1. Mahler's Theorem

F is a finite extension of $\mathbb{Q}_p$, then for any closed subring $A \subset F$, and $f \in C(\mathbb{Z}_p, A)$

- $$a_n(f) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k) \in A$$

$$\lim_{n\to\infty} a_n(f) = 0$$

- $$f(x) = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n} \quad \text{on} \quad \mathbb{Z}_p \qquad (*)$$

by $a_n(f) \to 0$, this sequence converges uniformly

Sketch of the proof:

build up $(*)$ on $\mathbb{Z}$, and use the fact that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, the main difficulty is to prove $\lim_{n\to\infty} a_n(f) = 0$

Rmk: We can define a norm on $C(\mathbb{Z}_p, A)$, by

$$|f|_p := \sup_{x \in \mathbb{Z}_p} |f(x)|$$

then it's easy to see that:

- $|f|_p = \sup_n |a_n(f)|$
- $C(\mathbb{Z}_p, A)$ is complete w.r.t. this norm

2. p-adic measures on $\mathbb{Z}_p$

$A \overset{\text{closed}}{\subset} F$, $\varphi : C(\mathbb{Z}_p, A) \longrightarrow A$ is A-linear

φ is p-adic bounded measure : $\exists B > 0$, s.t.

$$|\varphi(f)|_p \leq B \cdot |f|_p , \forall f$$

$\mathrm{Meas}(\mathbb{Z}_p ; A)$ = Space of p-adic bounded measure
$\quad \| \qquad$ (with an obvious A-mod structure)
$\mathcal{M}(\mathbb{Z}_p, A)$

$$|\varphi|_p = \sup_{|\phi|_p = 1} |\varphi(f)| \rightsquigarrow \mathcal{M}(\mathbb{Z}_p, A) \text{ is complete}$$

- Structure of $\mathcal{M}(\mathbb{Z}_p, A)$

$f \in C(\mathbb{Z}_p, A)$ has an expansion $f = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n}$

$$f_m := \sum_{n=0}^{m} a_n(f) \binom{x}{n}, \qquad f_m \rightrightarrows f$$

$$\varphi(f) = \lim_{m \to \infty} \varphi(f_m) = \lim_{m \to \infty} \sum_{n=0}^{m} a_n(f) \varphi\left(\binom{x}{n}\right)$$

$$= \sum_{n=0}^{\infty} a_n(f) \varphi\left(\binom{x}{n}\right) \qquad \left|\binom{x}{n}\right|_p = 1$$

these equalities hold: $\left|\varphi\left(\binom{x}{n}\right)\right| \leq |\varphi|_p$ bounded

φ is totally determined by the bounded sequence

$$\left\{ \varphi\left(\binom{x}{n}\right) \right\}_{n=0}^{\infty}, \text{ and } |\varphi|_p = \sup_n \left|\varphi\left(\binom{x}{n}\right)\right|$$

For any bounded sequence $\{b_n\}_{n=0}^{\infty} \rightsquigarrow \varphi_b \in \mathcal{M}(\mathbb{Z}_p, A)$

therefore $\mathcal{M}(\mathbb{Z}_p, A) \leftrightarrow$ bounded sequence space

- Power series associated to a measure

$$\varphi \in \mathcal{M}(\mathbb{Z}_p, A)$$

$$\Phi_\varphi(T) := \sum_{n=0}^{\infty} \left(\int \binom{x}{n} d\varphi \right) T^n$$

this gives: $\mathcal{M}(\mathbb{Z}_p, A) \hookrightarrow A[[T]]$

If A is already bounded (i.e. $A \subset \mathcal{O}_F$)

then this is an isomorphism, $\mathcal{M}(\mathbb{Z}_p, \mathcal{O}_F) \xrightarrow{\sim} \mathcal{O}_F[[T]]$

We will mainly focus on this case ($A = \mathcal{O}_F$)

then the ring structure of RHS gives multiplication on LHS

$$\varphi * \psi : f \longmapsto \iint f(x+y)\, d\varphi(x)\, d\psi(y)$$

3. Riemann zeta as a measure

- Main result: for $\forall a \geq 2$, $(a,p)=1$, $\exists !$ p-adic bounded measure ζ_a, s.t.

$$\int_{\mathbb{Z}_p} x^m \, d\zeta_a = (1-a^{m+1})\zeta(-m), \quad \text{for } \forall m \geq 0$$

Sketch of the proof:

every measure is determined by their effect on $\binom{x}{n}$, hence on x^n

therefore if ζ_a exists, then it is unique.

$$\binom{x}{n} = \sum_{k=0}^{n} C_{n,k} x^k$$

be careful, $C_{n,k} \notin A$ sometimes but the formula still holds

we only need to show: $\int_{\mathbb{Z}_p} \binom{x}{n} d\zeta_a$ is bounded

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\zeta_a = \sum_{k=0}^{n} C_{n,k} (1-a^{n+1}) \zeta(-n)$$

Chapter 1

$$= \sum_{k=0}^{n} C_{n,k} \left(t \frac{d}{dt}\right)^n \psi_a(t) \Big|_{t=1}$$

$$= \frac{1}{n!} \frac{d^n}{dt^n} \psi_a \Big|_{t=1}$$

where $\psi_a(t) = \dfrac{t}{1-t} - \dfrac{a t^a}{1-t^a} = - \dfrac{\sum_{b=1}^{a} \zeta(b)(1+t+\cdots+t^{b-1})}{1+t+\cdots+t^{a-1}}$

Why we need $p \nmid a$?

if we apply the same computation, we will finally ψ_a has the same form as before, then $\frac{d^n}{dt^n}\psi_a\Big|_{t=1}$ will give more p in the denominator.

By the formula

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\zeta_a = \frac{1}{n!} \frac{d^n}{dt^n} \psi_a(t) \Big|_{t=1}, \in \mathbb{Z}_p$$

we get the corresponding power series:

$$\Phi_{\zeta_a}(T) = \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{n} d\zeta_a \right) T^n = \psi_a(1+T) = a-1 - \frac{1}{T} + \frac{a}{(1+T)^a - 1} \in \mathbb{Z}_p[[T]]$$

- Can we remove a ?

If there exists $\Phi(T)$ (i.e. ζ) , s.t.

$$\int_{\mathbb{Z}_p} x^n d\zeta = \zeta(-n)$$

equivalently:

$$\left(t \frac{d}{dt}\right)^n \psi(t) \Big|_{t=1} = \zeta(-n) = -\frac{B_{n+1}}{n+1}$$

No ! consider $\tilde{\psi}(s) = \psi(e^s)$, then $\frac{d^n}{ds^n} \tilde{\psi} \Big|_{s=0} = -\frac{B_{n+1}}{n+1}$

$$\tilde{\psi}(s) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n \tilde{\psi}}{ds^n}(0) s^n = \frac{1}{s} - \frac{1}{e^s - 1}$$

$$\psi(1+T) = \frac{1}{\log(1+T)} - \frac{1}{T} \notin \mathbb{Z}_p[[T]]$$

4. Interpolation with Dirichlet character

Given $\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \simeq (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times \longrightarrow F^\times$

lift χ to a locally constant function $\chi : \mathbb{Z}_p \longrightarrow F^\times$ $(\chi(p\mathbb{Z}_p) = 0)$

We want to compute:

$$\int_{\mathbb{Z}_p} \chi(x) x^n \, d\zeta_a$$

- Theorem: set $\chi_0 = \chi$ if χ is non-trivial, $\chi_0 = 1$ if χ is trivial

$$\int_{\mathbb{Z}_p} \chi(x) x^n \, d\zeta_a = (1 - a^{n+1}\chi(a))(1 - \chi_0(p)p^n) L(-n, \chi_0)$$

if χ is primitive, and χ is even, then

$$\int_{\mathbb{Z}_p} \chi(x) x^{-1} \, d\zeta_a = \begin{cases} (1 - p^{-1}) \log_p a, & \chi = 1 \\ -(1 - \chi(a)) p^{-n} G(\chi) \sum\limits_{b \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi^{-1}(b) \log_p (1 - \zeta^b), & \chi \neq 1 \end{cases}$$

here $\log_p$ is p-adic logarithm defined over $\mathbb{C}_p^*$

ζ is primitive p^n-th root of unity

pf: for any locally constant $\phi \in C(\mathbb{Z}_p, A)$, it acts on ζ_a by:

$$\int f \, d([\phi]\zeta_a) := \int \phi f \, d\zeta_a$$

so we could derive an action formula of ϕ on $\Phi_{\zeta_a}(T) = \psi_a(t)$

$$([\phi]\Psi_a)(t) = \sum_{n=1}^{\infty} \phi(n) t^n - a \sum_{n=1}^{\infty} \phi(an) t^{an}$$

Define L-function: $L(s, \phi - a^{k+1}\phi_a) = \sum_{n=1}^{\infty} \left(\phi(n) - a^{k+1}\phi(an)\right) n^{-s}$

By Mellin transform:

$$L(-n, \phi - a^{n+1}\phi_a) = \left(t \frac{d}{dt}\right)^n \left([\phi]\Psi_a\right)(t) \Big|_{t=1}$$

$$= \int_{\mathbb{Z}_p} \phi(x) x^n \, d\zeta_a$$

replace ϕ by χ,

$$\int_{\mathbb{Z}_p} \chi(x) x^n \, d\zeta_a = L(-n, \chi - a^{n+1}\chi_a) = (1 - a^{n+1}\chi(a))(1 - \chi_0(p) p^n) L(-n, \chi_0)$$

Evaluation at -1 is more difficult

We have observed that: $[x] \longleftrightarrow t \frac{d}{dt} = (1+T)\frac{d}{dT}$

intuitively, $[x^{-1}] \longleftrightarrow [x]^{-1}$, and there do exists

$$\Phi_a(t) = \log \frac{1-t^a}{1-t}, \quad \text{s.t.} \quad [x]\Phi_a = \Psi_a$$

but x^{-1} is not a well-defined continuous function over $\mathbb{Z}_p$

and coefficient of Φ_a is p-adically unbounded, $\Phi_a \in \mathbb{Q}_p[[T]]$

We should combine χ and x^{-1} together

$$x \longmapsto \chi(x) x^{-1} \text{ is continuous, } 0 \text{ on } p\mathbb{Z}_p$$

then we can verify: $[\chi(x)x^{-1}]\Psi_a(t) = [\chi]\Phi_a(t) \quad \left([\chi]\Phi_a \in \mathcal{O}_F[[T]]\right)$

$$\int_{\mathbb{Z}_p} \chi(x) x^{-1} \, d\zeta_a = \left([\chi(x)x^{-1}]\Psi_a\right)(1) = \left([\chi]\Phi_a\right)(1)$$

By this theorem,

$$(1-a^{n+1}\chi(a))^{-1}\int_{\mathbb{Z}_p}\chi(x)x^n\,d\xi_a \quad \text{is independent of } a$$

To let n varies p-adically, we consider the decomposition

$$\mathbb{Z}_p^\times \simeq \mu_{\varphi(q)} \times (1+q\mathbb{Z}_p), \quad q=\begin{cases}4, & p=2\\ p, & p>2\end{cases}$$

$$x \mapsto (\omega(x), \ \langle x\rangle)$$

Def: p-adic Dirichlet L-function with character χ

$$L_p(s,\chi)=(1-\chi(a)\langle a\rangle^{1-s})^{-1}\int_{\mathbb{Z}_p}\chi\omega^{-1}(x)\langle x\rangle^{-s}\,d\xi_a$$

Rmk: the appearance of $\chi\omega^{-1}$ instead of χ is

because of we use $\langle a\rangle$ instead of a

values at negative integer:

$$L_p(-n,\chi)=(1-\chi(a)\langle a\rangle^{1+n})^{-1}\int_{\mathbb{Z}_p}\chi\omega^{-1}(x)\langle x\rangle^{n}\,d\xi_a$$

$$=(1-(\chi\omega^{-n-1})(a)\,a^{n+1})^{-1}\int_{\mathbb{Z}_p}\chi\omega^{-n-1}(x)\,x^n\,d\xi_a$$

$$=(1-(\chi\omega^{-n-1})_0(p)\,p^n)\,L(-n,(\chi\omega^{-n-1})_0)$$

$n\equiv m \mod p^q(p-1)$, $\quad \chi\omega^{-n-1}=\chi\omega^{-m-1}$

- For nontrivial χ, $L_p(s,\chi)$ is continuous on $\mathbb{Z}_p$

For $\chi = 1$, $\zeta_p(s) := L_p(s, 1)$ is continuous on $\mathbb{Z}_p - \{1\}$

Moreover, $L_p(s,\chi)$ is p-adic analytic on $\mathbb{Z}_p$

$\zeta_p(s)$ is p-adic meromorphic on $\mathbb{Z}_p - \{1\}$

if χ is odd, $L_p(n,\chi) = 0$ if n is odd $\Rightarrow L_p(s,\chi) = 0$

χ is even, $L_p(1,\chi) = -p^{-n} \sum_{b \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi^{-1}(b) \log_p(1-\zeta^b)$

$\zeta_p(s)$ has a pole with residue $1-p^{-1}$

$$\int_{\mathbb{Z}_p^\times} \chi(x) \langle x \rangle^{-s} d\zeta_a$$

5. pseudo-measure ζ_p

$F/\mathbb{Q}_p$ finite extension, $\mathcal{O}_F$: integral closure of $\mathbb{Z}_p$ in F

For any profinite group G, there is an isomorphism:

$$\mathcal{M}(G, \mathcal{O}_F) \simeq \varprojlim_{H \overset{\text{open}}{\in} G} \mathcal{O}_F[G/H] =: \Lambda(G)$$

$$\phi \longmapsto \Big(\sum_{a \in G/H} \phi(aH)\, aH \Big)$$

Iwasawa algebra of G

$\forall g \in G$ can be viewed as an element in $\Lambda(G)$

$$(gH)_H$$

therefore it corresponds to a measure $\delta_g \in \mathcal{M}(G, \mathcal{O}_F)$

$$\delta_g(f) = \int_G f\, d\delta_g = f(g) \quad \text{Dirac measure}$$

<u>Def</u>: a pseudo measure on G is

- an element $\mu \in Q(\Lambda(G))$ (total quotient ring)
- $(g-1)\mu \in \Lambda(G)$, $\forall g \in G$

Example:

- $G = \mathbb{Z}_p$

$\Lambda(\mathbb{Z}_p) \simeq \mathcal{O}_F[[T]]$, $Q(\Lambda(\mathbb{Z}_p)) \simeq \mathcal{O}_F((T)) \otimes_{\mathcal{O}_F} F$

$\forall a \in \mathbb{Z}_p$, δ_a corresponds to $(1+T)^a \in \mathcal{O}_F[[T]]$

$\mu \in \mathcal{O}_F((T)) \otimes_{\mathcal{O}_F} F$ is a pseudo-measure $\Leftrightarrow \mu \cdot \left((1+T)^a - (1+T)\right) \in \mathcal{O}_F[[T]], \forall a \in \mathbb{Z}_p$

$\Rightarrow \mu \in \frac{1}{T(T+1)} \mathcal{O}_F[[T]]$

- $G = \mathbb{Z}_p^\times$

there is an inclusion (but multiplication are different)

$$i : \Lambda(\mathbb{Z}_p^\times) \longrightarrow \Lambda(\mathbb{Z}_p)$$

$$\mu \longmapsto i(\mu) : f \mapsto \int_{\mathbb{Z}_p^\times} f|_{\mathbb{Z}_p^\times} d\mu$$

image of $\Lambda(\mathbb{Z}_p^\times)$ can be identified with

$$\Phi \in \mathcal{O}_F[[T]], \text{ s.t } \sum_{\xi \in \mu_p} \Phi(\xi(1+T)-1) = 0$$

example: $\Phi_a(T) = (1+T)^a$, $a \in \mathbb{Z}_p^\times \rightsquigarrow$ evaluation at a

then

$\mu \in Q(\Lambda(\mathbb{Z}_p^\times)) \subset F((T))$ is a pseudo-measure

$$\Leftrightarrow \underbrace{\mu \cdot (a-1)}_{\downarrow} \in \Lambda(\mathbb{Z}_p^\times)$$

this multiplication shouldn't be viewed as in $F((T))$

We have already constructed ζ_a, s.t.

$$\int_{\mathbb{Z}_p} x^m \, d\zeta_a = (1-a^{m+1})\zeta(-m)$$

- $\zeta_a \big|_{\mathbb{Z}_p^\times} : \quad x^m \longrightarrow (1-a^{m+1})(1-p^m)\zeta(-m) \in \Lambda(\mathbb{Z}_p^\times)$
- $\theta_a = [x]\cdot(1-a) \in \Lambda(\mathbb{Z}_p^\times)$ not a zero-divisor

$$\zeta_p := \frac{\zeta_a}{\theta_a}$$

independence of $a \in \mathbb{Z}_p^\times$ is easy to check $\Rightarrow$ ζ_p is pseudo-measure

the integral of $\mathrm{Hom}_{cts}(G, \mathcal{O}_L^\times)$ w.r.t pseudo-measure is well-defined, therefore we can consider $\int_{\mathbb{Z}_p^\times} x^n d\zeta_p$, $\int_{\mathbb{Z}_p^\times} \chi(x) x^n d\zeta_p$

$$\int_{\mathbb{Z}_p^\times} x^n \, d\zeta_p = (1-p^n)\zeta(-n)$$

integral on $\mathbb{Z}_p^\times$ instead of on $\mathbb{Z}_p$

if χ is primitive mod p^n, $n \geq 1$, then

$$\int_{\mathbb{Z}_p^\times} \chi(x) x^n \, d\zeta_p = L(-n, \chi)$$

and $$\zeta_p(s) = L_p(s, 1) = \int_{\mathbb{Z}_p^\times} \omega^{-1}(x)\langle x\rangle^{-s} \, d\zeta_p$$

$$\int_{\mathbb{Z}_p^\times} \omega^{-1}(x)\langle x\rangle^n \, d\zeta_p = \frac{\int_{\mathbb{Z}_p^\times} \omega^{-1-n}(x)\, x^n \, d\zeta_a}{\int \omega^{-1}(x)\langle x\rangle^n \, d\theta_a} = \frac{(1-p^n)\int_{\mathbb{Z}_p} \omega^{-n-1}(x) x^n \, d\zeta_a}{\langle a\rangle^{n+1} - 1} = \zeta_p(-n)$$