic isogeny (i.e., $\ker(x)$ is a cyclic group scheme over S) lifting $\rho' \circ x_0 \circ \rho^{-1}$.

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(1) X and X' ...
(2) $\lambda : X \to X^\vee$, $\lambda' : X' \to X'^\vee$...
(3) $x : X \to X'$ is a cyclic isogeny (i.e., $\ker(x)$ is a cyclic group scheme over S) lifting $\rho' \circ x_0 \circ \rho^{-1}$.

Theorem ([KM85])

The natural morphism $\mathcal{N}_0(N) \to \mathcal{N}$ is a closed immersion, and $\mathcal{N}_0(N)$ is regular.

An isomorphism

Recall that we have fixed a N-isogeny x_0 when we define $\mathcal{N}_0(N)$.

Theorem

There is an isomorphism between formal schemes,

$$\mathcal{D}(x_0) \xrightarrow{\sim} \mathcal{N}_0(N).$$

Remark

- By the isomorphism and Zink's windows theory, we can compute the special fiber $\mathcal{N}_0(p^n)_p$ as follows

$$\mathbb{F}[[t_1, t_2]] / \left((t_1 - t_2^{p^n}) \cdot (t_2 - t_1^{p^n}) \cdot \prod_{\substack{a+b=n \\ a,b \geq 1}} (t_1^{p^{a-1}} - t_2^{p^{b-1}})^{p-1} \right).$$

which coincides with Katz-Mazur's computation.

Special cycles on $\mathcal{N}_0(N)$

Let $\mathbb{W} = \{x_0\}^\perp \subset \mathbb{B}$.

Definition

For any subset $M \subset \mathbb{W}$, define the special cycle $\mathcal{Z}(M) \subset \mathcal{N}_0(N)$ to be the closed formal subscheme cut out by the condition,

$$\rho'^{\mathrm{univ}} \circ x \circ (\rho^{\mathrm{univ}})^{-1} \in \mathrm{Hom}(X^{\mathrm{univ}}, X'^{\mathrm{univ}}).$$

for all $x \in M$.

Let M be a rank 2 lattice with basis x_1 and x_2. Define the local arithmetic intersection number on $\mathcal{N}_0(N)$ to be

$$\mathrm{Int}(M) = \chi(\mathcal{N}_0(N), \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{N}_0(N)}} \mathcal{O}_{\mathcal{Z}(x_2)}).$$

Difference formula at the geometric side

By the isomorphism $\mathcal{D}(x_0) \simeq \mathcal{N}_0(N)$, we can prove the following theorem

Theorem

For any rank 2 lattice $M \subset \mathbb{W}$, the following identity holds,

$$\mathrm{Int}(M) = \mathrm{Int}^{\sharp}(M \oplus \mathbb{Z}_p \cdot x_0) - \mathrm{Int}^{\sharp}(M \oplus \mathbb{Z}_p \cdot p^{-1} x_0).$$

Local density

- For two quadratic lattice L and M, the local density is defined to be
$$\mathrm{Den}(M, L) = \lim_{d\to\infty} \frac{\#\mathrm{Rep}_{M,L}(\mathbb{Z}_p/p^d)}{p^{d\cdot \dim(\mathrm{Rep}_{M,L})_{\mathbb{Q}_p}}}.$$
- Let H be a rank 2 quadratic lattice given by $q_H(x, y) = xy$, define the local density polynomial to be (rank $L = 2n-1$)
$$\mathrm{Den}(X, L)\big|_{X=p^{-k}} = \frac{\mathrm{Den}(H^{k+n}, L)}{\mathrm{Nor}^{+}(p^{-k}, 2n-1)},$$
where $\mathrm{Nor}^{\varepsilon}(X, m) =$
$$(1 - \tfrac{1+(-1)^{m+1}}{2} \cdot \varepsilon q^{-(m+1)/2} X) \prod_{1\le i<(m+1)/2} (1 - q^{-2i} X^2).$$

Examples of local density

- When m is squarefree, then

$$\mathrm{Den}(\Delta(1) \otimes \mathbb{Z}_p, \langle m \rangle) = 1 - \chi_m(p) p^{-1}.$$

- When $\nu_p(N) = 0$ or 1, we have

$$\mathrm{Den}(H^k, \langle N \rangle) = \begin{cases} (1 - p^{-k})(1 + p^{1-k}), & \text{when } p \mid N; \\ 1 - p^{-k}, & \text{when } p \nmid N. \end{cases}$$

Difference formula at the analytic side

Let $\delta_p(N) = \Delta(N) \otimes \mathbb{Z}_p$, define the following local density function with level N,

$$\mathrm{Den}_{\Delta(N)}(X, M)\Big|_{X=p^{-k}} = \begin{cases} \dfrac{\mathrm{Den}(\delta_p(N) \oplus H^k, M)}{\mathrm{Nor}^{+}(p^{-k}, 1)}, & \text{when } p \mid N; \\ \dfrac{\mathrm{Den}(\delta_p(N) \oplus H^k, M)}{\mathrm{Nor}^{(N,p)_p}(p^{-k}, 2)}, & \text{when } p \nmid N. \end{cases}$$

Theorem

For any rank 2 lattice $M \subset \mathbb{W}$, the following identity holds,

$$\mathrm{Den}_{\Delta(N)}(X, M) = \mathrm{Den}(X, M \oplus \mathbb{Z}_p \cdot x_0) - X^2 \cdot \mathrm{Den}(X, M \oplus \mathbb{Z}_p \cdot p^{-1} x_0).$$

Difference formula at the analytic side

- We also define

$$\partial \mathrm{Den}(L) = -\frac{\mathrm{d}}{\mathrm{d}X}\Big|_{X=1} \mathrm{Den}(X, L).$$

$$\partial \mathrm{Den}_{\Delta(N)}(M) = -\frac{\mathrm{d}}{\mathrm{d}X}\Big|_{X=1} \mathrm{Den}_{\Delta(N)}(X, M).$$

Corollary

The lattice $M \oplus \mathbb{Z}_p \cdot x_0$ can't be isometrically embedded into the lattice H^2, hence $\mathrm{Den}(1, M \oplus \mathbb{Z}_p \cdot x_0) = 0$.

$$\partial \mathrm{Den}_{\Delta(N)}(M) = \partial \mathrm{Den}(M \oplus \mathbb{Z}_p \cdot x_0) - \partial \mathrm{Den}(M \oplus \mathbb{Z}_p \cdot p^{-1} x_0).$$

A theorem of Gross and Keating

Theorem ([GK93],[Rap07],[Wed07])

For any rank 3 lattice $L \subset \mathbb{B}$,

$$\mathrm{Int}^\sharp(L) = \partial\mathrm{Den}(L).$$

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For any rank 3 lattice $L \subset \mathbb{B}$,

$$\mathrm{Int}^\sharp(L) = \partial\mathrm{Den}(L).$$

Combing this with two difference formulas, we obtain

Theorem (KR conjecture for the RZ space $\mathcal{N}_0(N)$)

For any rank 2 lattice $M \subset \mathbb{W}$,

$$\mathrm{Int}(M) = \partial\mathrm{Den}_{\Delta(N)}(M).$$

Formal uniformization

- There is an isomorphism of formal stacks over W,

$$\hat{\mathcal{X}}_0(N)/_{(\mathcal{X}_0(N)^{ss}_{\mathbb{F}_p})} \xrightarrow[\sim]{\Theta_{\mathcal{X}_0(N)}} B^\times(\mathbb{Q})_0 \backslash [\mathcal{N}_0(N) \times \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)]$$

  where $B^\times(\mathbb{Q})_0$ is the subgroup of $B^\times(\mathbb{Q})$ consisting of elements whose norm has p-adic valuation 0.

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  where $B^\times(\mathbb{Q})_0$ is the subgroup of $B^\times(\mathbb{Q})$ consisting of elements whose norm has p-adic valuation 0.

  As a corollary, we have the formal uniformization of the special cycles,

$$\hat{\mathcal{Z}}^{ss}(T) = \sum_{\substack{\boldsymbol{x} \in B^\times(\mathbb{Q})_0 \backslash (\Delta(N)^{(p)})^2 \\ T(\boldsymbol{x}) = T}} \sum_{g \in B_{\boldsymbol{x}}^\times(\mathbb{Q})_0 \backslash \mathrm{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)} 1_{\Delta(N) \otimes \hat{\mathbb{Z}}^p}(g^{-1}\boldsymbol{x}) \cdot \Theta^{-1}_{\mathcal{X}_0(N)}(\mathcal{Z}(\boldsymbol{x}), g).$$

Proof strategy

Theorem

For any rank 2 lattice $M \subset \mathbb{W}$, the following identity holds,

$$\mathrm{Den}_{\Delta(N)}(X, M) = \mathrm{Den}(X, M \oplus \mathbb{Z}_p \cdot x_0) - X^2 \cdot \mathrm{Den}(X, M \oplus \mathbb{Z}_p \cdot p^{-1} x_0).$$

Key idea: First embed x_0 to the large self-dual lattice H^k, the depth of the an embedding is defined to be

$$x_0 \in p^t H^k, \text{ but } x_0 \notin p^{t+1} H^k.$$

then embed M into $\{x_0\}^\perp \subset H^k$, which is totally determined by the depth of x_0!

Lemma (Witt theorem for lattices, [Mor79])

Let H be a self-dual quadratic lattice, if x_1 and x_2 has the same depth and norm, then there exists $g \in \mathrm{O}(H)$ such that $g \cdot x_1 = x_2$.

Thank you!

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