Arithmetic Siegel-Weil formula on $\mathcal{X}_0(N)$

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Let $Y_0(1) := \text{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$ be the modular curve, it parameterize elliptic curves over $\mathbb{C}$ by

$$\tau \in \mathcal{H} \mapsto E_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau.$$ 

Let $X_0(1)$ be its compactification.

Given an integer $m > 0$, we consider the following moduli problem: for a $\mathbb{C}$-scheme $S$,

$$Z(m)(S) = \{(E, \alpha) : E/S \text{ is an elliptic curve, }$$

$$\alpha \in \text{End}_S(E) \text{ satisfying } \alpha^2 = -m \}.$$
Counting points on the modular curve $X_0(1)$

- The moduli problem $Z(m)$ parameterise elliptic curves with complex multiplication by the order $\mathcal{O}_m = \mathbb{Z} + \mathbb{Z} \cdot \sqrt{-m}$.
- It can be shown the set $Z(m)(\mathbb{C})$ consists of finitely many points.

A natural question: what’s $\#Z(m)(\mathbb{C})$?
Counting points on the modular curve $X_0(1)$

$$\#Z(m)(\mathbb{C}) = \sum_{E} \#\{\alpha \in \text{End}_{\mathbb{C}}(E) : \alpha^2 = -m.\} / \text{aut}(E).$$
Counting points on the modular curve $X_0(1)$

$$\# Z(m)(\mathbb{C}) = \sum_{E} \# \{ \alpha \in \text{End}_{\mathbb{C}}(E) : \alpha^2 = -m \} / \text{aut}(E).$$

If $E = E_\tau$ appears on the right hand side, then $\tau$ satisfies a quadratic equation

$$a\tau^2 + b\tau + c = 0.$$

where $a, b, c \in \mathbb{Z}$ and $\gcd(a, b, c) = 1$, the discriminant of this equation is $b^2 - 4ac$.

Recall that the discriminant of $O_m$ is $-4m$, then there exists an integer $k > 0$ such that

$$-4m = k^2(b^2 - 4ac).$$
Counting points on the modular curve $X_0(1)$

Then by the theory of complex multiplication, we have

$$
\#Z(m)(\mathbb{C}) = \sum_{E} \#\{\alpha \in \text{End}_\mathbb{C}(E) : \alpha^2 = -m.\}/\text{aut}(E).
$$

$$
= \sum_{k \geq 0 : k^2 | 4m} h\left(\frac{4m}{k^2}\right) = H(4m).
$$

Recall that for a positive integer $N$,

$$
H(N) = \#\text{SL}_2(\mathbb{Z})\text{-equivalence classes of positive definite binary quadratic form of disc } -N.
$$

$$
h(N) = \#\text{SL}_2(\mathbb{Z})\text{-equivalence classes of primitive positive definite binary quadratic form of disc } -N.
$$
The modular curve $X_0(1)$ is the (compactified) GSpin Shimura variety attached to the rank 3 quadratic lattice $V = M_2(\mathbb{Z})^{\text{tr}=0}$, because

$$\text{GSpin}(V_\mathbb{Q}) \cong \text{GL}_2, \quad \text{GSpin}(V) \cong \text{GL}_2(\mathbb{Z})$$

There is an Eisenstein series $E(z, s, 1_{V \otimes \hat{\mathbb{Z}}})$ associated to the lattice $V$ via Weil representation.
The modular curve $X_0(1)$ is the (compactified) GSpin Shimura variety attached to the rank 3 quadratic lattice $V = M_2(\mathbb{Z})^\text{tr}=0$, because

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There is an Eisenstein series $E(z, s, 1_{V \otimes \hat{\mathbb{Z}}})$ associated to the lattice $V$ via Weil representation.

**Theorem (Geometric Siegel-Weil formula on $Y_0(1)$)**

*Let $m > 0$ be an integer, then*

$$\#Z(m)(\mathbb{C}) \cdot q^m = \frac{1}{12} \cdot E_m(z, \frac{1}{2}, 1_{V \otimes \hat{\mathbb{Z}}}).$$
Computing the Eisenstein series

We can also compute the Fourier coefficients $E_m(z, \frac{1}{2}, 1_{V \otimes \hat{Z}})$ in the following way,

$$E_m(z, \frac{1}{2}, 1_{V \otimes \hat{Z}}) = 4\pi (1 + i) \sqrt{m} \cdot q^m \prod_p W_{m,p}(1, \frac{1}{2}, 1_{V \otimes \mathbb{Z}_p}).$$
Computing the Eisenstein series

We can also compute the Fourier coefficients $E_m(z, \frac{1}{2}, 1\cdot \mathbb{V} \otimes \mathbb{Z})$ in the following way,

$$E_m(z, \frac{1}{2}, 1\cdot \mathbb{V} \otimes \mathbb{Z}) = 4\pi (1 + i) \sqrt{m} \cdot q^m \prod_p W_m,p(1, \frac{1}{2}, 1\cdot \mathbb{V} \otimes \mathbb{Z}_p).$$

Assume $-m < -4$ is a fundamental discriminant, by the works of Kudla, Rapoport and Yang

$$E_m(z, \frac{1}{2}, 1\cdot \mathbb{V} \otimes \mathbb{Z}) = (3 - \chi_m(2)) \cdot \frac{\sqrt{m}}{\pi} L(1, \chi_m) \cdot q^m.$$
On the other hand,

\[ H(4m) = h(m) + h(4m) = (3 - \chi_m(2)) \cdot h(m). \]
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The geometric Siegel-Weil formula on \( Y_0(1) \) implies

**Theorem (Class number formula)**

*Let \(-m < -4\) be a fundamental discriminant, then*

\[ h(m) = \frac{\sqrt{m}}{\pi} L(1, \chi_m). \]
The modular curve $Y_0(N)$

Let $Y_0(N) = \Gamma_0(N) \backslash \mathcal{H}$, and $X_0(N) = Y_0(N) \cup \{\text{cusps}\}$. The modular curve $Y_0(N)$ parameterizes cyclic isogenies between elliptic curves over $\mathbb{C}$ by the following

$$\tau \in \mathcal{H} \longmapsto \left( E_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau \to E_{N \tau} \right).$$

Given an integer $m > 0$, we consider the following moduli problem: for a $\mathbb{C}$-scheme $S$,

$$Z(m)(S) = \{(E \xrightarrow{\pi} E', \alpha) : E \xrightarrow{\pi} E' \text{ is a cyclic isogeny between elliptic curves}, \alpha \in \text{Hom}_S(E, E') \text{ satisfies } \alpha^\vee \circ \pi + \pi^\vee \circ \alpha = 0 \text{ and } \alpha^\vee \circ \alpha = m \}. $$
Let $N > 0$ be an integer, and $\Delta(N)$ be the following rank 3 quadratic lattice over $\mathbb{Z}$,

$$\Delta(N) = \left\{ x = \begin{pmatrix} -Na & b \\ c & a \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

Then the geometric Siegel-Weil formula on $X_0(N)$ is proved by Tuoping and Tonghai \cite{DY19}, Theorem (Geometric Siegel-Weil formula on $X_0(N)$)
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Then the geometric Siegel-Weil formula on $X_0(N)$ is proved by Tuoping and Tonghai [DY19],

**Theorem (Geometric Siegel-Weil formula on $X_0(N)$)**

For an integer $m > 0$, we have

$$\#Z(m)(\mathbb{C}) \cdot q^m = \frac{\psi(N)}{12} E(z, \frac{1}{2}, 1_{\Delta(N)}(\hat{\mathbb{Z}})).$$

Here $\psi(N) = N \prod_{p|N} (1 + p^{-1})$.  

The stack $\mathcal{X}_0(N)$

Let $\mathcal{Y}_0(N)$ be the stack of $\Gamma_0(N)$-level structures on elliptic curves defined by Katz and Mazur in [KM85]: for a scheme $S$,

$\mathcal{Y}_0(N)(S) = \{ E \overset{\pi}{\to} E' : \pi \text{ is a cyclic isogeny and } \pi^\vee \circ \pi = N \}$. 
Let $\mathcal{Y}_0(N)$ be the stack of $\Gamma_0(N)$-level structures on elliptic curves defined by Katz and Mazur in [KM85]: for a scheme $S$,

$$\mathcal{Y}_0(N)(S) = \{ E \xrightarrow{\pi} E' : \pi \text{ is a cyclic isogeny and } \pi^\vee \circ \pi = N \}.$$ 

here $\pi$ is cyclic means that the order $N$ group scheme $G := \text{ker}(\pi)$ is a cyclic group scheme in the sense that there exists a section $P \in G(S)$ such that for any $f \in \mathcal{O}_G$,

$$\text{det}(T - f) = \prod_{a=1}^{N} (T - f(aP)).$$

Let $\mathcal{X}_0(N)$ be its compactification.
Special cycles on $\mathcal{X}_0(N)$

Given an integer $m > 0$, we consider the following moduli problem: for a scheme $S$,

$$Z(m)(S) = \{ (E \xrightarrow{\pi} E', \alpha) : \pi \text{ is a cyclic } N\text{-isogeny, } \alpha \text{ is an isogeny from } E \text{ to } E' \text{ satisfying } \alpha^\vee \circ \alpha = m \text{ and } \alpha^\vee \circ \pi + \pi^\vee \circ \alpha = 0. \}.$$

It is a generalized Cartier divisor on $\mathcal{X}_0(N)$ and has no intersections with cusps.
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It is a generalized Cartier divisor on $\mathcal{X}_0(N)$ and has no intersections with cusps.

- Given a $2 \times 2$ positive definite symmetric matrix $T$, we define the moduli problem $\mathcal{Z}(T)$ as follows: for a scheme $S$,

$$\mathcal{Z}(T)(S) = \{(E \xrightarrow{\pi} E', \alpha_1, \alpha_2) : \pi \text{ is a cyclic } N\text{-isogeny, } \alpha_i \text{ are isogenies from } E \text{ to } E' \text{ satisfying } \frac{1}{2}(\alpha_i, \alpha_j) = T \text{ and } \alpha_i^\vee \circ \pi + \pi^\vee \circ \alpha_i = 0.\}.$$
The special cycle $\mathcal{Z}(T)$

Let $T \in \text{Sym}_2(\mathbb{Q})$ be a nonsingular matrix, we define the following difference set

$$\text{Diff}(T, \Delta(N)) = \{ l \text{ is a finite prime} : T \text{ is not represented by } \Delta(N) \otimes \mathbb{Q}_l. \}$$
The special cycle $\mathcal{Z}(T)$

Let $T \in \text{Sym}_2(\mathbb{Q})$ be a nonsingular matrix, we define the following difference set

$$\text{Diff}(T, \Delta(N)) = \{ l \text{ is a finite prime : } T \text{ is not represented by } \Delta(N) \otimes \mathbb{Q}_l \}.$$

Lemma

Let $T \in \text{Sym}_2(\mathbb{Q})$ be a nonsingular matrix. If $\mathcal{Z}(T)(\overline{\mathbb{F}}_p) \neq \emptyset$ for some prime $p$, then $T$ is positive definite, and

$$\text{Diff}(T, \Delta(N)) = \{ p \}.$$  

Moreover, in this case, the special cycle $\mathcal{Z}(T)$ is supported in the supersingular locus of the special fiber $\mathcal{X}_0(N)_{\mathbb{F}_p}$. 
Arithmetic Siegel-Weil formula on $\mathcal{X}_0(N)$

Let $T$ be a $2 \times 2$ positive definite symmetric matrix with diagonal elements $m_1, m_2$, then we define

$$\deg(Z(T)) = \chi(Z(T), \mathcal{O}_{Z(m_1)} \otimes \mathcal{O}_{Z(m_2)}) \cdot \log p.$$ 

where $p \in \text{Diff}(T, \Delta(N))$. 

Arithmetic Siegel-Weil formula on $\mathcal{X}_0(N)$

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$$\deg(\mathcal{Z}(T)) = \chi(\mathcal{Z}(T), \mathcal{O}_{\mathcal{Z}(m_1)} \otimes^L \mathcal{O}_{\mathcal{Z}(m_2)}) \cdot \log p.$$ 

where $p \in \text{Diff}(T, \Delta(N))$.

Theorem (Arithmetic Siegel-Weil formula on $\mathcal{X}_0(N)$)

Let $T \in \text{Sym}_2(\mathbb{Q})$ be a positive definite symmetric matrix, then

$$\deg(\mathcal{Z}(T))q^T = \frac{\psi(N)}{24} \cdot E'(z, 0, 1_{(\Delta(N) \otimes \hat{\mathbb{Z}})^2}).$$

where $z = x + iy \in \mathcal{H}_2$ and $q^T = e^{2\pi i \text{tr}(Tz)}$. 
Key ingredients

- Formal uniformization of the supersingular locus of $\mathcal{X}_0(N)$. It connects intersection numbers on $\mathcal{X}_0(N)$ with local arithmetic intersection numbers on the RZ space associated to $\mathcal{X}_0(N)$. 

- Kudla-Rapoport conjecture for the RZ space associated to $\mathcal{X}_0(N)$. It connects local arithmetic intersection numbers on RZ space with Whittaker functions. Both are proved by embedding trick!
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Both are proved by embedding trick!
RZ space associated to $\mathcal{X}_0(1) \times \mathcal{X}_0(1)$

Let $X$ be a $p$-divisible group of dim 1, height 2. Consider the following functor: for every $S \in \text{Nilp}_W$, the set $\mathcal{N}(S)$ consists of $((X, X'), (\rho, \rho'), (\lambda, \lambda'))$, where

1. $X$ and $X'$ are two $p$-divisible group over $S$, $\rho$ and $\rho'$ are two height 0 quasi-isogenies between $p$-divisible groups $\rho : X \times \mathbb{F} S \rightarrow X \times_S S$, $\rho' : X \times \mathbb{F} S \rightarrow X' \times_S S$.
2. $\lambda : X \rightarrow X^\vee$, $\lambda' : X' \rightarrow X'^\vee$ are two principal polarizations, such that Zariski locally on $\overline{S}$, we have

\[
\rho^\vee \circ \lambda \circ \rho = c(\rho) \cdot \lambda_0, \quad \rho'^\vee \circ \lambda \circ \rho' = c(\rho') \cdot \lambda_0.
\]

for some $c(\rho) = c(\rho') \in \mathbb{Z}_p^\times$. 

\[\mathcal{N} \cong \text{Spf} W[[t_1, t_2]].\]
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Let $\mathfrak{X}$ be a $p$-divisible group of dim 1, height 2. Consider the following functor: for every $S \in \text{Nilp}_W$, the set $\mathcal{N}(S)$ consists of $((X, X'), (\rho, \rho'), (\lambda, \lambda'))$, where

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$$\rho^\vee \circ \lambda \circ \rho = c(\rho) \cdot \lambda_0, \quad \rho'^{\vee} \circ \lambda \circ \rho' = c(\rho') \cdot \lambda_0.$$ 

for some $c(\rho) = c(\rho') \in \mathbb{Z}_p^\times$.

$\mathcal{N} \simeq \text{Spf } W[[t_1, t_2]]$. 
Special cycles on \( \mathcal{N} \)

Let \( \mathbb{B} \) the unique division quaternion algebra over \( \mathbb{Q}_p \), it is isometric to \( \text{End}^0(X) \).

**Definition**

For any subset \( L \subset \mathbb{B} \), define the special cycle \( Z^\#(L) \subset \mathcal{N} \) to be the closed formal subscheme cut out by the condition,

\[
\rho'^\text{univ} \circ x \circ (\rho^\text{univ})^{-1} \in \text{Hom}(X^\text{univ}, X'^\text{univ}).
\]

for all \( x \in L \).
Special cycles on $\mathcal{N}$

Let $\mathcal{B}$ the unique division quaternion algebra over $\mathbb{Q}_p$, it is isometric to $\text{End}^0(X)$.

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for all $x \in L$.

Let $L$ be a rank 3 lattice with basis $x_1, x_2$ and $x_3$. Define the local arithmetic intersection number on $\mathcal{N}$ to be

$$\text{Int}^\#(L) = \chi(\mathcal{N}, \mathcal{O}_{Z^\#(x_1)} \otimes^L \mathcal{O}_{\mathcal{N}} \mathcal{O}_{Z^\#(x_2)} \otimes^L \mathcal{O}_{\mathcal{N}} \mathcal{O}_{Z^\#(x_3)}).$$
For any $x \in \mathbb{B}$, the special cycle $\mathcal{Z}^\#(x)$ is cut out by a single equation $f_x \in W[[t_1, t_2]]$, define $d_x = f_x/f_{p^{-1}x} \in W[[t_1, t_2]]$ and the difference divisor $\mathcal{D}(x) = \text{Spf} \ W[[t_1, t_2]]/(d_x)$.
Difference divisor on $\mathcal{N}$

For any $x \in \mathbb{B}$, the special cycle $\mathcal{Z}^\#(x)$ is cut out by a single equation $f_x \in \mathcal{W}[[t_1, t_2]]$, define $d_x = f_x/f_{p^{-1}x} \in \mathcal{W}[[t_1, t_2]]$ and the difference divisor $\mathcal{D}(x) = \text{Spf} \mathcal{W}[[t_1, t_2]]/(d_x)$

Theorem

The difference divisor $\mathcal{D}(x)$ is regular.

Recently we have proved that difference divisors on GSpin RZ spaces with hyperspecial level structure are regular, the formal scheme $\mathcal{N}$ is a special example of such RZ spaces.
RZ space associated to $\mathcal{X}_0(N)$

Fix a $N$-isogeny $\chi_0 : \mathbb{X} \to \mathbb{X}$. Consider the following functor: for every $S \in \text{Nilp}_W$, the set $\mathcal{N}_0(N)(S)$ consists of

$(\chi : X \to X', (\rho, \rho'), (\lambda, \lambda'))$, where

1. $X$ and $X'$...
2. $\lambda : X \to X^\vee$, $\lambda' : X' \to X'^\vee$...
3. $\chi : X \to X'$ is a cyclic isogeny (i.e., $\ker(\chi)$ is a cyclic group scheme over $S$) lifting $\rho' \circ \chi_0 \circ \rho^{-1}$. 

Theorem ([KM85])

The natural morphism $\mathcal{N}_0(N) \to \mathcal{N}$ is a closed immersion, and $\mathcal{N}_0(N)$ is regular.
Fix a $N$-isogeny $x_0 : \mathcal{X} \to \mathcal{X}$. Consider the following functor: for every $S \in \text{Nilp}_W$, the set $\mathcal{N}_0(N)(S)$ consists of $(X \to X', (\rho, \rho'), (\lambda, \lambda'))$, where

1. $X$ and $X'$ ...
2. $\lambda : X \to X^\vee$, $\lambda' : X' \to X'^\vee$ ...
3. $x : X \to X'$ is a cyclic isogeny (i.e., $\ker(x)$ is a cyclic group scheme over $S$) lifting $\rho' \circ x_0 \circ \rho^{-1}$.

**Theorem ([KM85])**

*The natural morphism $\mathcal{N}_0(N) \to \mathcal{N}$ is a closed immersion, and $\mathcal{N}_0(N)$ is regular.*
Recall that we have fixed a $N$-isogeny $x_0$ when we define $\mathcal{N}_0(N)$.

**Theorem**

*There is an isomorphism between formal schemes,*

$$\mathcal{D}(x_0) \overset{\sim}{\longrightarrow} \mathcal{N}_0(N).$$
Remark

- By the isomorphism and Zink’s windows theory, we can compute the special fiber $\mathcal{N}_0(p^n)_p$ as follows

$$\mathbb{F}[[t_1, t_2]]/ \left( (t_1 - t_2^{p^n}) \cdot (t_2 - t_1^{p^n}) \cdot \prod_{a+b=n \atop a,b \geq 1} (t_1^{p^{a-1}} - t_2^{p^{b-1}})^{p-1} \right).$$

which coincides with Katz-Mazur’s computation.
Special cycles on $\mathcal{N}_0(N)$

Let $W = \{x_0\} \perp \subset B$.

**Definition**

For any subset $M \subset W$, define the special cycle $Z(M) \subset \mathcal{N}_0(N)$ to be the closed formal subscheme cut out by the condition,

$$\rho'_{\text{univ}} \circ x \circ (\rho_{\text{univ}})^{-1} \in \text{Hom}(X_{\text{univ}}, X'_{\text{univ}}).$$

for all $x \in M$.

Let $M$ be a rank 2 lattice with basis $x_1$ and $x_2$. Define the local arithmetic intersection number on $\mathcal{N}_0(N)$ to be

$$\text{Int}(M) = \chi(\mathcal{N}_0(N), \mathcal{O}_{Z(x_1)} \otimes_{\mathcal{O}_{\mathcal{N}_0(N)}} \mathcal{O}_{Z(x_2)}).$$
Difference formula at the geometric side

By the isomorphism $\mathcal{D}(x_0) \simeq \mathcal{N}_0(N)$, we can prove the following theorem.

**Theorem**

*For any rank 2 lattice $M \subset \mathbb{W}$, the following identity holds,*

\[
\text{Int}(M) = \text{Int}^\#(M \oplus \mathbb{Z}_p \cdot x_0) - \text{Int}^\#(M \oplus \mathbb{Z}_p \cdot p^{-1}x_0).
\]
Local density

- For two quadratic lattice $L$ and $M$, the local density is defined to be

$$\text{Den}(M, L) = \lim_{d \to \infty} \# \text{Rep}_{M,L}(\mathbb{Z}_p/p^d) \cdot \frac{d^{\dim(\text{Rep}_{M,L})_{\mathbb{Q}_p}}}{p^d \cdot \dim(\text{Rep}_{M,L})_{\mathbb{Q}_p}}.$$ 

- Let $H$ be a rank 2 quadratic lattice given by $q_H(x, y) = xy$, define the local density polynomial to be (rank $L = 2n - 1$)

$$\text{Den}(X, L) \bigg|_{X=p^{-k}} = \frac{\text{Den}(H^{k+n}, L)}{\text{Nor}^+(p^{-k}, 2n - 1)},$$

where $\text{Nor}^\epsilon(X, m) = (1 - \frac{1+(-1)^{m+1}}{2} \cdot \epsilon q^{-(m+1)/2} X) \prod_{1 \leq i < (m+1)/2} (1 - q^{-2i} X^2)$.
Examples of local density

- When $m$ is squarefree, then
  \[ \text{Den}(\Delta(1) \otimes \mathbb{Z}_p, \langle m \rangle) = 1 - \chi_m(p)p^{-1}. \]

- When $\nu_p(N) = 0$ or $1$, we have
  \[ \text{Den}(H^k, \langle N \rangle) = \begin{cases} 
  (1 - p^{-k})(1 + p^{1-k}), & \text{when } p \mid N; \\
  1 - p^{-k}, & \text{when } p \nmid N.
\end{cases} \]
Difference formula at the analytic side

Let $\delta_p(N) = \Delta(N) \otimes \mathbb{Z}_p$, define the following local density function with level $N$,

$$\text{Den}_{\Delta(N)}(X, M) \mid_{X=p^{-k}} = \begin{cases} \frac{\text{Den}(\delta_p(N) \oplus H^k, M)}{\text{Nor}^+(p^{-k}, 1)}, & \text{when } p \mid N; \\ \frac{\text{Den}(\delta_p(N) \oplus H^k, M)}{\text{Nor}(N,p)^p(p^{-k}, 2)}, & \text{when } p \nmid N. \end{cases}$$

**Theorem**

*For any rank 2 lattice $M \subset \mathbb{W}$, the following identity holds,*

$$\text{Den}_{\Delta(N)}(X, M) = \text{Den}(X, M \oplus \mathbb{Z}_p \cdot x_0) - X^2 \cdot \text{Den}(X, M \oplus \mathbb{Z}_p \cdot p^{-1} x_0).$$
Difference formula at the analytic side

- We also define

\[ \partial \text{Den}(L) = - \frac{d}{dX} \bigg|_{X=1} \text{Den}(X, L). \]

\[ \partial \text{Den}_{\Delta(N)}(M) = - \frac{d}{dX} \bigg|_{X=1} \text{Den}_{\Delta(N)}(X, M). \]

**Corollary**

*The lattice* \( M \oplus \mathbb{Z}_p \cdot x_0 \) *can’t be isometrically embedded into the lattice* \( H^2 \), *hence* \( \text{Den}(1, M \oplus \mathbb{Z}_p \cdot x_0) = 0. \)

\[ \partial \text{Den}_{\Delta(N)}(M) = \partial \text{Den}(M \oplus \mathbb{Z}_p \cdot x_0) - \partial \text{Den}(M \oplus \mathbb{Z}_p \cdot p^{-1}x_0). \]
A theorem of Gross and Keating

\textbf{Theorem ([GK93],[Rap07],[Wed07])}

For any rank 3 lattice \( L \subset \mathbb{B}, \)

\[ \text{Int}^\#(L) = \partial \text{Den}(L). \]
A theorem of Gross and Keating

**Theorem ([GK93],[Rap07],[Wed07])**

For any rank 3 lattice \( L \subset \mathbb{B} \),

\[
\text{Int}^\#: (L) = \partial \text{Den}(L).
\]

Combing this with two difference formulas, we obtain

**Theorem (KR conjecture for the RZ space \( \mathcal{N}_0(N) \))**

For any rank 2 lattice \( M \subset \mathcal{W} \),

\[
\text{Int}(M) = \partial \text{Den}_{\Delta(N)}(M).
\]
Formal uniformization

There is an isomorphism of formal stacks over $W$,

$$
\hat{X}_0(N)/(x_0(N)_{s_{s}}) \xrightarrow{\Theta_{x_0(N)}} B^\times(\mathbb{Q})_0 \backslash [\mathcal{N}_0(N) \times \text{GL}_2(\mathbb{A}_f^p)/\Gamma_0(N)(\hat{\mathbb{Z}}^p)]
$$

where $B^\times(\mathbb{Q})_0$ is the subgroup of $B^\times(\mathbb{Q})$ consisting of elements whose norm has $p$-adic valuation 0.
There is an isomorphism of formal stacks over \( W \),

\[
\hat{\mathcal{X}}_0(N)/(x_0(N)_{\hat{\mathbb{F}}_p}) \xrightarrow{\Theta_{\mathcal{X}_0(N)}} B^{\times}(\mathbb{Q})_0 \backslash [\mathcal{N}_0(N) \times \text{GL}_2(\mathbb{A}_f^p) / \Gamma_0(N)(\hat{\mathbb{Z}}^p)]
\]

where \( B^{\times}(\mathbb{Q})_0 \) is the subgroup of \( B^{\times}(\mathbb{Q}) \) consisting of elements whose norm has \( p \)-adic valuation 0.

As a corollary, we have the formal uniformization of the special cycles,

\[
\hat{\mathcal{Z}}^{ss}(T) = \sum_{x \in B^{\times}(\mathbb{Q})_0 \backslash (\Delta(N)(p))^2} \sum_{g \in B^{\times}_x(\mathbb{Q})_0 \backslash \text{GL}_2(\mathbb{A}_f^p) / \Gamma_0(N)(\hat{\mathbb{Z}}^p)} 1_{\Delta(N) \otimes \hat{\mathbb{Z}}_p} (g^{-1}x) \cdot \Theta_{\mathcal{X}_0(N)}^{-1}(\mathcal{Z}(x), g).
\]
Proof strategy

Theorem

*For any rank 2 lattice* \( M \subset \mathbb{W} \), the following identity holds,

\[
\text{Den}_{\Delta(N)}(X, M) = \text{Den}(X, M \oplus \mathbb{Z}_p \cdot x_0) - X^2 \cdot \text{Den}(X, M \oplus \mathbb{Z}_p \cdot p^{-1} x_0).
\]

Key idea: First embed \( x_0 \) to the large self-dual lattice \( H^k \), the depth of the an embedding is defined to be

\[ x_0 \in p^t H^k, \text{ but } x_0 \notin p^{t+1} H^k. \]

then embed \( M \) into \( \{ x_0 \}^\perp \subset H^k \), which is totally determined by the depth of \( x_0 \)!

Lemma (Witt theorem for lattices, [Mor79])

*Let* \( H \) *be a self-dual quadratic lattice, if* \( x_1 \) *and* \( x_2 \) *has the same depth and norm, then there exists* \( g \in O(H) \) *such that* \( g \cdot x_1 = x_2 \).
Thank you!
References


Torsten Wedhorn. Calculation of representation densities.