· Kirillov model & Multiplicity One theorem

Whittaker functional gives us a realization of an infinite dim'l irr smooth admissible repn π

$$J_\psi(\pi)^\vee \simeq \mathrm{Hom}_N(\pi, \psi) \simeq \mathrm{Hom}_M(\pi, \mathrm{Ind}_N^M \psi) \simeq \mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G \psi)$$

the morphism is given by

$$\Lambda \rightsquigarrow \Lambda: v \mapsto (m \mapsto \Lambda(\pi(m)v)) \rightsquigarrow \underbrace{\Lambda: v \mapsto (g \mapsto \Lambda(\pi(g)v))}$$

$$\begin{tikzcd} \pi \arrow[r] \arrow[d] & \mathbb{C} \\ J_\psi(\pi) \arrow[ur] & \end{tikzcd}$$

$\Lambda: \pi \to \mathbb{C}$ smooth

$$\Lambda\left(\pi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} v\right) = \psi(x)\Lambda(v)$$

Since π is injective, this map must be injective, i.e. $\Lambda(\pi(g)v) = 0, \forall g \in G \Rightarrow v = 0$

Actually, this is even true for the middle map, i.e.

$$\Lambda\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix} v\right) = 0, \forall y \in F^\times, x \in F \Rightarrow v = 0$$

or $\quad \Lambda\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} v\right) = 0, \forall y \in F^\times \Rightarrow v = 0 \qquad (*)$

or $\quad \pi \xrightarrow{f} \mathrm{Ind}_N^M \psi$ is injective

pf: consider $\ker f$, we know that

$$\pi \xrightarrow{f} \mathrm{Ind}_N^M \psi \to \psi \Rightarrow \pi_\psi \simeq \psi \Rightarrow (\ker f)_\psi = 0$$

i.e. $(\ker f(N))_\psi = 0$, but $(\ker f(N))_N = 0 \Rightarrow \ker f(N) = 0 \Rightarrow N$ acts trivially on $\ker f$

now if $\ker f \neq 0 \Rightarrow \pi$ must be ϕ-dim, impossible

Therefore we have realized π in the function space $\mathrm{Ind}_N^M \psi$ in a unique way

Note! $\mathrm{Ind}_N^M \psi \subsetneq C^\infty(F^\times)$! hence we realized π in $C^\infty(F^\times)$. M acts on $C^\infty(F^\times)$ as:

$$\left(\begin{pmatrix} a & b \\ & 1 \end{pmatrix}\right) f(y) = f\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ & 1 \end{pmatrix}\right) = f\begin{pmatrix} ay & by \\ & 1 \end{pmatrix} = \psi(by) f(ay)$$

<u>Claim</u>: image of π in $C^\infty(F^\times)$ contains $C_c^\infty(F^\times)$

pf: $\quad \mathrm{im}\,\pi(N) \subset \mathrm{Ind}_N^M \psi(N) = c\text{-}\mathrm{Ind}_N^M \psi$

$$\begin{array}{ccc} \mathrm{Ind}_N^M \psi(N) & & c\text{-}\mathrm{Ind}_N^M \psi \\ \cap & & \| \\ (C^\infty(F^\times))(N) & & C_c^\infty(F^\times) \end{array}$$

$$\tilde{f} = \pi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) f - f$$

$$\tilde{f}(y) = (\psi(xy) - 1) f(y)$$

· $f \in \mathrm{Ind}_N^M(N)$, $nf = f$ when n sufficiently small, i.e. $\exists N_j$, s.t. $f(y)\psi(xy) = f(y)$, $\forall y \in F^\times, x \in N_j$ then when $|y| \to +\infty \Rightarrow f(y) = 0$, i.e. $f \in C^\infty(F^\times)$ vanishes when $|y| \gg 0$

· when y small enough, $\tilde{f}$ vanishes

Actually, we get more. Let's state them together

- $\mathrm{Ind}_N^M \psi \subsetneq C^\infty(F^\times)$, it consists of all $f \in C^\infty(F^\times)$ which vanishes at ∞, i.e. $f(y)=0$ when $|y| >> 0$
- $c\text{-}\mathrm{Ind}_N^M \psi = \mathrm{Ind}_N^M \psi(N) = C_c^\infty(F^\times)$.
- $c\text{-}\mathrm{Ind}_N^M \psi \simeq C_c^\infty(F^\times)$ is irreducible under the action of M $\rightsquigarrow$ technical
- if π is a cuspidal repn, then the Kirillov model of π, $\mathcal{K}(\pi) = C_c^\infty(F^\times)$

  pf: π cuspidal $\Leftrightarrow$ $\pi = \pi(N)$ $\rightsquigarrow$ $\mathrm{im}\,\pi = \mathrm{im}\,\pi(N) \subset C_c^\infty(F^\times)$

  now since $C_c^\infty(F^\times)$ is irr as M-repn $\Rightarrow$ $\mathrm{im}\,\pi = C_c^\infty(F^\times)$

Now we look back on the theory we have developed.

- First suppose $\pi \simeq \mathrm{Ind}_B^G \chi$ is irr

$$0 \to V \to \mathrm{Ind}_B^G \chi \to \chi \to 0 \qquad ①$$

$$0 \to V(N) \to V \to \chi^w \cdot \delta_B^{-1} \to 0 \qquad ②$$

$V \simeq C_c^\infty(F)$

$f \longmapsto f_N : x \mapsto f\left(w\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right)$

$$\left(\pi\begin{pmatrix}a & b\\ & 1\end{pmatrix} f_N\right)(x) = \left(\pi\begin{pmatrix}a & b\\ & 1\end{pmatrix} f\right)_N(x) = \left(\pi\begin{pmatrix}a & b\\ & 1\end{pmatrix} f\right)\left(w\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right)$$
$$= f\left(w\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}a & b\\ & 1\end{pmatrix}\right) = f\left(w\begin{pmatrix}a & x+b\\ & 1\end{pmatrix}\right)$$

$$\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\begin{pmatrix}a & \\ & 1\end{pmatrix}\begin{pmatrix}1 & a^{-1}(x+b)\\ & 1\end{pmatrix} = \begin{pmatrix}1 & \\ & a\end{pmatrix}\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\begin{pmatrix}1 & a^{-1}(x+b)\\ & 1\end{pmatrix}$$

$\Rightarrow \left(\pi\begin{pmatrix}a & b\\ & 1\end{pmatrix} f_N\right)(x) = \chi_2(a) f_N(a^{-1}(x+b))$

but we know $V(N)$ as M-repn, has

$$V(N) \simeq c\text{-}\mathrm{Ind}_N^M \psi \simeq C_c^\infty(F^\times)$$

we consider the Kirillov model of π, say $\pi_K \subseteq C^\infty(F^\times)$, then

$$0 \to C_c^\infty(F^\times) \to \pi_K \to W \to 0$$

$\hookrightarrow$ essentially, $W \simeq \pi_N$ the Jacquet module

Where $\dim_{\mathbb{C}} W = 2$, as B-repn, we have $\chi_1 \cdot |\cdot|^{-\frac{1}{2}} \cdot (\chi_2 \cdot |\cdot|^{\frac{1}{2}})^{-1}$

$$0 \to \chi^w \cdot \delta_B^{-1} \to W \to \chi \to 0$$

so when $\chi^w \cdot \delta_B^{-1} \neq \chi \Leftrightarrow \chi_1(x_2)\chi_2(x_1)\left|\frac{x_1}{x_2}\right| \neq \chi_1(x_1)\chi_2(x_2) \Leftrightarrow \chi_1\chi_2^{-1}(t) \neq t$

then $W \simeq \chi^w \cdot \delta_B^{-1} \oplus \chi$ as T-repn

We consider the following function:

- If $\begin{pmatrix}a & \\ & a\end{pmatrix}$ acts as $\chi_1 \cdot \chi_2(a)$, then which one corresponds to χ?

$$\left(\begin{pmatrix}a & \\ & b\end{pmatrix} f\right)(y) = f\left(\begin{pmatrix}ay & \\ & b\end{pmatrix}\right) = \chi_1 \cdot \chi_2(b) \cdot f\left(\begin{pmatrix}ab^{-1}y & \\ & 1\end{pmatrix}\right) = \chi_1 \cdot \chi_2(b) f(ab^{-1}y) = \chi_1(a)\chi_2(b) f(y)$$

$$\Rightarrow \quad f(ab^{-1}y) = \chi_1(ab^{-1}) f(y)$$

$$\Rightarrow f(y) = C_1 \chi_1(y)$$

also for $\chi^w \cdot \delta_B^{-1} \rightsquigarrow f(y) = C_2 \cdot \chi_2(y)|y|$

Now when y is very small, $C_c^\infty(F^\times)$ vanishes, then near 0, $f \in \pi_K$ is linear combination of $\chi_1(y)$ & $|y| \cdot \chi_2(y)$

Now what happenes when $\chi_1 \chi_2^{-1} = |\cdot|$

We consider the $W \simeq \pi_N$ essentially, then

$$\mathrm{Hom}_T(W, \chi) \simeq \mathrm{Hom}_B(\pi, \chi) \simeq \mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \chi),$$ it is one-dim !

because π & $\mathrm{Ind}_B^G \chi$ are isomorphic irreducible

hence $0 \to \chi \to W \to \chi \to 0$ doesn't split (*)

we have a homomorphism

$$T \longrightarrow GL_2(\mathbb{C})$$

$$t \longmapsto \begin{pmatrix} \chi(t) & * \\ & \chi(t) \end{pmatrix} = \chi(t) \cdot \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$$ we denote it by $\varphi(t)$

turn out to be $c \cdot v\left(\frac{t_2}{t_1}\right)$

then $\varphi : T \longrightarrow \mathbb{C}^+$ is a homomorphism !

$\varphi : F^\times \times F^\times \to \mathbb{C}$

$(x, 1) \longmapsto \varphi_1(x)$ $\quad \varphi_1|_{\mathcal{O}^\times}$ must be 0 !

$(1, x) \longmapsto \varphi_2(x)$ $\quad \varphi_2|_{\mathcal{O}^\times}$ must be 0 !

Claim: (*) Splits when viewed as a representation of $\begin{pmatrix} a & \\ & a \end{pmatrix} \simeq F^\times$ $\quad$ $\chi_1 = |\cdot|^{\frac{1}{2}}$, $\chi_2 = |\cdot|^{-\frac{1}{2}}$

pf: $0 \to V \to \mathrm{Ind}_B^G \delta_B^{-\frac{1}{2}} \to \delta_B^{-\frac{1}{2}} \to 0$ ① $\overset{\text{take dual}}{\rightsquigarrow}$ $0 \to \delta_B^{\frac{1}{2}} \to \mathrm{Ind}_B^G \delta_B^{-\frac{1}{2}} \to \check{V} \to 0$ ①$^\vee$

$0 \to V(N) \to V \to \delta_B^{-\frac{1}{2}} \to 0$ ②

when restrict to $Z \Rightarrow$ ① becomes $0 \to V \to \mathrm{Ind}_B^G \delta_B^{-\frac{1}{2}} \to 1_Z \to 0$

①$^\vee$ becomes $0 \to 1_Z \to \mathrm{Ind}_B^G \delta_B^{-\frac{1}{2}} \to V^\vee \to 0$ $\quad \Rightarrow$ splits & $\check{V} \simeq V$

② $\Rightarrow$ $0 \to V(N) \to V \to 1_Z \to 0$

②$^\vee$ $\Rightarrow$ $0 \to 1 \to \underset{\wr\wr}{V^\vee} \to V(N)^\vee \to 0$ $\quad$ ② splits

hence (*) splits $\Rightarrow$ therefore $\varphi(t) = c \cdot v\left(\frac{t_2}{t_1}\right)$, i.e.

then near 0: we have $f_1(y) = c_1 \chi_1(y) = c_1 |y| \chi_2(y)$

$$\left(\begin{pmatrix} a & \\ & b \end{pmatrix} f\right)(y) = c_1 v\left(\frac{b}{a}\right) \chi_2(ab) |a| \cdot |y| \cdot \chi_2(y) + |a| \chi_2(ab) f(y)$$

$\overset{\shortparallel}{|b| \chi_2(ab) f(ab^{-1}y)}$ $\quad \Rightarrow f\left(\frac{a}{b} y\right) = -c_1 v\left(\frac{a}{b}\right) \left|\frac{a}{b}\right| \cdot |y| \cdot \chi_2(y) + \left|\frac{a}{b}\right| f(y)$

$\Rightarrow f(ay) = -c_1 v(a) |a| \cdot |y| \cdot \chi_2(y) + |a| f(y)$

$$\Rightarrow f(ay) = -c_1 v(a)|a|\cdot|y|\cdot \chi_2(y) + |a| f(y) \qquad (*)$$

let $a = y^{-1}$, then

$$f(1) = c_1 v(y)|y|^{-1}|y|\cdot\chi_2(y) + |y|^{-1} f(y) \Rightarrow f(y) = |y|\cdot f(1) - c_1 v(y)|y|\chi_2(y).$$

then put it back in $(*)$

$$\underbrace{|ay|\cdot f(1)} - \underbrace{c_1 v(ay)|ay|\chi_2(ay)} = -c_1 v(a)|a|\cdot|y|\cdot\chi_2(y) + \underbrace{|a|\cdot}\left(|y| f(1) - c_1 v(y)|y|\chi_2(y)\right)$$

Let's stop for a moment and explain the following identification

First suppose $\pi \simeq \mathrm{Ind}_B^G \chi$ is irr

$$0 \to V \to \mathrm{Ind}_B^G \chi \to \chi \to 0$$

$$0 \to V(N) \to V \to \chi^w \cdot \delta_B^{-1} \to 0$$

but we know $V(N)$ as M-repn, has

$$V(N) \simeq c\text{-}\mathrm{Ind}_N^M \psi \simeq C_c^\infty(F^\times)$$

Question: We know before that $V \simeq C_c^\infty(F)$ how do we conclude $V(N) \simeq C_c^\infty(F^\times)$?

- Now suppose $\chi_1\chi_2^{-1} = |\cdot|^2$, then $0 \to St_G \to \mathrm{Ind}_B^G \delta_B^{-1} \to 1_G \to 0$

i.e. we choose $\chi_1 = |\cdot|$, $\chi_2 = |\cdot|^{-1}$, then

$$0 \to C_c^\infty(F^\times) \to St_G \to \underset{\substack{\| \\ 1_G}}{\chi^w \cdot \delta_B^{-1}} \to 0 \qquad \chi^w \cdot \delta_B^{-1}\begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix} = \chi_2(x_1)\cdot\chi_1(x_2)\cdot\left|\frac{x_1}{x_2}\right|$$
$$= |x_1|^{-1}\,|x_2|\cdot\left|\frac{x_1}{x_2}\right| = 1$$

then locally near 0,

$$f(y) = \begin{pmatrix} a & \\ & b\end{pmatrix} f(y) = f\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\begin{pmatrix} a & \\ & b\end{pmatrix}\right) = f\left(\begin{pmatrix} ay & \\ & b\end{pmatrix}\right) = \chi_1(b)\chi_2(b) f(ab^{-1}y) = f(ab^{-1}y)$$

$\Rightarrow$ f locally near 0 is just a constant. i.e. $St_G \simeq C_c^\infty(F)$

- If we use the standard language. $\qquad \mathrm{Ind}\ \chi_1\cdot|\cdot|^{\frac{1}{2}} \otimes \chi_2\cdot|\cdot|^{-\frac{1}{2}}$

  - $\pi \simeq B(\chi_1, \chi_2)$ irr, then $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$

$\pi_K = f \in C^\infty(F)$: When $\chi_1 \neq \chi_2$

- $|t| \to +\infty$, $f(t) = 0$
- $|t| \to 0$, $f(t) = C_1|t|^{\frac{1}{2}}\chi_1(t) + C_2|t|^{\frac{1}{2}}\chi_2(t)$

When $\chi_1 = \chi_2 = \chi$

- $|t| \to +\infty$, $f(t) = 0$
- $|t| \to 0$, $f(t) = C_1|t|^{\frac{1}{2}}\chi(t) + C_2|t|^{\frac{1}{2}}v(t)\chi(t)$

only works when $\phi = 1$. $\quad St_G = \sigma(|\cdot|^{-\frac{3}{2}}, |\cdot|^{-\frac{1}{2}})$

- $\pi \simeq \phi\cdot St_G$, $\pi_K \simeq C_c^\infty(F)$

Theorem 4.7.3 *Let $(\pi, V) = \sigma(\chi_1, \chi_2)$ be a special representation of $GL(2,F)$, where $(\chi_1\chi_2^{-1})(t) = |t|^{-1}$. Then the space of the Kirillov model of V consists of the functions ϕ on $F^\times$ that are locally constant and that vanish for large values of t and such that there exists a constant C such that for $|t|$ small*

$$\phi(t) = C\,t^{1/2}\chi_2(t). \tag{7.7}$$

Proof The proof is similar to Theorem 4.7.2(i), but makes use of Exercise 4.5.4. ■

- π cuspidal, $\pi_K \simeq C_c^\infty(F^\times)$

The Multiplicity One Theorem we are going to State is the following

<u>Thm</u>: Suppose (π, V) is an irr smooth admissible repn of $GL_2(F)$, then for $\forall \chi: F^\times \to \mathbb{C}^\times$, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{GL_1}(\pi, \chi) = 1$$

except when $\pi = \phi \cdot \det$ for some $\phi \neq \chi$

or equivalently, $L: \pi \to \mathbb{C}$ satisfying

$$L\left(\pi\begin{pmatrix} y & \\ & 1 \end{pmatrix} v\right) = \chi(y) L(v), \ \forall y \in F^\times, v \in V$$

is unique up to a scalar constant

pf: We consider the Kirillov model of π

- π cuspidal, then L is a functional: $C_c^\infty(F^\times) \to \mathbb{C}$, s.t.

$$L(\rho(a) f) = \chi(a) f, \quad (\rho(a) f)(y) = f(ay)$$

L must be unique by the uniqueness of Haar measure

- $\pi \simeq St_G$: this is essentially the case $C^\infty(F)$! Which is dealt with Tate's thesis

- $\pi \simeq B(\chi_1, \chi_2)$ irr

  - $\chi_1 \neq \chi_2$, then we only need to consider

Now let's see how this theorem imply local functional equation:
We consider zeta integral, for an irr smooth admissible repn π, W denotes the Whittaker model, then we consider "zeta integral", $\xi: F^\times \to \mathbb{C}^\times$ character

$$Z(s, \xi, W) = \int_{F^\times} W\begin{pmatrix} y & \\ & 1 \end{pmatrix} \xi(y) |y|^{s-\frac{1}{2}} d^\times y$$

- π cuspidal, since $K_\pi \simeq C_c^\infty(F^\times)$, this integral converges for $\forall s \in \mathbb{C}$, defines an entire function
- $\pi \simeq B(\chi_1, \chi_2)$ irr, then we know, when $|y|$ is very small & $\chi_1 \neq \chi_2$

$$W\begin{pmatrix} y & \\ & 1 \end{pmatrix} = C_1 |t|^{\frac{1}{2}} \chi_1(t) + C_2 |t|^{\frac{1}{2}} \chi_2(t), \text{ then}$$

$$C_1 \int_{|y| \leq \varepsilon} |t|^{\frac{1}{2}} \chi_1 \xi(t) |t|^{s-\frac{1}{2}} d^\times t + C_2 \int_{|y| \leq \varepsilon} |t|^{\frac{1}{2}} \chi_2 \xi(t) |t|^{s-\frac{1}{2}} d^\times t$$

$$c_1 \int_{|y|\le\varepsilon} |t|^{\frac12}\chi_1\xi(t)\,|t|^{s-\frac12}\, d^\times t + c_2 \int_{|y|\le\varepsilon} |t|^{\frac12}\chi_2\xi(t)\,|t|^{s-\frac12}\, d^\times t$$

$$= c_1 \int_{|y|\le\varepsilon} \chi_1\xi(t)\,|t|^{s}\, d^\times t + c_2 \int_{|y|\le\varepsilon} \chi_2\xi(t)\,|t|^{s}\, d^\times t$$

- When $\chi_1\xi$ & $\chi_2\xi$ are both unramified, then

$$\int_{|y|\le\varepsilon} \chi_i\xi(t)\,|t|^s\, d^\times t = \sum_{n=N_0}^{\infty}\int_{v(y)=n} \chi_i\xi(t)\,|t|^s\, d^\times t = \sum_{n=N_0}^{\infty} \alpha_i^n q^{-ns}\cdot 1 = \frac{\alpha_i^{N_0} q^{-N_0 s}}{1-\alpha_i q^{-s}}$$

then $\dfrac{Z(s,\xi,W)}{L(s,\xi,\pi)}$ is entire for $\forall s\in\mathbb{C}$

we consider the functional:

$$z_0(s,\xi) = \frac{Z(s,\xi,-)}{L(s,\xi,\pi)} : W_\pi \to \mathbb{C}$$

$$z_0(s,\xi)\left(\pi\begin{pmatrix} y & \\ & 1\end{pmatrix} W\right) \underset{\substack{\text{when}\\ Re(s)\gg 0}}{=\!=} L(s,\xi,\pi)^{-1}\int_{F^\times} W\begin{pmatrix} ty & \\ & 1\end{pmatrix}\xi(t)\,|t|^{s-\frac12}\, d^\times t$$

$$= L(s,\xi,\pi)^{-1}\int_{F^\times} W\begin{pmatrix} t & \\ & 1\end{pmatrix}\xi(t)\,\xi(y)^{-1}\,|t|^{s-\frac12}\,|y|^{\frac12-s}\, d^\times y$$

$$= \xi(y)^{-1}|y|^{\frac12-s} z_0(s,\xi)(W)$$

since both sides are entire functions $\Rightarrow$ $z_0(s,\xi)\left(\pi\begin{pmatrix} y & \\ & 1\end{pmatrix} W\right) = \xi(y)^{-1}|y|^{\frac12-s} z_0(s,\xi)(W)$, $\forall s\in\mathbb{C}$ $\quad(*)$

When $\chi_1=\chi_2=\chi$, the "essential integral" is the following

$$c_1 \int_{|y|\le\varepsilon} |t|^{\frac12}\chi\xi(t)\,|t|^{s-\frac12}\, d^\times t + \underbrace{c_2 \int_{|y|\le\varepsilon} |t|^{\frac12}\chi\xi(t)\,|t|^{s-\frac12}\,v(t)\, d^\times t}_{\sum_{n=N_0}^{\infty}\int_{v(t)=n}\chi\xi(t)\,|t|^s\,v(t)\,d^\times t}$$

When $\chi\xi$ is unramified, then

$$\sum_{n=N_0}^{\infty}\int_{v(t)=n}\chi\xi(t)\,|t|^s\,v(t)\,d^\times t = \sum_{n=N_0}^{\infty}\alpha^n q^{-ns} n$$

$$= \frac{\text{polynomial in } q^{-s}}{(1-\alpha q^{-s})^2}$$

$S = \beta^{N_0}N_0 + \beta^{N_0+1}(N_0+1) + \cdots$

$\beta S = \beta^{N_0+1}N_0 + \beta^{N_0+2}(N_0+1) + \cdots$

$(1-\beta)S = \beta^{N_0}N_0 + \beta^{N_0+1} + \beta^{N_0+2} \cdots$

$= \beta^{N_0}N_0 + \dfrac{\beta^{N_0+1}}{1-\beta}$

$\Rightarrow$ $\dfrac{Z(s,\xi,W)}{L(s,\xi,\pi)}$ is entire, belongs to $\mathbb{C}[q^{-s}]$, it also satisfies $(*)$

$\shortparallel$

$z_0(s,\xi)(W)$

Therefore, in all cases, we have defined:

$$Z_0(s,\xi,\pi) \in \mathrm{Hom}_{GL_1}(\pi, \xi^{-1}\cdot|\cdot|^{\frac{1}{2}-s}) \sim 1\text{-dimensional}$$

Now we consider $Z_0(s,\hat{\xi},\pi)$ it is defined by

$$Z_0(s,\hat{\xi},\pi)(W) = Z_0(s,\xi)(\pi(w_0)W),$$

then

$$Z_0(s,\hat{\xi},\pi)\left(\pi\begin{pmatrix} y & \\ & 1\end{pmatrix}W\right) = Z_0(s,\xi)\left(\pi(w_0)\pi\begin{pmatrix} y & \\ & 1\end{pmatrix}W\right) \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} y & \\ & 1\end{pmatrix} = \begin{pmatrix} 1 & \\ & y\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$$

$$= Z_0(s,\xi)\left(\pi\begin{pmatrix} 1 & \\ & y\end{pmatrix}\pi(w_0)W\right)$$

$$= \omega_\pi(y)\, Z_0(s,\xi)\left(\pi\begin{pmatrix} y^{-1} & \\ & 1\end{pmatrix}\pi(w_0)W\right)$$

$$= \omega_\pi(y)\,\xi(y)\,|y|^{s-\frac{1}{2}}\, Z_0(\hat{s},\xi)(W)$$

$$\Rightarrow Z_0(s,\hat{\xi},\pi) \in \mathrm{Hom}_{GL_1}(\pi, \omega_\pi\cdot\xi\cdot|\cdot|^{s-\frac{1}{2}})$$

now we consider $Z_0(1-s,\widehat{\xi^{-1}\cdot\omega_\pi^{-1}},\pi) = \epsilon(s,\xi,\psi)\, Z_0(s,\xi,\pi)$

or

$$Z_0(1-s,\xi^{-1}\omega_\pi^{-1},\pi,\pi(w_0)W) = \epsilon(s,\xi,\psi)\, Z_0(s,\xi,\pi,W)$$

$$\| $$

$$\frac{Z(1-s,\xi^{-1}\omega_\pi^{-1},\pi,\pi(w_0)W)}{L(1-s,\xi^{-1}\omega_\pi^{-1},\pi)} = \epsilon(s,\xi,\psi)\,\frac{Z(s,\xi,\pi,W)}{L(s,\xi,\pi)}$$

both sides should be understood as after meromorphic continuation

i.e.

- the initial zeta integral admits meromorphic continuation

Now we consider the global zeta integral, initially

$$Z(s,\xi,\pi,W) = \int_{\mathbb{A}_F^\times} W\begin{pmatrix} y & \\ & 1\end{pmatrix}\xi(y)\,|y|^{s-\frac{1}{2}}\,d^\times y$$

$$\overset{\text{formally}}{=\!=} \prod_v \underbrace{\int_{F_v^\times} W_v\begin{pmatrix} y_v & \\ & 1\end{pmatrix}\xi_v(y_v)\,|y_v|^{s-\frac{1}{2}}\,d^\times y_v}$$

local term will be denominated by $\mathbb{C}[q^{-s}]\cdot L_v(s,\xi_v,\pi_v)$

with almost all $= L_v(s,\xi_v,\pi_v)$ $\qquad (1-\alpha_1 q^{-s})^{-1}(1-\alpha_2 q^{-s})^{-1}$

$\alpha_1 = \xi_v\cdot\chi_{1v}(p_v),\ \alpha_2 = \xi_v\cdot\chi_{2v}(p_v)$

$W = \prod W_v$, and almost all W_v is the unique spherical whittaker function at v

then $Z_0(s, \xi_v, \pi_v, W_v) = 1$ for almost all places (ξ_v almost all unramified)

we define

$$Z_0(s, \xi, \pi, W) := \prod_v Z_0(s, \xi_v, \pi_v, W_v)$$

then

$$Z_0(s, \xi, \pi)\left(\pi\begin{pmatrix} y & \\ & 1 \end{pmatrix} W\right) = \prod_v Z_0(s, \xi_v, \pi_v)\left(\pi_v\begin{pmatrix} y_v & \\ & 1 \end{pmatrix} W_v\right)$$

$y \in \mathbb{A}_F^\times$

$$= \prod_v \xi_v^{-1}(y_v) \cdot |y_v|^{\frac{1}{2}-s} Z_0(s, \xi_v, \pi_v)(W_v)$$

$$= \xi^{-1}(y) \cdot |y|^{\frac{1}{2}-s} Z_0(s, \xi, \pi)(W)$$

$$Z_0(s, \xi, \pi, W) \underset{\forall s \in \mathbb{C}}{=} \prod_v Z_0(s, \xi_v, \pi_v, W_v) \underset{\forall s \in \mathbb{C}}{=} \prod_v \frac{Z(s, \xi_v, \pi_v)(W_v)}{L_v(s, \xi_v, \pi_v)}$$

$$\underset{\mathrm{Re}\, s \gg 0}{=} \prod_v \frac{\int_{F_v^\times} W_v\begin{pmatrix} y_v & \\ & 1 \end{pmatrix} \xi_v(y_v) |y_v|^{\frac{1}{2}-s} d^\times y_v}{L_v(s, \xi_v, \pi_v)}$$

this one to guarantee $\int_{F_v^\times} W_v\begin{pmatrix} y_v & \\ & 1 \end{pmatrix} \xi_v(y_v) |y_v|^{\frac{1}{2}-s} d^\times y_v$ converges

we should have a uniform bound on $|\chi_{1,v}|$ & $|\chi_{2,v}|$, & $|\xi_v|$

$$\underset{\mathrm{Re}(s) \gg 0}{=} \frac{\prod_v \int_{F_v^\times} W_v\begin{pmatrix} y_v & \\ & 1 \end{pmatrix} \xi_v(y_v) |y_v|^{\frac{1}{2}-s} d^\times y_v}{\prod_v L_v(s, \xi_v, \pi_v)}$$

then two to guarantee global zeta integral $= \prod_v$ local zeta integral

$$\underset{\mathrm{Re}(s) \gg 0}{=} \frac{Z(s, \xi, \pi)(W)}{L(s, \xi, \pi)}$$

$\Rightarrow$ analytic continuation of global zeta integral $\rightsquigarrow$ analytic continuation of L

<u>Point</u>: $Z(s, \xi, \pi)(W)$ is more easier to deal with than $L(s, \xi, \pi)$

Suppose that we proved $Z(s, \xi, \pi)(W)$ can be meromorphically continued to $\mathbb{C}$, then

$$Z_0(1-s, \xi^{-1}\omega_\pi^{-1}, \pi, \pi(w_0)W) = \prod_v Z_0(1-s, \xi_v^{-1}\omega_{\pi_v}^{-1}, \pi_v, \pi_v(w_0)W_v)$$

$$\parallel$$

$$\frac{Z(1-s, \xi^{-1}\omega_\pi^{-1}, \pi)(\pi(w_0)W)}{L(1-s, \xi^{-1}\omega_\pi^{-1}, \pi)} \qquad = \prod_v \epsilon(s, \xi_v, \psi_v) Z_0(s, \xi_v, \pi_v, W_v) = \epsilon(s, \xi, \psi) \frac{Z(s, \xi, \pi)(W)}{L(s, \xi, \pi)}$$

$$z(1-s, \omega_\pi^{-1}\xi^{-1}, \pi)(\pi(w_0)W)$$

$$= \int_{F^\times\backslash\mathbb{A}_F^\times} (\pi(w_0)\phi)\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right) \omega_\pi^{-1}\xi^{-1}(y)|y|^{\frac{1}{2}-s} d^\times y \quad = \int_{F^\times\backslash\mathbb{A}_F^\times} \phi\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right) \omega_\pi^{-1}\xi^{-1}(y)|y|^{\frac{1}{2}-s} d^\times y$$

$$= \int_{F^\times\backslash\mathbb{A}_F^\times} \phi\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & \\ & y\end{pmatrix}\right) \omega_\pi^{-1}\xi^{-1}(y)|y|^{\frac{1}{2}-s} d^\times y \quad = \int_{F^\times\backslash\mathbb{A}_F^\times} \phi\left(\begin{pmatrix} 1 & \\ & y\end{pmatrix}\right) \omega_\pi^{-1}\xi^{-1}(y)|y|^{\frac{1}{2}-s} d^\times y$$

$$= \int_{F^\times\backslash\mathbb{A}_F^\times} \phi\left(\begin{pmatrix} y^{-1} & \\ & 1\end{pmatrix}\right) \xi^{-1}(y)|y|^{\frac{1}{2}-s} d^\times y \quad = \int_{F^\times\backslash\mathbb{A}_F^\times} \phi\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right) \xi(y)|y|^{s-\frac{1}{2}} d^\times y = z(\phi, \xi, \pi)(W)$$

$$\Rightarrow \quad L(s, \xi, \pi) = \epsilon(s, \xi, \psi)\, L(1-s, \omega_\pi^{-1}\xi^{-1}, \pi)$$

Therefore our next step is to understand the following integral,

$$z(s, \pi, \xi, \phi) = \int_{F^\times\backslash\mathbb{A}_F^\times} \phi\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right) \xi(y)|y|^{s-\frac{1}{2}} dy$$

if $F=\mathbb{Q}$, then $\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times = (0,+\infty)\times\prod_p \mathbb{Z}_p^\times$

$$= \int_{(0,+\infty)} \int_{\prod_p \mathbb{Z}_p^\times} \phi\left(\begin{pmatrix} tk & \\ & 1\end{pmatrix}\right) \xi(tk)|tk|^{s-\frac{1}{2}} dt\, dk$$

we suppose ξ is unitary, (if it's not, it only affect the $s-\frac{1}{2}$ part)

$$= \int_{(0,+\infty)} \tilde{\phi}\begin{pmatrix} t & \\ & 1\end{pmatrix} |t|^{s-\frac{1}{2}} dt \qquad (*)$$

here $\tilde{\phi}\left(\begin{pmatrix} t & \\ & 1\end{pmatrix}\right) = \int_{\prod_p \mathbb{Z}_p^\times} \phi\left(\begin{pmatrix} tk & \\ & 1\end{pmatrix}\right) dk$

when $|t| \gg 0$, we have $\phi\left(\begin{pmatrix} tk & \\ & 1\end{pmatrix}\right) \le C_N|t|^{-N} \Rightarrow \tilde{\phi}\left(\begin{pmatrix} t & \\ & 1\end{pmatrix}\right) \le C\cdot t^{-N}$

$|t|\to 0$, $\phi\left(\begin{pmatrix} tk & \\ & 1\end{pmatrix}\right) \ll C_N|t|^N \Rightarrow \tilde{\phi}\left(\begin{pmatrix} t & \\ & 1\end{pmatrix}\right) \le C' t^N$

hence $(*)$ must converges! for ϕ a cusp form

For just Automorphic forms, we have

$\tilde{\phi}\left(\begin{pmatrix} t & \\ & 1\end{pmatrix}\right) \le C\cdot t^N \quad t\to+\infty$

$\tilde{\phi}\left(\begin{pmatrix} t & \\ & 1\end{pmatrix}\right) \le C\cdot t^{-N}, \quad t\to 0$

$\Rightarrow$ Mellin transform admits analytic continuation

Question: Why we use the zeta integral?

For $\phi \in$ cuspidal automorphic form

① Fourier expansion

$$\phi\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g\right) = \sum_{\alpha\in F^\times}\left(\int_{\mathbb{A}/F}\phi\left(\begin{pmatrix}1 & t\\ & 1\end{pmatrix}g\right)\psi(-\alpha t)\,dt\right)\psi(\alpha x)$$

$$W_\alpha(g) = \int_{\mathbb{A}/F}\phi\left(\begin{pmatrix}1 & t\\ & 1\end{pmatrix}g\right)\psi(-\alpha t)\,dt$$

$$= \int_{\mathbb{A}/F}\phi\left(\begin{pmatrix}1 & \alpha^{-1}t\\ & 1\end{pmatrix}g\right)\psi(-t)\,dt$$

$$= \int_{\mathbb{A}/F}\phi\left(\begin{pmatrix}\alpha^{-1} & \\ & 1\end{pmatrix}\begin{pmatrix}1 & t\\ & 1\end{pmatrix}\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}g\right)\psi(-t)\,dt$$

$$= \int_{\mathbb{A}/F}\phi\left(\begin{pmatrix}1 & t\\ & 1\end{pmatrix}\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}g\right)\psi(-t)\,dt$$

$$\Rightarrow \quad \phi(g) = \sum_{\alpha\in F^\times} W_1\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}g\right), \quad W_1(g) = \int_{\mathbb{A}/F}\phi\left(\begin{pmatrix}1 & t\\ & 1\end{pmatrix}g\right)\psi(-t)\,dt$$

Now we consider the value of $W_1(g)$

$$G = B \sqcup N w B$$

$$\Rightarrow \quad W_1\begin{pmatrix}b_1 & x\\ & b_2\end{pmatrix} = \omega(b_2)\, W_1\begin{pmatrix}b_1b_2^{-1} & b_2^{-1}x\\ & 1\end{pmatrix} = \omega(b_2)\,\psi(b_2^{-1}x)\, W_1\begin{pmatrix}b_1b_2^{-1} & \\ & 1\end{pmatrix}$$

$$W_1\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix} w \begin{pmatrix}b_1 & x'\\ & b_2\end{pmatrix}\right) = \psi(x)\, W_1\begin{pmatrix}b_1 & x'\\ & b_2\end{pmatrix} = \omega(b_2)\,\psi(x + b_2^{-1}x')\, W_1\begin{pmatrix}b_1b_2^{-1} & \\ & 1\end{pmatrix}$$

$\Rightarrow$ W_1 is essentially determined by $W_1\begin{pmatrix}y & \\ & 1\end{pmatrix}$ $y \in \mathbb{A}_F^\times$

· Those local factors:

π_v principal series $\Rightarrow$ $\pi_v \simeq B(\chi_1, \chi_2)$, then $L_v(s, \pi_v) = L_v(s, \chi_1) \cdot L_v(s, \chi_2)$

π_v is steinberg $\Rightarrow$ $\pi_v \simeq B(\chi, \chi)$, then $L_v(s, \pi) = L_v(s, \chi)^2$

π_v is supercuspidal $\Rightarrow$ $L_v(s, \pi_v) = 1$

$$L_v(s, \xi, \pi_v) := L_v(s, \xi \pi_v) \longrightarrow \pi_v \simeq B(\chi_1, \chi_2) \Rightarrow \xi \cdot \pi_v \simeq B(\xi\chi_1, \xi\chi_2)$$

We have shown the following theorem

Thm: If π is a cuspidal automorphic repn, then

$$L(s,\pi)$$

admits an analytic continuation to $s \in \mathbb{C}$, and satisfies a functional equation

$$L(s,\xi,\pi) = \epsilon(s,\pi,\psi,\xi)\, L(1-s, \omega_\pi^{-1}\xi^{-1}, \pi)$$

$$L(s,\xi\cdot\pi) = \epsilon(s,\pi,\psi,\xi)\, L(1-s, \omega_\pi^{-1}\cdot\xi^{-1}\cdot\pi)$$

or for any automorphic form we can associate the $L(s,\pi)$

π may not be automorphic because f may not be square-integrable s.t. Eisenstein series

the strategy we use is the following:

- local functional equation $\Leftarrow$ local Multiplicity One Theorem
- convergence of global zeta integral $\Leftarrow$ rapid decay of ϕ

Local repns : Whittaker model at Archimedean place

According to the existence of Whittaker model, over Archimedean field $\mathbb{R}$, there also exists a Whittaker model for an irr unitary repn π

↳ only confined in this case, $(\mathfrak{g}, K)$-isomorphism implies $G(\mathbb{R})^+$-isomorphism

Recall that generally, π is of the following form:

$$\pi = \left\{ f: GL_2(\mathbb{R})^+ \to \mathbb{C} \;\middle|\; f\left(\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} g\right) = |y_1|^{s_1+\frac{1}{2}} |y_2|^{s_2-\frac{1}{2}} f(g) \right\}$$

and if we denote, $s = \frac{1}{2}(s_1 - s_2 + 1)$, $\lambda = s(1-s)$, $\mu = s_1 + s_2$,

eigenvalue of Δ eigenvalue of Z

so the Whittaker model of π we are looking for, should be made up of the following functions:

a) $W\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = e^{\frac{ix}{2}} W(g)$

b) $W(g\kappa_\theta) = W(g) e^{ik\theta}$, $k \in \mathbb{Z}$

c) $\Delta W = \lambda W$

d) $ZW = \mu W$

e) W is of moderate growth

The uniqueness result of the Whittaker model is the following:

Prop: For $\lambda, \mu \in \mathbb{C}$. $\mathcal{W}(\lambda, \mu, k)$ denotes the functions on $G(\mathbb{R})^+$ satisfying a) - e), then

$$\dim_{\mathbb{C}} \mathcal{W}(\lambda, \mu, k) = 1$$

Actually, if we ignore e), then there are two functions, one grows exponentially another may decays fast and is analytic

pf: Iwasawa decomposition

$$g = \begin{pmatrix} u & \\ & u \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} \kappa_\theta. \text{ then}$$

$$W(g) = u^{\mu} \cdot e^{\frac{ix}{2}} \cdot e^{ik\theta}\, w(y), \quad w: \mathbb{R}_+^\times \to \mathbb{C}$$

Recall the Δ operator:

$$w'' + \left\{ -\frac{1}{4} + \frac{k}{2y} + \frac{\lambda}{y^2} \right\} w = 0.$$

Interlude on the theory over the Archimedean place

The account of Bump is the following

consider the space $L^2(\Gamma\backslash\mathcal{H}, \chi, k)$: $f: \mathcal{H} \to \mathbb{C}$ satisfying $f(\gamma z) = \chi(\gamma)(cz+d)^k f(z)$

we also consider an operator Δ_k on this space:

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}$$

the astonishing fact is that

spectral decomposition of Δ_k $\rightsquigarrow$ decomposition of rpn $L^2(\Gamma\backslash G, \chi)$ of $GL_2(\mathbb{R})^+ = G$

How this work?. And What is $L^2(\Gamma\backslash G, \chi)$?

- Archimedean lift of f:

For $f \in L^2(\Gamma\backslash\mathcal{H}, \chi, k)$, we consider the following:

$$\phi_f(g) = f(gi)\cdot j(g,i)^{-k}$$

then we have:

- $\phi_f(\gamma g) = \chi(\gamma)\cdot\phi_f(g)$, $\forall \gamma \in \Gamma$

$$\phi_f(\gamma g) = f(\gamma g i)\cdot j(\gamma g, i)^{-k} = \chi(\gamma)\, j(\gamma, gi)^k f(gi)\cdot j(\gamma, gi)^{-k}\cdot j(g,i)^{-k} = \phi_f(g)$$

- $\phi_f(g\kappa_\theta) = \phi_f(g)e^{ik\theta}$

$$\phi_f(g\kappa_\theta) = f(gi)\, j(g\kappa_\theta, i)^{-k} = f(gi)\cdot j(g,i)^{-k}\cdot j(\kappa_\theta, i)^{-k} \qquad j(\kappa_\theta, i) = (-\sin\theta\, i + \cos\theta) = e^{-i\theta}$$

$$= \phi_f(g)\, e^{ik\theta}$$

Note: $j(g,i)^{-k}$ should be understood as the automorphy factor of line bundle $\Omega^{\otimes\frac{k}{2}}$

Δ_k should be understood as the Laplacian of this line bundle!

the Hermitian inner product on this space is

$$(f, g) := \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)}(\operatorname{Im} z)^k \frac{dx\wedge dy}{y^2} \qquad z = gi = \frac{ai+b}{ci+d} \Rightarrow \operatorname{Im} z = \frac{\det g}{|ci+d|^2}$$

$$= \int_{\Gamma\backslash G} \phi_f(g)\overline{\phi_g(g)}\, dg = \int_{Z\Gamma\backslash G} f(gi)\overline{g(gi)}\left|j(g,i)\right|^{-2k} dg = \int_{\underset{\cdot\, PGL_2(\mathbb{R})^+ \xleftarrow{2:1} SL_2(\mathbb{R})}{Z\Gamma\backslash G}} f(gi)\overline{g(gi)}\,(\operatorname{Im} gi)^k (\det g)^{-k} dg$$

$$= \int_{\Gamma\backslash\mathcal{H}\times SO(2)} f(z)\overline{g(z)}(\operatorname{Im} z)^k\cdot\frac{dx\wedge dy}{y^2} = \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)}(\operatorname{Im} z)^k\frac{dx\wedge dy}{y^2}$$

Archimedean lift of Δ:

We have defined: $L^2(\Gamma\backslash\mathcal{H}, \chi, k) \to L^2(\Gamma\backslash G, \chi, k)$

Does this admit a reserve map ? Set: $j(g,z) = (cz+d)\cdot(\det g)^{-\frac{1}{2}}$

Let's consider $\phi \in L^2(\Gamma\backslash G, \chi, k)$, then we define

$$f(z) = \phi(g)\cdot j(g,i)^k \quad , \text{ for } \forall g \in GL_2(\mathbb{R})^+, \ gi = z$$

· Well-definedness: $\phi(g')\cdot j(g',i)^k = \phi(gK_\theta)\cdot j(gK_\theta, i)^k$

$$= \phi(g)e^{ik\theta}\cdot j(g,i)^k\cdot j(K_\theta,i)^k = \phi(g)\cdot j(g,i)^k$$

and $f(\gamma z) = \phi(\gamma g)\, j(\gamma g, i)^k = \chi(\gamma)\phi(g)\cdot j(\gamma,z)^k\cdot j(g,i)^k = \chi(\gamma)(cz+d)^k\cdot f(z)$

So actually, we get an isomorphism $L^2(\Gamma\backslash\mathcal{H}, \chi, k) \xrightarrow{\sim} L^2(\Gamma\backslash G, \chi, k)$

<u>Question</u>: What's the corresponding operator Δ for Δ_k ?

The answer is very ample: Δ = the casimir element in $U(\mathfrak{g}_{\mathbb{C}})$, $\mathfrak{g} = \mathfrak{gl}$

Generally speaking, for a semi-simple complex Lie algebra $\mathfrak{g}$, the Casimir element is defined as follows: Fix a basis $/\mathbb{C}$ for $\mathfrak{g}$, say X_i, $\tilde{X}_i$ is the dual basis w.r.t. $B: \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$

$$\Delta := \sum B(X_i, X_j)\,\tilde{X}_i\cdot\tilde{X}_j$$

For $\mathfrak{sl}_2(\mathbb{C})$: consider H, R, L:

$$[R,L] = H,\quad [H,R] = 2R,\quad [H,L] = -2L$$

$$\mathrm{ad}H = \begin{pmatrix} 0&0&0\\ 0&2&0\\ 0&0&-2 \end{pmatrix},\quad \mathrm{ad}R = \begin{pmatrix} 0&0&1\\ -2&0&0\\ 0&0&0 \end{pmatrix},\quad \mathrm{ad}L = \begin{pmatrix} 0&-1&0\\ 0&0&0\\ 2&0&0 \end{pmatrix}$$

$B(H,H) = \mathrm{Tr}(\mathrm{ad}H\cdot\mathrm{ad}H) = 8$, $B(H,R) = 0$, $B(H,L) = 0$, $B(R,L) = 4$, $B(R,R) = 0$, $B(L,L) = 0$

Hence $\tilde{H} = \frac{1}{8}H$, $\tilde{R} = \frac{1}{4}L$, $\tilde{L} = \frac{1}{4}R$. th

$$\Delta = 8\cdot\tilde{H}\cdot\tilde{H} + 4\tilde{R}\cdot\tilde{L} + 4\tilde{L}\cdot\tilde{R} = \frac{1}{8}H^2 + \frac{1}{4}RL + \frac{1}{4}LR \ ,\quad 8\Delta = H^2 + 2RL + 2LR$$

the most commonly used basis for $sl_2(\mathbb{C})$ is the following

$$\hat{H} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} & 1 \\ & \end{pmatrix}, \quad \hat{L} = \begin{pmatrix} & \\ 1 & \end{pmatrix}$$

& the Iwasawa decomposition of $GL_2(\mathbb{R})^+$

$$\begin{pmatrix} u & \\ & u \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} \kappa_\theta$$

we have the following:

Proposition 2.2.5 *We have, using the coordinates Eq. (1.19)*

$$d\hat{R} = y\cos(2\theta)\frac{\partial}{\partial x} + y\sin(2\theta)\frac{\partial}{\partial y} + \sin^2(\theta)\frac{\partial}{\partial \theta}, \tag{2.28}$$

$$d\hat{L} = y\cos(2\theta)\frac{\partial}{\partial x} + y\sin(2\theta)\frac{\partial}{\partial y} - \cos^2(\theta)\frac{\partial}{\partial \theta}, \tag{2.29}$$

$$d\hat{H} = -2y\sin(2\theta)\frac{\partial}{\partial x} + 2y\cos(2\theta)\frac{\partial}{\partial y} + \sin(2\theta)\frac{\partial}{\partial \theta}. \tag{2.30}$$

but in Bump we use a different basis:

$$H = -i\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\mathrm{Lie}(SO(2))}, \quad R = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

eigenvalues of H will correspond to $\mathbb{Z}$

$$dR = e^{2i\theta}\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial \theta}\right), \tag{2.31}$$

$$dL = e^{-2i\theta}\left(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial \theta}\right), \tag{2.32}$$

and

$$dH = -i\frac{\partial}{\partial \theta}. \tag{2.33}$$

$$RLf = e^{2i\theta}\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial \theta}\right)\left(e^{-2i\theta}\left(-iy\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} - \frac{1}{2i}\frac{\partial f}{\partial \theta}\right)\right)$$

$$= e^{2i\theta}\Bigg(e^{-2i\theta}\, iy\left(-iy\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x\partial y} - \frac{1}{2i}\frac{\partial^2 f}{\partial x\partial \theta}\right) + ye^{-2i\theta}\left(-i\frac{\partial f}{\partial x} - iy\frac{\partial^2 f}{\partial x\partial y} + \frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2} - \frac{1}{2i}\frac{\partial^2 f}{\partial y\partial \theta}\right)$$

$$+ \frac{1}{2i}(-2i)e^{-2i\theta} + \frac{1}{2i}e^{-2i\theta}\left(-\frac{1}{2i}\frac{\partial^2 f}{\partial \theta^2}\right)\Bigg)$$

$$= iy\left(-iy\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x\partial y} - \frac{1}{2i}\frac{\partial^2 f}{\partial x\partial \theta}\right) + y\left(-i\frac{\partial f}{\partial x} - iy\frac{\partial^2 f}{\partial x\partial y} + \frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2} - \frac{1}{2i}\frac{\partial^2 f}{\partial y\partial \theta}\right) - 1 + \frac{1}{4}\frac{\partial^2 f}{\partial \theta^2}$$

$$= y^2\frac{\partial^2 f}{\partial x^2} + iy^2\frac{\partial^2 f}{\partial x\partial y} - \frac{y}{2}\frac{\partial^2 f}{\partial x\partial \theta} - iy\frac{\partial f}{\partial x} - iy^2\frac{\partial^2 f}{\partial x\partial y} + y\frac{\partial f}{\partial y} + y^2\frac{\partial^2 f}{\partial y^2} - \frac{y}{2i}\frac{\partial^2 f}{\partial y\partial \theta} - 1 + \frac{1}{4}\frac{\partial^2 f}{\partial \theta^2}$$

$$= y^2\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + y\left(\frac{\partial f}{\partial y} - i\frac{\partial f}{\partial x}\right) + \frac{y}{2}\left(\frac{i\,\partial^2 f}{\partial y\partial \theta} - \frac{\partial^2 f}{\partial x\partial \theta}\right) + \frac{1}{4}\frac{\partial^2 f}{\partial \theta^2} - 1$$

• Let's go back to the decomposition or Spectral problem, we want to understand the following space

$$L^2(\Gamma\backslash G, \chi)$$

<u>Prop</u> : If $\Gamma\backslash\mathcal{H}$ is compact, then all the operator $\rho(\phi)$ are compact, then

$L^2(\Gamma\backslash G, \chi)$ decomposes into Hilbert direct sum of irr repns:

$$L^2(\Gamma\backslash G, \chi) = \hat{\bigoplus_i} \pi_i$$

and $$L^2(\Gamma\backslash G, \chi, k) = \hat{\bigoplus_i} \pi_i(k)$$

If $\Gamma\backslash\mathcal{H}$ is not compact, i.e. it has cusps, then $\rho(\phi)$ is not compact in general, but

$\rho(\phi)$ is compact when restricted to $L^2_0(\Gamma\backslash G, \chi)$

and we have:

$$L^2_0(\Gamma\backslash G, \chi) = \hat{\bigoplus_i} \pi_i$$

and $$L^2_0(\Gamma\backslash G, \chi, k) = \hat{\bigoplus_i} \pi_i(k)$$

Moreover

As we have already seen before, irr unitary repn of $GL_2(\mathbb{R})^+$ are of the following four types

1. One-dimensional . $g \mapsto |\det(g)|^\mu$
2. Principal series : $P_\mu(\lambda, \varepsilon)$, here μ is purely imaginary, $\lambda \geq \frac{1}{4}$. $\varepsilon = 0$ or 1 A
3. Complentary series : $P_\mu(\lambda, 0)$, here $0 < \lambda < \frac{1}{4}$. μ purely imaginary B1
4. Discrete series : $D^{\pm}_\mu(k)$, for $k \geq 1$, μ purely imaginary B2

· Maass form

we know that $L^2(\Gamma\backslash\mathcal{H}, \chi, k) \simeq L^2(\Gamma\backslash G, \chi, k) = \hat{\bigoplus_i} \pi_i(k)$

When $\pi_i \simeq \pi_k$, then $\pi_i(k)$ consists of holomorphic functions

if $\pi_i \simeq \pi_n$, $n \neq k$ or $\pi_i \simeq \pi_s, \pi_s^c \Rightarrow \pi_i(k)$ consists of non-holomorphic functions.

We take $f \in L^2(\Gamma\backslash\mathcal{H}, \chi, k)$. we know $\phi_f(g) = f(gi)\, j(g,i)^{-k}$ is a "reasonable" function

it makes us to consider the following function:

$$f^*(z) = f(z)\cdot(\mathrm{Im}\, z)^{\frac{k}{2}} \qquad z = \frac{ai+b}{ci+d} \Rightarrow \mathrm{Im}\, z = \frac{\det g}{|ci+d|^2} = |j(g,i)|^{-2}\cdot\det g$$

$$= f(gi)\cdot|j(g,i)|^{-k}\cdot(\det g)^{\frac{k}{2}}$$

$$f^*(\gamma z) = f(\gamma z)\,(\mathrm{Im}\,\gamma z)^{\frac{k}{2}} \qquad \mathrm{Im}\,\gamma z = \frac{\det\gamma\cdot\mathrm{Im}\, z}{|cz+d|^2}$$

$$= \chi(\gamma)\,(cz+d)^k f(z)\cdot\frac{(\det\gamma)^{\frac{k}{2}}(\mathrm{Im}\, z)^{\frac{k}{2}}}{|cz+d|^k}$$

$$= f^*(z)\cdot(\chi(\gamma)\det\gamma^{\frac{k}{2}})\cdot\left(\frac{cz+d}{|cz+d|}\right)^k$$

$f \in \Omega^{\otimes\frac{k}{2}}$, $j(g,i)$ is automorphy factor of $\bar{\Omega}^{\otimes\frac{k}{2}} \Rightarrow f^* \in (\Omega\otimes\bar{\Omega})^{\otimes\frac{k}{2}}$

Strong Multiplicity One Theorem

Kirillov model & Weil representation

We have stated that, for NA v, irr infinite-dimensional smooth admissible π, and nontrivial character ψ of F_v, we have:

$$\mathrm{Hom}_N(\pi|_N, \psi) \simeq \mathbb{C}$$

Whittaker model is nothing but Frobenius Reciprocity:

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G \psi) \simeq \mathrm{Hom}_N(\pi|_N, \psi) \simeq \mathbb{C}$$

hence we get a unique (up to scalar) embedding $\pi \hookrightarrow \mathrm{Ind}_N^G \psi$, the image we denote by $\mathcal{W}(\pi)$

Now we consider a smaller group,

$$M = \left\{ \begin{pmatrix} y & x \\ & 1 \end{pmatrix} \middle| \; y \in F_v^\times, \; x \in F_v \right\}$$

then By FR: $\quad \mathrm{Hom}_M(\pi|_M, \mathrm{Ind}_N^M \psi) \simeq \mathrm{Hom}_N(\pi|_N, \psi) \simeq \mathbb{C}$

We hence get a unique (up to scalar) M-homomorphism $\pi \to \mathrm{Ind}_N^M \psi \simeq C^\infty(F_v^\times)$

<u>Prop</u>: 1. This M-homomorphism is injective

2. The image contains $c\text{-}\mathrm{Ind}_N^M \psi \simeq C_c^\infty(F_v^\times)$, in particular, if π is cuspidal, then image $= c\text{-}\mathrm{Ind}_N^M \psi$

<u>pf</u>: 1. $\pi \xrightarrow{f} \mathrm{Ind}_N^M \psi \to \psi$ (the last map is restriction to N), if $\pi \to \psi$ is zero, then $f = 0$, because $f(v)\begin{pmatrix} y & x \\ & 1 \end{pmatrix} = f(\pi\begin{pmatrix} y & \\ & 1 \end{pmatrix} v)\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = 0$

therefore $\pi \to \psi$ is nonzero $\Rightarrow \pi_\psi \simeq \psi$, then $(\ker f)_\psi = 0 \Rightarrow ((\ker f)(N))_\theta = 0, \; \forall \theta \Rightarrow (\ker f)(N) = 0 \Rightarrow \ker f$ is N-invariant

but $\ker f \subseteq \pi$ (π irr), then $\pi = \chi \circ \det \Rightarrow$ contradiction, hence $\ker f = 0 \Rightarrow$ injective

2. $f(\pi(N)) = c\text{-}\mathrm{Ind}_N^M \psi \Rightarrow f(\pi) \supset f(\pi(N)) = c\text{-}\mathrm{Ind}_N^M \psi$

We also have an obvious map:

$$\begin{aligned} \mathrm{Ind}_N^G \psi &\longrightarrow \mathrm{Ind}_N^M \psi \\ f &\longmapsto f|_M \end{aligned}$$

<u>Claim</u>: Under this restriction map, we have $\mathcal{W}(\pi) \simeq \mathcal{K}(\pi)$ (just as an M-representation)

pf: again FR: $\quad \mathrm{Hom}_M(\pi|_M, \mathrm{Ind}_N^M \psi) \simeq \mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G \psi)$.

<u>Rmk</u>: Now we have a model of π in the locally constant function space on $F_v^\times$, these functions are essentially

$$W \in \mathcal{W}(\pi) \longmapsto \widetilde{W}(y) = W\begin{pmatrix} y & \\ & 1 \end{pmatrix}, \; y \in F_v^\times, \text{ here } W \in \mathcal{W}(\pi)$$

We consider how does π acts on this function space $C^\infty(F_v^\times)$

$$\left(\pi\begin{pmatrix} a & b \\ & 1 \end{pmatrix} W\right)(y) = W\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ & 1 \end{pmatrix}\right) = W\left(\begin{pmatrix} ay & by \\ & 1 \end{pmatrix}\right) = W\left(\begin{pmatrix} 1 & by \\ & 1 \end{pmatrix}\begin{pmatrix} ay & \\ & 1 \end{pmatrix}\right) = \psi(by)\, W(ay)$$

$$\left(\pi\begin{pmatrix} a & \\ & a \end{pmatrix} W\right)(y) = W\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & a \end{pmatrix}\right) = \omega_\pi(a)\, W(y)$$

the only remaining element we still don't know is $w_0 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$

Explicit description of Kirillov model

In this section, we aim to give an explicit description of Kirillov model of an infinite-dimensional irreducible smooth admissible representation π. We try to describe the underlying function space and how the $w_0 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ acts

1. π cuspidal
   - $K(\pi) = C_c^\infty(F_v^\times)$
   - w_0: the action of w_0 is not easy to describle, we can find an answer after the functional equation part

2. $\pi \simeq \pi(\chi_1, \chi_2)$

   - $K(\pi)$

Prop: 1. Assume $\chi_1 \neq \chi_2$, then $\phi \in K(\pi)$ satisfies:

when $|t| \to +\infty$, $\phi(t) = 0$

when $|t| \to 0$, $\exists C_1, C_2$, s.t.

$$\phi(t) = C_1 |t|^{\frac{1}{2}} \chi_1(t) + C_2 |t|^{\frac{1}{2}} \chi_2(t)$$

2. Assume $\chi_1 = \chi_2 = \chi$, then $\phi \in K(\pi)$ satisfies:

when $|t| \to +\infty$, $\phi(t) = 0$

when $|t| \to 0$, $\exists C_1, C_2$, s.t.

$$\phi(t) = C_1 |t|^{\frac{1}{2}} \chi(t) + C_2 |t|^{\frac{1}{2}} v(t) \chi(t)$$

pf: This description is anything but easy

We first consider the structure of the Jacquet module of $\pi(\chi_1, \chi_2)$.

Case 1: V_N as a T-repn is isomorphic to:

$$\begin{pmatrix} \delta^{\frac{1}{2}}\chi & \\ & \delta^{\frac{1}{2}}\chi' \end{pmatrix}$$

key observation: $\mathrm{Hom}_T(\pi_N, \eta) \simeq \mathrm{Hom}_B(\pi, \eta) \simeq \mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \eta)$ is at most 1-dimensional

Case 2: V_N as a T-repn is isomorphic to:

$$\delta^{\frac{1}{2}}\chi \cdot \begin{pmatrix} 1 & v(\frac{t_1}{t_2}) \\ & 1 \end{pmatrix}$$

Now we consider $\bar{\phi} \in V_N$, s.t. $\pi_N(t)\bar{\phi} = (\delta^{\frac{1}{2}}\chi(t)) \cdot \bar{\phi}$, then $\pi\begin{pmatrix} t & \\ & 1 \end{pmatrix}\phi - \delta^{\frac{1}{2}}\chi_1(t)\phi \in V(N)$

then since $V(N)$ is mapped to $C_c^\infty(F_v^\times)$, we get $\pi_K\begin{pmatrix} t & \\ & 1 \end{pmatrix}\phi = \delta^{\frac{1}{2}}\chi_1(t)\phi$ when we look at nbd of 0

now we choose $t_0 \in w_v \mathcal{O}_v^\times$, then $\exists\, \epsilon(t_0) > 0$, s.t. $\forall |t| < \epsilon(t_0)$, we have

$$\phi(t_0 t) = \delta^{\frac{1}{2}}\chi_1(t_0)\phi(t)$$

since ϕ is smooth, we could find a universal bound ϵ, s.t. $\forall t \in w_v \mathcal{O}_v^\times$, $\forall |t'| < \epsilon$, we have

$$\phi(tt') = \delta^{\frac{1}{2}}\chi_1(t)\,\phi(t')$$

then since $|w| < 1$, $\phi(tt') = \delta^{\frac{1}{2}}\chi(t)\phi(t')$, $\forall |t| < 1$, $|t'| < \epsilon \Rightarrow \phi(t) = C \cdot \delta^{\frac{1}{2}}\chi_1(t)$, when $|t| < \epsilon q^{-1}$

Case 1 can be solved immediately, now we consider Case 2, now we pick $\bar{\phi}$, $\bar{\varphi} \in V_N$, s.t.

$$\pi_N(t)\,\bar{\phi} = \delta^{\frac{1}{2}}\chi(t)\,\bar{\phi}, \quad \pi_N(t)\,\bar{\varphi} = \delta^{\frac{1}{2}}\chi(t)\,\bar{\varphi} + \delta^{\frac{1}{2}}\chi(t)\,v\left(\tfrac{t_1}{t_2}\right)\bar{\phi}$$

then we know, near 0, $\phi(t) = C\cdot|t|^{\frac{1}{2}}\chi(t)$, now $\pi_N(t)\,\bar{\varphi} - \delta^{\frac{1}{2}}\chi(t)\,\bar{\varphi} = \delta^{\frac{1}{2}}\chi(t)v\left(\frac{t_1}{t_2}\right)\bar{\phi}$, we get for fixed t_0, $\exists\, \epsilon(t_0)>0$, s.t. when $|t|<\epsilon(t_0)$,

$$\left(\pi_N\begin{pmatrix} t_0 & \\ & 1\end{pmatrix}\varphi\right)(t) - |t_0|^{\frac{1}{2}}\chi(t_0)\,\varphi(t) = C\,|t_0t|^{\frac{1}{2}}\,\chi(t_0t)\,v(t_0), \text{ i.e.}$$

$$\varphi(t_0t) - |t_0|^{\frac{1}{2}}\chi(t_0)\,\varphi(t) = C\cdot|t_0t|^{\frac{1}{2}}\,\chi(t_0t)\,v(t_0) \quad (*)$$

Use the same argument, for $\exists\, \varepsilon>0$, s.t. for $\forall |t_0|<1$, $|t|<\varepsilon$, $(*)$ is right, then

$$\varphi(t) = D\cdot|t|^{\frac{1}{2}}\chi(t) + C\cdot|t|^{\frac{1}{2}}\chi(t)\,v(t)$$

3. $\pi \simeq \tilde{\chi}\, St_G$ $\left(\tilde{\chi} = \chi_2\cdot\delta^{-\frac{1}{2}}\right)$

$$\tilde{\chi}\cdot St_G\left(\begin{pmatrix} a & x\\ & b\end{pmatrix}g\right) = \tilde{\chi}(ab)\cdot\tilde{\chi}(\det g)\,St_G(g)$$
$$= \chi_1(a)\chi_2(b)\left|\tfrac{a}{b}\right|^{\frac{1}{2}} f(g)$$
$$= \chi_2(ab)\,|ab|^{-\frac{1}{2}}$$

- $K(\pi)$

<u>Prop</u>: Assume $\chi_1\chi_2^{-1}(t) = |t|^{-1}$, then $\phi \in K(\tilde{\chi}\, St_G)$ satisfies

when $|t|\to+\infty$, $\phi(t)=0$

when $|t|\to 0$, $\phi(t) = C\cdot|t|^{\frac{1}{2}}\chi_2(t)$

pf: We should understand the Jacquet module of $\tilde{\chi}\cdot St_G$, it is one-dimensional, because

$$0 \to 1_G \to \mathrm{Ind}_B^G 1 \longrightarrow St_G \longrightarrow 0$$
$$\Rightarrow\ 0 \to \chi_2\delta^{-\frac{1}{2}} \longrightarrow \mathrm{Ind}_B^G\,\chi \longrightarrow \tilde{\chi}\cdot St_G \longrightarrow 0$$
$$\Rightarrow\ 0 \to \chi_2\,\delta^{-\frac{1}{2}} \to \left(\mathrm{Ind}_B^G\,\chi\right)_N \to \left(\tilde{\chi}\cdot St_G\right)_N \longrightarrow 0$$

hence $\left(\tilde{\chi}\cdot St_G\right)_N \simeq \chi_2\cdot\delta^{\frac{1}{2}}$ as T-representation, then use the same argument

$$\phi(t) = C\cdot|t|^{\frac{1}{2}}\chi_2(t) \quad \text{when } |t|\to+\infty$$

<u>Rmk</u>: Let's finally show the "vanishing at ∞", for (π, V) an infinite dimensional irr smooth admissible repn, then for $\forall\, \phi\in K(\pi)$, $\exists\, C>0$, s.t. $\forall\,|y|>C$, $\phi(y)=0$

<u>pf</u>: $\left(\begin{pmatrix} 1 & b\\ & 1\end{pmatrix}\phi\right)(x) = \psi(bx)\,\phi(x)$, since the repn is smooth, $\exists\, k>0$, s.t. $\forall\, b\in p^k$, $\begin{pmatrix} 1 & b\\ & 1\end{pmatrix}\phi = \phi$, i.e.

$$\phi(x) = \psi(bx)\,\phi(x), \quad \forall\, b\in p^k$$

then when $|x|\to+\infty$, $\psi(bx)\neq 1$ for $\forall\, x$ sufficiently large $\Rightarrow \phi(x)=0$

<u>Rmk</u>: The Kirillov model is determined by 2 actions:

1. Center (acts as a scalar)
2. w_0 (mistery!)

Let's now try to derive the action of w_0 on $K(\pi)$

We know that we have an explicit description of $W(\pi)$ in this case, $Re(s_1-s_2)>0$

$$W_f(g) = \int_{F_v} f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g)\,\psi(-x)\,dx$$

then

$$W_f(y) = \int_{F_v} f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y & \\ & 1 \end{pmatrix})\,\psi(-x)\,dx = \int_{F_v} f(w_0 \begin{pmatrix} y & x \\ & 1 \end{pmatrix})\,\psi(-x)\,dx$$

apply $\pi(w_0)$, we get:

$$\left(\pi(w_0) W_f\right)(y) = \int_{F_v} f(w_0 \begin{pmatrix} y & x \\ & 1 \end{pmatrix} w_0)\,\psi(-x)\,dx = \int_{F_v} f(\begin{pmatrix} x^{-1}y & -1 \\ & x \end{pmatrix} w_0 \begin{pmatrix} 1 & -x^{-1}y \\ & 1 \end{pmatrix})\,\psi(-x)\,dx$$

$$= |y|^{\frac{1}{2}} \chi_1(y) \int_{F_v} \chi_1^{-1}\chi_2(x)\,|x|^{-1} f(w_0 \begin{pmatrix} 1 & -x^{-1}y \\ & 1 \end{pmatrix})\,\psi(-x)\,dx$$

$$\overset{x'y=x}{=\!=} |y|^{\frac{1}{2}} \chi_1(y) \int_{F_v} \chi_1^{-1}\chi_2(xy)\,|xy|^{-1} f(w_0 \begin{pmatrix} 1 & -x^{-1} \\ & 1 \end{pmatrix})\,\psi(-xy)\,|y|\,dx$$

$$= |y|^{\frac{1}{2}} \omega_\pi(y) \int_{F_v} \chi_1^{-1}\chi_2(x)\,|x|^{-1} f(w_0 \begin{pmatrix} 1 & -x^{-1} \\ & 1 \end{pmatrix})\,\psi(-xy)\,dx, \quad \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & -x^{-1} \\ & 1 \end{pmatrix} = \begin{pmatrix} -x & -1 \\ & -x^{-1} \end{pmatrix}\begin{pmatrix} 1 & \\ -x & 1 \end{pmatrix}$$

$$= |y|^{\frac{1}{2}} \omega_\pi(-y) \int_{F_v} f(\begin{pmatrix} 1 & \\ -x & 1 \end{pmatrix})\,\psi(-xy)\,dx \qquad g(x) = f(\begin{pmatrix} 1 & \\ -x & 1 \end{pmatrix})$$

to keep consistence:

$$W_f(y) = \int_{F_v} \chi_2(y)\,|y|^{-\frac{1}{2}} f(w_0 \begin{pmatrix} 1 & xy^{-1} \\ & 1 \end{pmatrix})\,\psi(-x)\,dx$$

$$= \chi_2(y)\,|y|^{\frac{1}{2}} \int_{F_v} f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix})\,\psi(-xy)\,dx, \quad h(x) = f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix})$$

$g(x)$ & $h(x)$ are related by:

$$h(x) = f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}) = \chi_1^{-1}\chi_2(x)\,g(x^{-1}), \text{ for } x \neq 0, \quad h(0) = f(w_0), \quad g(0) = f(1)$$

and $\lim_{|x|\to+\infty} h(x) = 0, \quad \lim_{|x|\to+\infty} g(x) = \lim_{|x|\to 0} g(x^{-1}) = 0$

Weil representations

In this section, we try to give a complete treatment of Weil representation over local field. We will try to discribe the Jacquent-Langlands transfer.

Prelimnaries

- Witt's theorem

Since we will work over vector spaces with skew-symmetric / symmetric bilinear forms. The following thm is useful

<u>Witt's theorem</u>: (V, B) is a vector space over k with char $k \neq 2$ and B: non-degenerate skew-symmetric / symmetric bilinear forms, then if $f: U \to U'$ is an isometry between two subspaces of V, then f extends to an isometry of V

Corollary: Witt's index: maximal isotropic subspace all have the same dimension.
or, langlavians

- Heisenberg group

Suppose F is a local field char $F \neq 2$, W is a $2n$-dimensional vector space over F with non-degenerate symplectic form $\langle \cdot, \cdot \rangle$.
→ allow finite fields

<u>Def</u>: Heisenberg group associated to W is $H(W) = W \oplus F$, with group law

$$(w_1, x_1) + (w_2, x_2) = (w_1 + w_2, x_1 + x_2 + \tfrac{1}{2}\langle w_1, w_2\rangle)$$

The importance of Heisenberg group is that the irr repn of $H(W)$ is easy to describe. First, $\forall \rho$ an irr admissible repn of $H(W)$ restricts to $Z(H(W)) = 0 \oplus F \simeq F$ gives a character ψ of F, we will see that given any character ψ of F. we obtain an irr admissible repn of $H(W)$, by the following:

<u>Stone-von Neumann Theorem</u>: Given an character ψ of F, then there exists (up to isomorphism) a unique irreducible admissible repn (ρ, S) of $H(W)$ with central character ψ

Let's give a sketch of the proof, which will also gives us an important model of this representation. Take V to be a maximal isotropic subspace of W, then consider $V_H = V \oplus F \subset H(W)$, extend ψ to V_H by being trivial on V, denote by ψ_V, then consider

$$S_V = \mathrm{Ind}_{V_H}^{H(W)} \psi_A = \{ f: H(W) \to \mathbb{C} \mid f(ah) = \psi_A(a) f(h), \forall a \in V_H \}$$

Claim: S_V is irreducible under the action of $H(W)$, with central character ψ_A

<u>Rmk</u>: Usually in the situations we shall see, we will get two transversal maximal isotropic subspaces, i.e. $W = V_1 \oplus V_2$, with V_1 & V_2 are both isotropic, then we will have

$$S_{V_1} \xrightarrow{\sim} S(V_2) \qquad S_{V_2} \xrightarrow{\sim} S(V_1)$$
$$f \longmapsto f|_{V_2} \qquad g \longmapsto g|_{V_1}$$

these are two models for the irr repn with central character ψ, the intertwing map can be given by

$$S(V_1) \longrightarrow S(V_2)$$
$$f \longmapsto \hat{f}: x \mapsto \int_{V_1} f(y) \psi(\langle y, x\rangle) dy$$

- Weil representation

With all these preliminaries, we could finally define the Weil repn associated to W. Let's consider the symplectic group $Sp(W)$. $Sp(W)$ has a natural action on $H(W)$: $\forall g \in Sp(W)$, $g(v, x) = (gv, x)$, i.e. $Sp(W) \to Aut_F(H(W))$

Let's now consider a twist of ρ_ψ

$$\rho_\psi^g((v,x)) = \rho_\psi((gv, x))$$

Obviously, ρ_ψ^g & ρ_ψ are both irreducible with central character ψ, hence by uniqueness property, there exists $W(g)$ s.t.

- $W(g): S \longrightarrow S$
- $W(g) \circ \rho_\psi^g = \rho_\psi \circ W(g)$

We expect $W(g)$ to be a representation of $Sp(W)$, but by Schur's lemma, $W(g)$ is only determined up to a scalar multiple of $\mathbb{C}^\times$ hence we only got a reasonable map $\omega: Sp(W) \longrightarrow PGL(S)$, this is a homomorphism, because

$$\omega(g)\omega(h) \circ \rho_\psi^{gh} = W(g)\rho_\psi^g \circ W(h) = \rho_\psi \circ W(g)W(h) \Rightarrow \omega(gh) \doteq \omega(g)\cdot\omega(h) \text{ up to scalar } \in \mathbb{C}^\times$$

Now the question comes: How does one got a "true" representation? We are led naturally to the following diagram:

$$\begin{array}{ccc} \widetilde{Sp(W)} & \xrightarrow{\tilde{\omega}} & GL(S) \\ \downarrow & & \downarrow \\ Sp(W) & \xrightarrow{\omega} & PGL(S) \end{array}$$

Here $\widetilde{Sp(W)}$ is the fiber product of $Sp(W)$ & $GL(S)$ over $PGL(S)$

<u>Rmk</u>: Sometimes we will require $W(g)$ to be unitary: for example, in the model $S_{V_1} \simeq S(V_2)$, the representation space has a natural inner product to make it into a Hermitian inner product space. ρ is automatically unitary, since it is essentially right multiplication, a non-trival factor is that $W(g)$ can be unitary. Hence the extension of $Sp(W)$ can be taken to be extending by $T \simeq S^1$, not simply in our case: by $\mathbb{C}^\times$

Fact:
- For $F = \mathbb{C}$, $Sp(W)$ is simply connected, then ω automatically gives rise to a true representation
- For local fields $F \neq \mathbb{C}$, $Sp(W)$ admits a non-trivial two-fold covering $M_p(W)$
  $\tilde{\omega}$ descends to a true representation / ω extends to a true representation of S
- For $F = \mathbb{R}$, $\dim_{\mathbb{R}} W = 2$, $M_p(W)$ can be described by:

$$M_p(W) = \{ (g, f) \mid g \in Sp(W) \cong SL_2(\mathbb{R}),\ f^2 = c\tau + d \text{ is a function } \mathcal{H} \to \mathbb{C} \}$$

- $M_p(W)$ is not algebraic, hence $M_p(W)(\mathbb{Q})$ doesn't make sense, when we want to define a "global" version of $M_p(W)$, i.e. over $\mathbb{A}$, we proceed by

  Suppose F is a global field, consider $M_p'(W) = \prod_v' M_p(W_v)$, here the restricted product ranges over all places v of F, with complex place $M_p(W_v) = S_p(W_v)$, w.r.t. the maximal compact subgroup of each place, then define $Z = \left(\bigoplus_{v \neq \mathbb{C}} \mu_2^v\right)^1 = \{ (z^v) \in \bigoplus_{v \neq \mathbb{C}} \mu_2^v \mid \prod z^v = 1 \}$,

$$M_p(W)(\mathbb{A}) := M_p'(W)/Z$$

  hence we have the following

$$1 \to \mu_2 \to M_p(W)(\mathbb{A}) \to Sp(W)(\mathbb{A}) \to 1$$

  and the discrete subgp is given by $Sp(W)(F) \hookrightarrow M_p(W)(\mathbb{A})$ diagonally

Fact (continued) :
- W dependends on ψ, but we can show $\widetilde{Sp(W)}$ is independent of ψ, hence the non-trivial two-fold covering $M_p(W)$ is also independent of ψ
- For NA local field F, there are exactly $F^\times/F^{\times 2}$ inequivalent W.
  Let's explain this. for a different choice of ψ', we have $\psi'(x) = \psi(ax)$ for some $a \in F^\times$
  we can show that $\rho_{\psi_a} \simeq \rho_{\psi_{ab^2}}$ by considering $s(b): (w, x) \mapsto (bw, b^2 x)$ of $H(W)$

· Schrödinger Model

In this section, we introduce an explicit model for local Weil repn over NA local fields.
The idea has been stated before, suppose we have a decomposition

$$W = V_1 \oplus V_2$$

where V_1, V_2 are maximal isotropic subspace of W, then S_{V_i} is irr under the action of $H(W)$
here S_{V_1} can be taken to be the underlying space of Weil representation. Let's first consider finding a projective repn of $Sp(W)$ on S_{V_i}, we have the following diagram of repns of $H(W)$

$$\begin{tikzcd} S_{V_1} \arrow[r, "\text{restricts to } V_2", "\sim"'] \arrow[d, "(x,t)"'] & S(V_2) \arrow[d, "(x,t)"] \\ S_{V_1} \arrow[r, "\text{restricts to } V_2", "\sim"'] & S(V_2) \end{tikzcd}$$

right regular representation

then we could be able to derive a formula of $\rho(x,t)$ on $S(V_2)$, then consider $\rho^g(x,t)$, we could find the following projective repn would work:

$$\left(\omega\left(\begin{pmatrix} A & \\ & {}^tA^{-1}\end{pmatrix}\right) f\right)(X) = |\det A|^{\frac{1}{2}} f({}^tA X)$$

$$\left(\omega\left(\begin{pmatrix} I & B \\ & I\end{pmatrix}\right) f\right)(X) = \psi\left(\frac{{}^tXBX}{2}\right) f(X)$$

$$\left(\omega\left(\begin{pmatrix} & I \\ -I & \end{pmatrix}\right) f\right)(X) = \hat{f}(X)$$

now the question is: how to derive a formula of Weil repn on the group $M_p(W)$? recall that we have:

$$\begin{tikzcd} M_p(W) \arrow[r] \arrow[d, "\omega"'] & S_p(W) \arrow[d, "{[\omega]}"] \\ GL(S) \arrow[r] & PGL(S) \end{tikzcd}$$

- Reductive dual pairs & Howe duality

<u>Def</u>: a pair (G_1, G_2) of subgps of $Sp(W)$ is called a reductive dual pair if

- G_1, G_2 are both reductive groups
- $\mathrm{Cent}_{Sp(W)}(G_1) = G_2$, $\mathrm{Cent}_{Sp(W)}(G_2) = G_1$

Rmk: G_1 & G_2 centralize each other, hence they act on the Weil repn (ω, S) of $Sp(W)$ $(Mp(W))$ commutatively hence in principle the repn space S should decompose nicely with the action of $G_1 \times G_2$

Let's see some examples:

- V is a quadratic space of finite dimension, non-degenerate
- W is a symplectic space of finite dimension, non-degenerate

then $V \otimes_F W$ becomes a symplectic space of finite dimension, non-degenerate. then

<u>Claim</u>: $(O(V), Sp(W))$ is an irreducible reductive pair of $Sp(W)$
Moreover, $O(V) \times Sp(W)$ splits in the projection map $Mp(W) \to Sp(W)$, i.e. they can be viewed as (in a canonical way) subgroups of $Mp(W)$

Then Claim tells us that we can have the group repn $O(V) \times Sp(W)$ on the Weil repn ω_ψ of $Mp(W)$
Let's consider our previous model. suppose W is good, i.e. $W = W_1 \oplus W_2$ is a decomposition into Langarions, then $V \otimes_F W = V \otimes_F W_1 \oplus V \otimes_F W_2$ is a decomposition into langarians. so our ω_ψ can be realized on

$$S(V \otimes_F W_1) \quad \text{or} \quad S(V \otimes_F W_2)$$

in many situations, we would directly assume $W = F^{2n} = F^n \oplus F^n$, with the standard symplectic form, then for each $n \geq 1$, we get a repn of $O(V)$ is $S(V^n)$, this is also called the weil repn.

one group is fixed, another one can vary a lot!

Our next question is, how to derive a repn formula for these kinds of repns?
To understand the repn of $O(V)$ & $Sp(W)$ on $S(V^n)$, we should first understand their embedding as subgp of $Sp(W \otimes_F V)$
Hence we fix basis of W & V to be

- W: $e_1, \dots, e_n, f_1, \dots, f_n$. s.t. $\langle e_i, f_i \rangle = 1$
- V: $v_1, \dots, v_m$.

$\Big\}$ $W \otimes_F V$ has basis $e_i \otimes v_j$, $f_i \otimes v_j$

now $\forall g \in O(V)$, $(g, 1) \in O(V) \times Sp(W) \hookrightarrow Sp(W \otimes_F V)$, hence g takes the form

$$\begin{array}{cc} & \begin{array}{cc} mn & mn \end{array} \\ \begin{array}{c} mn \\ mn \end{array} & \left(\begin{array}{ccc|ccc} g & & & & & \\ & \ddots & & & & \\ & & g & & & \\ \hline & & & g & & \\ & & & & \ddots & \\ & & & & & g \end{array}\right) \end{array}$$

now $\forall h \in Sp(W)$, $(1, h) \in O(V) \times Sp(W) \hookrightarrow Sp(W \otimes_F V)$, h takes the form

Suppose h takes the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

i.e. $h(e_i) = \sum_{k=1}^{r} a_{ki} e_i + \sum_{k=1}^{r} c_{ki} f_i$

$h(f_i) = \sum_{k=1}^{r} b_{ki} e_i + \sum_{k=1}^{r} d_{ki} f_i$

$$h(e_1 \otimes v_1, e_2 \otimes v_1, \dots, e_n \otimes v_1, e_1 \otimes v_2, \dots, e_n \otimes v_2, \dots) = \left(\begin{array}{ccc|ccc} A & & & B & & \\ & A & & & B & \\ & & \ddots A & & & \ddots B \\ \hline C & & & D & & \\ & C & & & D & \\ & & \ddots C & & & \ddots D \end{array}\right)$$

therefore, combining previous results on explicit formula for Weil repns, we can get

$$\left(\omega\left(\begin{pmatrix} A & \\ & {}^tA^{-1}\end{pmatrix}\right)f\right)(X_1,\dots,X_n) = |\det A|^{\frac{m}{2}} f\left(\begin{pmatrix} {}^tA & & \\ & {}^tA & \\ & & \ddots & \\ & & & {}^tA\end{pmatrix}(X_1,\dots,X_n)\right) = |\det A|^{\frac{m}{2}} f\left(\sum_{i=1}^n a_{i1}X_i, \dots, \sum_{i=1}^n a_{in}X_i\right)$$

$X_i \in V$

$$\left(\omega\left(\begin{pmatrix} I & B \\ & I\end{pmatrix}\right)f\right)(X_1,\dots,X_n) = \psi\left(\frac{(X_1,\dots X_n)^t\begin{pmatrix} B & & \\ & B & \\ & & \ddots & \\ & & & B\end{pmatrix}(X_1,\dots X_n)}{2}\right) f(X_1,\dots,X_n)$$

$$\left(\omega\left(\begin{pmatrix} & I \\ -I & \end{pmatrix}\right)f\right)(X_1,\dots X_n) = \hat{f}(X_1,\dots,X_n)$$

this is a little bit mysterious, because actually we haven't identified much terms
For example, in the first equality, the situation should be

$$X = e_1 \otimes X_1 + \cdots + e_n \otimes X_n$$

$$\begin{pmatrix} {}^tA & & & \\ & {}^tA & & \\ & & \ddots & \\ & & & {}^tA\end{pmatrix} X = {}^tAe_1 \otimes X_1 + \cdots + {}^tAe_n \otimes X_n$$

$$= \sum_{i=1}^n \sum_{j=1}^n a_{ij}e_j \otimes X_i = \sum_{j=1}^n e_j \otimes \left(\sum_{i=1}^n a_{ij} X_i\right)$$

- Except from this kind of embedding, there is another important & intuitive embedding

$$Sp_{2i}(F) \hookrightarrow Sp_{2n}(F), \quad \text{for } 1 \le i \le n$$

this is very simple to describle, since we have a standard symplectic basis $\{e_1,\dots,e_n, f_1,\dots,f_n\}$
For any choice of i numbers, say $\{1,\dots,i\}$. $\{e_1,\dots e_i, f_1,\dots,f_i\}$ forms a standard symplectic space, then

$$\begin{pmatrix} A & B \\ C & D\end{pmatrix} \in Sp_{2i}(F) \hookrightarrow \begin{pmatrix} A & 0 & B & 0 \\ 0 & I_{n-i} & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & I_{n-i}\end{pmatrix} \in Sp_{2r}(F)$$

· Howe duality principle

Now suppose (G_1, G_2) is a reductive dual pair of $Sp(W)$, for a fixed ψ. consider the Weil repn (ω_ψ, S_ψ) of $Mp(W)$
Let $\hat{G}_1$ & $\hat{G}_2$ denote the inverse image of G_1 & G_2 in $Mp(W)$. moreover, $\hat{G}_1$ & $\hat{G}_2$ commutes with each other in $Mp(W)$. hence we have a homomorphism

$$\hat{G}_1 \times \hat{G}_2 \longrightarrow \hat{G}_1 \cdot \hat{G}_2 \subset Mp(W)$$

Now we pick $\pi \in irr(\tilde{G}_1)$, then we consider the π-isotypic quotient of S_ψ

$$S_\psi(\pi) = S_\psi \Big/ \bigcap_{\lambda \in Hom_{\tilde{G}_1}(S_\psi, \pi)} \ker\lambda$$

then $S_\psi(\pi)$ admits an action of $\tilde{G}_2$, because $\tilde{G}_2$ stabilizes each $\ker\lambda$ for $\lambda \in Hom_{\tilde{G}_1}(S_\psi, \pi)$
hence $S_\psi(\pi)$ is a repn of $\tilde{G}_1 \times \tilde{G}_2$, by a standard fact about repns of product of two gps, we got

$\exists\, \Theta_\psi(\pi)$ is a repn of $\tilde{G}_2$. s.t. $\quad S_\psi(\pi) \simeq \pi \otimes \Theta_\psi(\pi)$

The following is a fundamental result of Howe:
<u>Thm</u>: Assume the residue char of F is not 2, then for $\forall$ irr admissible repn π of $\tilde{G}_1$

- Either $\Theta_\psi(\pi) = 0$ or $\Theta_\psi(\pi)$ is an admissible repn of $\tilde{G}_2$ of finite length
- If $\Theta_\psi(\pi) \neq 0$, then there exists a unique $\tilde{G}_2$-invariant subspace $\Theta^\circ_\psi(\pi)$, s.t.

$$\theta_\psi(\pi) = \Theta_\psi(\pi) \Big/ \Theta^\circ_\psi(\pi)$$

  is irreducible, and $\theta_\psi(\pi)$ is uniquely determined by π
- If $\theta_\psi(\pi_1)$ & $\theta_\psi(\pi_2)$ are nonzero and isomorphic, then $\pi_1 \simeq \pi_2$

Obviously, we could define θ_ψ from irr repns of $\tilde{G}_2$, but actually this will be the "inverse" of the $\tilde{G}_1$ one
We define $\quad Howe_\psi(\tilde{G}_1, \tilde{G}_2) = \{\pi \in Irr(\tilde{G}_1) \mid \theta_\psi(\pi) \neq 0\}$, similarly for $Howe_\psi(\tilde{G}_2, \tilde{G}_1)$, then the above thm says

$$\begin{array}{ccc} Howe_\psi(\tilde{G}_1, \tilde{G}_2) & \xrightarrow{\sim} & Howe_\psi(\tilde{G}_2, \tilde{G}_1) \\ \pi & \longmapsto & \theta_\psi(\pi) \\ \theta_\psi(\sigma) & \longleftarrow & \sigma \end{array}$$

local theta correspondence

· An Example

In this section, we introduce another version of Weil representation, this kind of repn has deep relationship with Siegel modular forms of higher genus. Now suppose our discussion is over $\mathbb{Q}$

We need an additional fact about the group $Sp_{2g}(\mathbb{Z})$ & $\widetilde{Sp_{2g}}(\mathbb{Z})$ (inverse image of $Sp_{2g}(\mathbb{Z})$ in $Mp_{2g}(\mathbb{R})$)

· Generators of $Sp_{2g}(\mathbb{Z})$

parabolic : $\begin{pmatrix} A & \\ & {}^tA^{-1} \end{pmatrix}$ $A \in GL_g(\mathbb{Z})$

unipotent : $\begin{pmatrix} 1 & B \\ & 1 \end{pmatrix}$ $B \in Sym_g(\mathbb{Z})$

evil : $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

actually, in [Ki] article. he proved we can replace the evil one to be $\omega_{i,(j_1,\dots,j_i)}\left(\begin{pmatrix} & I \\ -I & \end{pmatrix}\right)$
therefore, when we want to show the modularity of something, we only need to check these three types of elements

Rmk: the generators for $\widetilde{Sp_{2g}}(\mathbb{Z})$ can also be written like this
A remaining question is: Does this still hold for $Sp_{2g}(\mathcal{O}_F)$ for $F/\mathbb{Q}$ finite extension ?

· A sub-repn of Weil repn

We consider a quadratic space (V, q) over $\mathbb{Q}$ of finite dimension. Then we know that, by tensoring with a standard symplectic space W over $\mathbb{Q}$, we can get a local, hence global Weil repn

$$Mp_n(\mathbb{A}_f) \curvearrowright S(V(\mathbb{A}_f)^n) \quad \text{every } n \geq 1$$

local split to $Sp_{2n}(\mathbb{Q}_p)$, but globally can't be splitted ?

Now we fix an integral lattice L of V, s.t. $L_{\mathbb{Q}} = V$, and consider $L^\vee = \{x \in V \mid (x, L) \subset \mathbb{Z}\}$
then clearly $L \subset L^\vee$. We consider $\hat{L} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, $\hat{L}^\vee = L^\vee \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, and the following function space

$$S_L = \{\varphi \in S(V(\mathbb{A}_f)^n) \mid \operatorname{Supp}\varphi \in \hat{L}^{\vee n}, \text{ and invariant under } \hat{L}^n\}$$

Clearly, we have the following isomorphism:

$$S_L \simeq \mathbb{C}[\hat{L}^{\vee n}/\hat{L}^n] \cong \mathbb{C}[(L^\vee/L)^n]$$

<u>Claim</u>: $\widetilde{Sp_{2n}(\hat{\mathbb{Z}})}$ preserves S_L

pf: this can be checked directly by using our previous results of embedding, the parabolic & unipotent elements are easily seen by the formula to perserve this subspace S_L. we consider the evil one

$$\left(\omega\left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right)\varphi\right)(X) = \int_{V(\mathbb{A}_f)^n} \psi(\langle X, Y\rangle)\varphi(-Y)\,dY$$

$$= \sum_{\lambda \in L^\vee/L} \varphi(-\lambda)\psi(\langle X, \lambda\rangle) \int_{\hat{L}} \underbrace{\psi(\langle X, Y\rangle)}\,dY$$

If $X \notin \hat{L}^n$, then this term will be nontrivial, with Integration $= 0$

Therefore we know there is a well-defined repn $\widetilde{Sp_{2n}}(\hat{\mathbb{Z}}) \curvearrowright S_L \simeq \mathbb{C}[(L^\vee/L)^n]$ as a subrepn of $\widetilde{\text{Weil}}$ repn
Now we consider $\forall g \in Sp_{2n}(\mathbb{Z})$, $\exists!\, g_f \in Sp_{2n}(\hat{\mathbb{Z}})$, s.t. $g \cdot g_f \in$ diagonal embedding of $\widetilde{Sp_{2n}(\mathbb{Q})}$

$$\rho_{L,n}(g)\cdot \varphi = \omega_\psi(g_f)\cdot\varphi$$

We hence obtained a repn of $\widetilde{Sp_{2n}(\mathbb{Z})}$ on $\mathbb{C}[(L^\vee/L)^n]$, it can be described explicitly by the following formula

<u>Prop</u>: Suppose $L_\mathbb{Q}$ has signature (p,q), then the repn $\rho_{L,n}$ of $\widetilde{Sp_{2n}(\mathbb{Z})}$ can be described by

$$\rho_{L,n}\left(\left(\begin{matrix} A & \\ & {}^tA^{-1}\end{matrix}\right), \sqrt{\det A}\right)\varphi_{\underline{\lambda}} = \left(\sqrt{\det A}\right)^{q-p}\varphi_{\underline{\lambda}A^{-1}} \quad \text{for } A \in GL_r(\mathbb{Z})$$

$$\rho_{L,n}\left(\left(\begin{matrix} I & B\\ & I\end{matrix}\right), \ 1\right)\varphi_{\underline{\lambda}} = e\left(\mathrm{tr}(Q(\underline{\lambda})B)\right)\varphi_{\underline{\lambda}}$$

$$\rho_{L,n}\left(\left(\begin{matrix} & -I\\ I & \end{matrix}\right), \sqrt{\det\tau}\right)\varphi_{\underline{\lambda}} = \frac{e\left(-\frac{n(p-q)}{8}\right)}{|L^\vee/L|^{\frac{n}{2}}}\sum_{\underline{\delta}\in(L^\vee/L)^n} e(\langle -\underline{\delta}, \underline{\lambda}\rangle)\varphi_{\underline{\delta}}$$

Let's now switch a little bit to the theory of Siegel modular forms

<u>Def</u>: Siegel modular form

Suppose we are given a finite-dim repn $\rho: GL_g(\mathbb{C}) \to GL(V)$, then a holomorphic map $f: \mathcal{H}_g \to V$ is said to be a Siegel modular form of weight ρ if

$$f(\gamma\tau) = \rho(C\tau+D)f(\tau) \quad \text{for } \gamma = \left(\begin{matrix} A & B\\ C & D\end{matrix}\right) \in Sp_{2g}(\mathbb{Z}),\ \forall \tau\in\mathcal{H}_g$$

when $g=1$, we need the additional condition that f is holomorphic at ∞

Rmk:
- this is a direct generalization of the usual modular form, we can of course do something much general, such as use $Mp(2g,\mathbb{Z})$ instead of $Sp(2g,\mathbb{Z})$
- when $g \geq 2$, the holomorphicity of f on $\mathcal{H}_g$ will guarantee holomorphicity at "∞", or, the Fourier expansion will only have positive part, this is called Koecher's principle.
- the well-definedness of this relation can be viewed as the following cocycle relation, if $j(g,\tau) = C\tau + D$, then

$$j(g_1g_2, \tau) = j(g_1, g_2\tau)\cdot j(g_2,\tau)$$

We can visualize f as a global section of the following line bundle

$$Sp_{2g}(\mathbb{Z})\backslash \mathcal{H}_g \times V$$

here the action is $\gamma\,(\tau, v) = (\gamma\tau,\ \rho(j(\gamma,\tau))v)$
then any global section $f: \mathcal{H}_g \to V$ should satisfy:

$$[\tau, f(\tau)] = [\gamma\tau, \underbrace{\rho(j(\gamma,\tau))f(\tau)}_{\overset{\shortparallel}{f(\gamma\tau)}}] \qquad \text{i.e. } f(\gamma\tau) = \rho(j(\gamma,\tau))f(\tau)$$

Now we give a slight generalization of this notion, we see from the definition that. ρ only depends on C & D, has nothing to do with A & B, but sometimes we need them, therefore we define

- Consider a repn $\rho: Mp(2g, \mathbb{Z}) \to GL(V_\rho)$, Siegel modular form of wt $k \in \frac{1}{2}\mathbb{Z}$, type ρ is
  - holomorphic map $f: \mathcal{H}_g \to V_\rho$
  - $f(\gamma\tau) = \omega(\tau)^{2k}\rho((\gamma, \omega))f(\tau)$, $\forall (\gamma, \omega) \in Mp(2g, \mathbb{Z})$, $\tau \in \mathcal{H}_g$

For example, we will consider $\rho = \rho_{L,n}$ in previous example
Usually, we will require ρ to factor through a finite quotient of $Mp(2g, \mathbb{Z})$
We can also realize this as a vector bundle over $\Gamma \backslash \mathcal{H}_g$, we consider:

$$Mp(2g, \mathbb{Z}) \backslash \mathcal{H}_g \times V_\rho$$

- $(\gamma, \omega) \cdot (\tau, v) = (\gamma\tau, \omega(\tau)^{2k}\rho((\gamma, \omega))v)$

Now consider a global section $f: \mathcal{H}_g \to V_\rho$, we have

$$[\tau, f(\tau)] = [\gamma\tau, \underbrace{\omega(\tau)^{2k}\rho((\gamma,\omega))f(\tau)}_{= f(\gamma\tau)}]$$

Tate thesis: Theory of Automorphic forms on GL_1

Main goal: give a Fourier-analytic explanation for Hecke's theory of L-functions for Dirichlet series from the view point of automorphic reprs of the reductive group GL_1

We first consider $(GL_1)_{\mathbb{Q}}$.

$$GL_1(\mathbb{A}) = \mathbb{A}^\times = \mathbb{Q}^\times \times (0,+\infty) \times \prod_p \mathbb{Z}_p^\times \qquad \text{Class number} = 1$$

$\forall$ automorphic form $f \in GL_1(\mathbb{A})$ is

- $f(\gamma g) = f(g)$, $\forall \gamma \in \mathbb{Q}^\times$
- $f(gk) = f(g)$, $\forall k \in K \subset \mathbb{A}_f^\times \Rightarrow K$ will be contained in $1+N\hat{\mathbb{Z}}$ for some $N \in \mathbb{Z}$

$$\hat{\mathbb{Z}}^\times / 1+N\hat{\mathbb{Z}} = \prod_{p \mid N} \mathbb{Z}_p^\times / 1+p^{v_p(N)}\mathbb{Z}_p$$

$$\mathbb{Z}_p^\times / 1+p^{v_p(N)}\mathbb{Z}_p \simeq \left(\mathbb{Z}/p^{v_p(N)}\mathbb{Z}\right)^\times$$

$$\simeq \left(\mathbb{Z}/N\mathbb{Z}\right)^\times$$

then $f(g) = f(\gamma \cdot \|g\| \cdot k) = f(\|g\| \cdot k) = \bar{k} \cdot f(\|g\|)$, here $\bar{k} \in (\mathbb{Z}/N\mathbb{Z})^\times$ acts on f
$= \bar{k} \cdot f_\infty(\|g\|)$

thus we care about those eigen forms under the action of $(\mathbb{Z}/N\mathbb{Z})^\times$.

$$f(gk) = \chi(k) \cdot f(g) \text{ for some Dirichlet character } \chi: (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$$

- f is $Z(U(\mathfrak{g}_\infty)_{\mathbb{C}})$-finite

$$\mathfrak{g}_\infty = \mathbb{R} \xrightarrow{\exp} \mathbb{R}^\times = GL_1(\mathbb{R})$$

$$\frac{d}{dt} = X$$

$U(\mathfrak{g}_\infty)_{\mathbb{C}} = \mathbb{C}[X]$.

$$X f_\infty(t) = f_\infty'(t), \quad X^2 f_\infty(t) = f_\infty''(t), \quad \dots \quad X^n f_\infty(t) = f_\infty^{(n)}(t)$$

then this condition tells us: $\exists\, c_0, \dots, c_n$, s.t. not all zero, and

$$\sum_{k=0}^{n} c_k f^{(k)}(t) = 0 \Rightarrow f(t) = \text{linear combinations of } e^{\lambda t}, \ \lambda \in \mathbb{C}$$

- moderate growth: $|f(t)| \le C \cdot \max\{t, t^{-1}\}$. $\sim \mathrm{Re}(\lambda) \le 0$

• $f(g) = \omega(g) f(1)$. $\omega: \mathbb{A}^\times \to \mathbb{C}^\times$ a character

$$(0,+\infty) \times \hat{\mathbb{Z}}^\times \xrightarrow{\omega} \mathbb{C}^\times$$

$$\omega(t k) = \omega_\infty(t) \cdot \chi(k), \quad \omega_\infty(t) = e^{\lambda t}, \quad \mathrm{Re}(\lambda) \le 0$$

Let's consider classical Hecke character:

$$\chi: (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$$

then we know $\hat{\mathbb{Z}}^\times \twoheadrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$, we still denote this by χ

and we moreover chose ω_∞ is trivial, this realized χ as a character on $\mathbb{A}^\times$

• Conductor

Suppose we have $\omega: \mathbb{A}^\times \to \mathbb{C}^\times$, then we have seen $\omega = \omega_\infty \cdot \chi$

and $\chi: \hat{\mathbb{Z}}^\times \to \mathbb{C}^\times$ is a character, then it factors through some $1+N\hat{\mathbb{Z}}$, we view ω_χ as $\mathbb{A}^\times \to \mathbb{C}^\times$

<u>Claim</u>: ω_χ has a decomposition into product of "local" characters

$$\omega_\chi = \bigotimes_p \omega_{\chi_p}, \quad \omega_{\chi_p}: \mathbb{Q}_p^\times \to \mathbb{C}^\times$$

and if χ factors through $1+N\hat{\mathbb{Z}}$, then for $p \nmid N$, ω_{χ_p} is unramified at p

pf: $\omega_{\chi_p}: \mathbb{Q}_p^\times \to \mathbb{Q}^\times \backslash \mathbb{A}^\times \xrightarrow{\omega_\chi} \mathbb{C}^\times$, the decomposition is very easy to see.

now if χ factors throug $1+N\hat{\mathbb{Z}}$, now pick $p \nmid N$, then

$$\forall x \in \mathbb{Z}_p^\times, \quad \omega_{\chi_p}(x) = \omega_\chi(\underset{\underset{\infty}{\uparrow}}{1}, \ldots, \underset{\underset{p}{\uparrow}}{x}, 1 \ldots) = 1.$$

<u>Claim</u>: If ω_χ is unramified at p, then χ factors through $1+(p,N)\hat{\mathbb{Z}}$

pf: consider $x \in \mathbb{Z}_p^\times$, and $\iota_p(x) \in \prod_v \mathbb{Z}_v^\times$, $v=p$, it is x, $v \neq p$, it is 1

$$\chi(\iota_p(x)) = \omega_\chi((\underset{\underset{\infty}{\uparrow}}{1}, \ldots \underset{\underset{p}{\uparrow}}{x}, \ldots)) = \omega_{\chi_p}(x) = 1 \Rightarrow \chi \text{ trivial in } \mathbb{Z}_p^\times \hookrightarrow \prod \mathbb{Z}_v^\times$$

then obviously χ factors throught $1+(p,N)\hat{\mathbb{Z}}$

<u>Claim</u>: ω_χ factors throught $1+N\hat{\mathbb{Z}}$ & $1+M\hat{\mathbb{Z}} \Rightarrow \omega_\chi$ factors through $1+(M,N)\hat{\mathbb{Z}}$

pf: for every place p $\quad \omega_{\chi_p}|_{\mathbb{Z}_p^\times} = \chi|_{\mathbb{Z}_p^\times}: \mathbb{Z}_p^\times \to \mathbb{C}^\times$

then ω_χ factors through $1+p^{v_p(N)}\mathbb{Z}_p$ & $1+p^{v_p(M)}\mathbb{Z}_p \Rightarrow$ factors through $1+p^{\min\{v_p(N), v_p(M)\}}\mathbb{Z}_p$

i.e. factors through $1+(M,N)\hat{\mathbb{Z}}$

From this, we can get $c(\chi) = c(\omega_\chi) = \prod_p c(\chi_p)$

· General theory & L-factors

Now we consider arbitrary # fields F, then $\mathbb{A}_F$,

$$\omega : \mathbb{A}_F^\times \to \mathbb{C}^\times \quad \text{is a character}$$

we will mainly focus on quasi-character, i.e. ω trivial on $F^\times \hookrightarrow \mathbb{A}_F^\times$
character, i.e. ω trivial on $F^\times \hookrightarrow \mathbb{A}_F^\times$ & $|\omega| = 1$

By CFT. $C_F = F^\times \backslash \mathbb{A}_F^\times$, open finite index subgp of C_F will correspond to Abelian extension of F

and consider $K = \prod_{v<\infty} \mathcal{O}_v^\times$, then we no longer have $\mathbb{A}_F^\times = F^\times \cdot (0, +\infty) \cdot K$.

we have:
$$1 \to K \cdot F_\infty \to \mathbb{A}_F^\times \longrightarrow \text{fractional ideals of } \mathcal{O}_F \to 1$$

then
$$F^\times \cdot F_\infty \backslash \mathbb{A}_F^\times / K \simeq Cl(F)$$

<u>Claim</u>: ω has decomposition $\omega = \bigotimes_v \omega_v$, including infinite places

<u>Def</u>: For NA place v,

ω_v is unramified, $L_v(s, \omega_v) = (1 - \omega(\mathfrak{p}_v) N\mathfrak{p}_v^{-s})^{-1}$
ω_v is ramified $\quad L_v(s, \omega_v) = 1$

For archimedean place, we will work out later

and we define

$$L^S(s, \omega) = \prod_{v \notin S} L_v(s, \omega_v) \qquad S \text{ is a finite set containing } \infty \text{ \& ramified places}$$

<u>Claim</u>: $L^S(s, \omega)$ is convergent for $Re(s) > 1$.

pf: $Re(s) > 1 \Rightarrow \left| \sum_{v \notin S} \log |1 - \omega(\mathfrak{p}_v) q_v^{-s}| \right| \leq \sum_{v \notin S} \frac{1}{q_v^s}$

Our main goal is to show that $L^S(s, w)$ can be analytically continued to $\mathbb{C}$
and this continuation has a functional equation!

The main goals are to prove the analytic properties of these functions, i.e.,

1. to "complete" the partial L-function $L^S(s, \omega)$ by including additional local L-factors for the primes $v \in S$, for example, for the archimedean places, and

2. to prove the meromorphic analytic continuation and functional equation of the completed L-function.

· Distribution theory

In this section, we explain how the local & global L-factors appear naturally as a scalar constant between two distributions on the representation model

· Def: For F is a local field, consider

Let

$S(F)$ = the space of Schwartz-Bruhat functions on F. $\rightsquigarrow$ $\begin{cases} \text{for NA } F, \text{ just locally constant functions} \\ \text{for A } F, F=\mathbb{R}/\mathbb{C}, \text{ Schwartz space} \end{cases}$

$S(F)'$ = the space of tempered distributions on F, i.e.,

the space of continuous linear functionals $\lambda : S(F) \to \mathbb{C}$.

λ is called a distribution

Especially, we focus on those eigen distributions

- $F^\times$ acts on $S(F)$ by $(\rho(x)f)(b) = f(bx)$
- hence $F^\times$ acts on $S(F)'$: $(\rho'(x)\lambda)(f) = \lambda(\rho(x^{-1})f)$

For $\omega: F^\times \to \mathbb{C}^\times$, we are interested in

$$S'(\omega) = \{ \lambda \in S'(F) \mid \rho'(x)\lambda = \omega(x)\lambda \}$$

Interlude: Characters of local fields:

F NA: $\omega: F^\times \to \mathbb{C}^\times$

$F^\times \simeq \mathbb{Z} \times \mathcal{O}_F^\times$, then

$\omega(x) = |x|^s \, \tilde{\omega}(x \cdot \pi_F^{-\mathrm{ord}(x)})$ $\quad$ $\tilde{\omega}: \mathcal{O}_F^\times \to \mathbb{C}^\times$ must factor through $1+\pi^n\mathcal{O}_F$ for some n

$F = \mathbb{R}$

$$\begin{aligned} \omega: \mathbb{R}^\times &\longrightarrow \mathbb{C}^\times \\ x &\longmapsto (\mathrm{sgn}\, x)^\epsilon \cdot |x|^s, \text{ for some } s \in \mathbb{C} \end{aligned}$$

$F = \mathbb{C}$

$$\begin{aligned} \omega: \mathbb{C}^\times &\longrightarrow \mathbb{C}^\times \\ re^{i\theta} &\longmapsto r^s \cdot e^{in\theta} \\ z &\longmapsto |z|^s \cdot \left(\frac{z}{|z|}\right)^n \quad \text{for some } s \in \mathbb{C}, n \in \mathbb{Z} \end{aligned}$$

Main result:

Theorem 3.4. *For any quasicharacter ω of $F^\times$,*

$$\dim S'(\omega) = 1.$$

Next, we will consider respectively in each type. NA (unramified / ramified), A ($\mathbb{R}$/$\mathbb{C}$), find a "canonical element" in the eigendistribution space,

- Unramified local theory

Suppose F is NA, $\omega: F^\times \to \mathbb{C}^\times$ is unramified, we pick $\Lambda \in S'(\omega)$
we first consider $\Lambda|_{S(F^\times)}$, it's by definition that

$$\Lambda(\rho(x^{-1})f) = \omega(x)\Lambda(f)$$

<u>Claim</u>: Λ is of the following form (up to a constant multiple)

$$\Lambda(f) = \int_{F^\times} f(y)\,\omega(y)\,d^\times y$$

(verify: $\Lambda(\rho(x^{-1})f) = \int_{F^\times} f(x^{-1}y)\,\omega(y)\,d^\times y \overset{y \to xy}{=} \int_{F^\times} f(y)\,\omega(xy)\,d^\times y = \omega(x)\Lambda(f)$)

it must take this form because if we apply the following operator on $S(F^\times)$

$$f \mapsto \omega^{-1}f : y \mapsto \omega^{-1}(y)\cdot f(y)$$

then define $\tilde{\Lambda}(f) = \Lambda(\omega^{-1}f)$, we get

$$\tilde{\Lambda}(\rho(x^{-1})f) = \Lambda(\omega^{-1}\rho(x^{-1})f) = \omega^{-1}(x)\Lambda(\rho(x^{-1})\cdot(\omega^{-1}f))$$
$$= \omega^{-1}(x)\cdot\omega(x)\,\Lambda(\omega^{-1}\cdot f) = \tilde{\Lambda}(f)$$

$\Rightarrow \tilde{\Lambda}$ is the unique Haar measure on $F^\times$, therefore Λ is "unique"

Now the problem is: how to extend it to $S(F)$?

naive way: Suppose $\omega(x) = |x|^s$, $s \in \mathbb{C}$, then

$$\int_F f(x)\,\omega(x)\,d^\times x \overset{\text{formally}}{=\!=} \sum_{n \in \mathbb{Z}} \int_{v(x)=n} f(x)\,\omega(x)\,d^\times x$$

actually we only care $v(x)$ very large! i.e. $|x|$ very small,
then f must be a constant, i.e. we care about the convergence of

$$\sum_{n>>0} q^{-ns}\int_{v(x)=n} d^\times x = \sum_{n>>0} q^{-ns}$$

this converges iff $\mathrm{Re}(s) > 0$

i.e. Now the answer seems to be that we can only extend it when $Re(s) > 0$
at least we know, when $Re(s) > 0$, $S'(\omega)$ is not empty, so it should also be true for $Re(s) \leq 0$

the summation is $\sum_{n \gg 0} q^{-ns} = \frac{q^{-n_0 s}}{1-q^{-s}}$

so it seems that, $(1-q^{-s})\Lambda$ is a naturally element to consider

<u>Claim</u>. Consider the following operator in $S(F)$

$$\begin{aligned} S(F) &\xrightarrow{\ \tau\ } S(F^\times) \\ f &\longmapsto (\tau f)(x) = f(x) - f(\varpi^{-1}x) \end{aligned}$$

then we consider

$$\Lambda_0(f) = \Lambda(\tau f)$$, easy to see that τ commutes with $\rho(x^{-1})$, hence

$$\Lambda_0(\rho(x)^{-1} f) = \omega(x)\Lambda_0(f)$$

Moreover Λ_0 is defined for $\forall f \in S(F)$, and it is non-zero, especially for $f \in S(F^\times)$

$$\Lambda_0(f) = \int_{F^\times} (f(x) - f(\varpi^{-1}x))\,\omega(x)\, d^\times x = (1-q^{-s})\Lambda(f) = (1-\omega(p))\Lambda(f)$$

same argument applies to $\omega \cdot \omega_s$: $\omega_s(x) = |x|^s$, then we get

$$\Lambda_0(s, f) = (1-\omega(p)q^{-s})\Lambda(s,f) \quad \text{where } \Lambda(s,f) = \int_{F^\times} f(x)\omega(x)|x|^s d^\times x$$

is well-defined for $\forall f \in S(F)$, and

$$\Lambda_0(s, \rho(x)^{-1} f) = \omega(x)|x|^s \Lambda_0(s,f)$$

$$\Lambda(s, f) = L(s, \omega)\Lambda_0(s, f)$$

· Ramified local theory

if ω is ramified, the theory is much more simpler! actually

$$\Lambda(s, f) = \int_{F^\times} f(x)\omega(x)|x|^s d^\times x$$ is already well-defined for $\forall s \in \mathbb{C}$, $f \in S(F)$

because when $|x|$ is small, f is constant, then $\int_{v(x)=n} \omega(x)|x|^s d^\times x = 0$ since $\omega: F^\times \to \mathbb{C}^\times$ is non-trivial on $\mathcal{O}_F^\times$

· $\mathbb{R}$ & $\mathbb{C}$

$\mathbb{R}$: the basic idea is the same, we consider the following

$$\omega: \mathbb{R}^\times \longrightarrow \mathbb{C}^\times$$
$$x \longmapsto (\operatorname{sgn} x)^\epsilon \cdot |x|^s, \quad \text{for some } s \in \mathbb{C}$$

and $f \in S(\mathbb{R})$, the zeta integral is the following:

$$\int_{\mathbb{R}^\times} f(x) \operatorname{sgn}(x)^\epsilon |x|^s \frac{dx}{x} = \int_{\mathbb{R}_{>0}} \left(f(x) + (-1)^\epsilon f(-x)\right) |x|^s \frac{dx}{x} \quad (*)$$

Prop: · When $f \in S(\mathbb{R})$, $(*)$ has a meromorphic continuation to $s \in \mathbb{C}$

· $\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \cdot (*)$ is entire on $\mathbb{C}$

pf: $\Lambda(s, f) = \int_0^\infty f(x) x^s \frac{dx}{x}$

we only need to focus near 0, $f(x) = \sum_{n \geq 0} a_n x^n$, $0 < x < 2\delta$, $x \geq 2\delta$, $f(x) = 0$

$$\int_0^\infty f(x) x^s \frac{dx}{x} = \int_0^\delta \sum_{n=0}^\infty a_n x^{n+s} \frac{dx}{x} + \underbrace{\int_\delta^{2\delta} \cdots}_{\text{always holomorphic}} = \sum_{n=0}^\infty a_n \int_0^\delta x^{n+s-1} dx = \sum_{n=0}^\infty \frac{a_n}{n+s} \cdot \delta^{n+s}$$

since $\lim_{n \to \infty} a_n \left(\frac{3}{2}\delta\right)^n = 0 \Rightarrow \left|\frac{a_n}{n+s}\right| \cdot |\delta^{n+s}| \leq |a_n| \cdot \left|\left(\frac{3}{2}\delta\right)^{n+s}\right| \cdot \left(\frac{2}{3}\right)^{n+s} \ll \left(\frac{2}{3}\right)^{n+\mathrm{Re}(s)}$

i.e. when $s \notin -\mathbb{Z}_{\geq 0}$, this gives a holomorphic function (absolute convergence)

Rmk: Actually, when $\epsilon = 0 \Rightarrow a_n = 0$ when n is odd. $\rightsquigarrow \Gamma(\frac{s}{2})$ (poles at $-2n$, $n \geq 0$)

when $\epsilon = 1 \Rightarrow a_n = 0$ when n is even $\rightsquigarrow \Gamma(\frac{s+1}{2})$ (poles at $-2n-1$, $n \geq 0$)

$\mathbb{C}$: $\omega: \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$

$$re^{i\theta} \longmapsto r^s \cdot e^{in\theta}$$
$$z \longmapsto |z|^s \cdot \left(\frac{z}{|z|}\right)^n \quad \text{for some } s \in \mathbb{C}, n \in \mathbb{Z}$$

$$\int_{\mathbb{C}^\times} f(x) |x|^s \cdot \left(\frac{z}{|z|}\right)^n \frac{dx}{x} = \int_0^{2\pi} \int_0^\infty f(r, \theta) r^s \cdot e^{i(n-1)\theta} \, dr \, d\theta \quad \rightsquigarrow \Gamma(s)$$

$$f(r, \theta) = \sum_{n \geq 0} a_n(\theta) r^n$$

Calculation of some examples

Proposition 3.5.

(i) *The distribution*

$$z_o(s,\omega) := L(s,\omega)^{-1}\, z(s,\omega)$$

has an entire analytic continuation to the whole s plane and, for all s, defines a basis vector for the space $S'(\omega\omega_s)$.

(ii) *If*

$$f^o(x) = \begin{cases} f_a(x) := x^a e^{-\pi x^2} & \text{if } F = \mathbb{R}, \text{ and} \\ f_{a,b}(x) := x^a\, \bar{x}^b\, e^{-2\pi x \bar{x}} & \text{if } F = \mathbb{C}, \end{cases}$$

then

$$\langle z_o(s,\omega), f^o \rangle = 1.$$

Note that we are using the measure $d^\times x = |x|^{-1}\, dx$, where $|x|$ is the usual absolute value (resp., $|x| = x\bar{x}$) and dx is Lebesque measure (resp., twice Lebesque measure) when $F = \mathbb{R}$ (resp., $F = \mathbb{C}$). For (ii), see [13, Lemma 8, p. 127].

· Fourier transformation & Local Functional equation

We consider the Fourier transform on $S(F)$, fix additive character $\psi: F \to \mathbb{C}^*$, and

$$\hat{f}(x) = \int_F f(y)\psi(xy)\,dy$$

then this induces a "transform" on distributions

$$\hat{\Lambda}(f) := \Lambda(\hat{f})$$

the starting point is the following:

<u>Claim</u>: Suppose $\Lambda \in S'(\omega)$, then $\hat{\Lambda} \in S'(\omega^{-1}\cdot\omega_1)$, $\omega_1(x) = |x|$

pf: $\hat{\Lambda}(\rho(x^{-1})f) = \Lambda(\widehat{\rho(x)^{-1}f})$

$$\left(\widehat{\rho(x^{-1})f}\right)(y) = \int_F \left(\rho(x^{-1})f\right)(z)\psi(yz)\,dz = \int_F f(x^{-1}z)\psi(yz)\,dz$$

$$\overset{z\to xz}{=} |x|\int_F f(z)\psi(xyz)\,dz = |x|\cdot\hat{f}(xy)$$

$$\Rightarrow \left(\rho(x^{-1})f\right)^{\wedge} = |x|\cdot\rho(x)\hat{f}$$

hence $\hat{\Lambda}(\rho(x^{-1})f) = \Lambda(|x|\cdot\rho(x)\hat{f}) = |x|\,\omega(x^{-1})\,\hat{\Lambda}(f)$

this claim tells us that, $\exists$ constant $\epsilon(s,\omega,\psi)$, s.t.

$$\begin{array}{ccc} \hat{\Lambda}_0(1-s,\omega^{-1}) = & \epsilon(s,\omega,\psi)\cdot & \Lambda_0(s,\omega) \\ \Big\downarrow & & \Big\downarrow \\ \dfrac{Z(1-s,\omega^{-1},\hat{f})}{L(1-s,\omega^{-1})} = & \epsilon(s,\omega,\psi)\cdot & \dfrac{Z(s,\omega,f)}{L(s,\omega)} \end{array}$$

when both zeta integrals converges, i.e.

$0 < \mathrm{Re}(s) < 1$

actually, if we view $Z(s,\omega,f)$ as already the analytic continuation of the original "converge" one, this equality holds for all $s \in \mathbb{C}$

Global Theory

Now we consider an automorphic repn ω of $GL_1(\mathbb{A}_F) = \mathbb{A}_F^\times$

$$\omega : \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times$$

ω has the decomposition $\omega = \bigotimes_v \omega_v$, $\omega_v : F_v^\times \to \mathbb{C}^\times$

hence ω can be viewed as a function on the Schwartz–Bruhat functions space

$$S(\mathbb{A}_F) = \bigotimes_v{}' S(F_v)$$

restricted tensor product, w.r.t. 1_{O_v} for v NA

We are interested in $S'(\mathbb{A}_F)$: continuous linear functionals $S(\mathbb{A}_F) \to \mathbb{C}$

Thm: $\dim_{\mathbb{C}} S'(\omega) = 1$, & $\lambda \in S'(\omega)$ is (up to a multiple) of the following form

$$\lambda = \bigotimes_v \lambda_v, \text{ where } \lambda_v \in S'(\omega_v), \ \& \ \lambda_v(1_{O_v}) = 1 \text{ for NA}$$

pf: such $\bigotimes_v \lambda_v$ obviously satisfies $\lambda \in S'(\omega)$. now we pick f factorizable, and $\lambda(f) = 1$

then $\forall v$, $f = f_v \otimes f^v$, $f_v \in S(F_v)$. then we obtain

$$\begin{array}{ccccc} S(F_v) & \longrightarrow & S(\mathbb{A}_F) & \xrightarrow{\lambda} & \mathbb{C} \\ g_v & \longmapsto & g_v \otimes f^v & \to & \lambda(g_v \otimes f^v) \end{array} \Rightarrow \underset{\forall \text{ place}}{\lambda_v \in S'(\omega_v)} \ \& \ \underset{\text{NA places}}{\lambda_v(1_{O_v}) = 1 \text{ for}}$$

Now our goal is to prove $\lambda = \bigotimes_v \lambda_v$

Choose $g \in S(\mathbb{A}_F)$ factorizable, $g = \bigotimes_v g_v$, suppose for $v \notin S$, $g_v = 1_{O_v}$ (S finite, contains all the A places)

$$= \bigotimes_{v \in S} g_v \otimes 1_S$$

$$\begin{array}{ccccc} \bigotimes_{v \in S} S(F_v) & \longrightarrow & S(\mathbb{A}_F) & \xrightarrow{\lambda} & \mathbb{C} \\ h & \longmapsto & h \otimes 1_S & \longmapsto & \lambda(h \otimes 1_S) \end{array}$$

then $\lambda(h \otimes 1_S) = \left(\bigotimes_{v \in S} \lambda_v\right)(h)$ ~ finitely many places, the result can be proved easily

$\Rightarrow \quad \lambda(g) = \left(\bigotimes_v \lambda_v\right)(g)$

· global "naive" zeta integral

$$z(s, \omega, f) = \int_{\mathbb{A}_F^\times} f(x)\,\omega(x)\,|x|^s\, d^\times x$$

$f \in S(\mathbb{A}_F)$

$$f = \otimes_v f_v \overset{\text{formally}}{=\!=} \prod_v \int_{F_v^\times} f_v(x)\,\omega_v(x)\,|x|_v^s\, d^\times x = \prod_v z(s, \omega_v, f_v)$$

when $Re(s) > 0$, $z(s, \omega_v, f_v)$ converges. and at almost all places,

$$z(s, \omega_v, f_v) = \int_{\mathcal{O}_v - \{0\}} \omega_v(x)\,|x|^s\, d^\times x = \sum_{n=0}^{\infty} \int_{v(x)=n} \omega_v(p_v)^n q^{-ns}\, d^\times x = \frac{1}{1 - \omega_v(p_v) q^{-s}} = L(s, \omega_v)$$

then the product converges absolutely for $Re(s) > 1$, i.e.

$$z(s, \omega, f) = \prod_v z(s, \omega_v, f_v), \quad \text{when } Re(s) > 1$$

both sides understand to be "zeta integral"

but obviously $z_0(s, \omega, f) = \prod_v z_0(s, \omega_v, f_v)$ is valid for $\forall s \in \mathbb{C}$

$$\underset{\forall s}{z_0(s, \omega, f)} = \underset{\forall s}{\prod_v z_0(s, \omega_v, f_v)} = \prod_v \frac{z(s, \omega_v, f_v)}{L(s, \omega_v)} \underset{Re(s)>1}{=\!=} \frac{\prod_v z(s, \omega_v, f_v)}{L(s, \omega)} \underset{Re(s)>1}{=\!=} \frac{z(s, \omega, f)}{L(s, \omega)}$$

analytic continuation of L $\Leftrightarrow$ analytic continuation of global zeta integral

Now we take another view point of the zeta integral $z(s, \omega, f) = \int_{\mathbb{A}_F^\times} f(x)\,\omega(x)\,|x|^s\, d^\times x$

We have noted that, the convergence of zeta integral is a problem only happens near "0"

Although local zeta integral may not converge, but the global one is almost convergent everywhere!!

We will say that a meromorphic function f is *essentially bounded in vertical strips* if whenever $\sigma_1 < \sigma_2$ are real numbers and U is an open set containing all poles of f in the region $\{\sigma_1 < \mathrm{re}(s) < \sigma_2\}$, then f is bounded in the region $\{\sigma_1 < \mathrm{re}(s) < \sigma_2\} - U$.

Proposition 3.1.6 *The function $\zeta(s, \chi, \Phi)$ has meromorphic continuation to all s, and is entire unless the restriction of χ to the subgroup $\mathbb{A}_1^\times$ of ideles of norm one is trivial. If this is the case, then there exists a purely imaginary complex number λ such that $\chi(x) = |x|^\lambda$. In this case, $\zeta(s, \chi, \Phi)$ can have poles at $s = 1 - \lambda$ and $s = -\lambda$ if F is a number field, and at $s = 1 - \lambda + 2\pi ni/\log(q)$ and $s = -\lambda + 2\pi ni/\log(q)$ $(n \in \mathbb{Z})$ if F is a function field and q is the cardinality of the finite ground field. These are the only possible poles. We have the functional equation*

$$\zeta(s, \chi, \Phi) = \zeta(1 - s, \chi^{-1}, \hat{\Phi}). \qquad (1.17)$$

The function $\zeta(s, \chi, \Phi)$ is essentially bounded in vertical strips.

pf: $z(s, w, f) = \int_{\mathbb{A}_F^\times} f(x)\,\omega(x)\,|x|^s\, d^\times x$

$$= \underbrace{\int_{|x|<1} f(x)\,\omega(x)\,|x|^s\, d^\times x}_{z^0(s,w,f)} + \underbrace{\int_{|x|>1} f(x)\,\omega(x)\,|x|^s\, d^\times x}_{z^1(s,w,f)}$$

Claim: $z^1(s, w, f)$ is well-defined for $\forall s \in \mathbb{C}$

pf: we know when $Re(s)>1$, the integral converges.

when $Re(s)$ decrease, $|x|^s$ also decrease, the integral still converges

Now we focus on z^0, then

$$\int_{|x|<1} f(x)\,\omega(x)\,|x|^s d^\times x = \sum_{\alpha\in F^\times} \int_{\substack{x\in A_F^\times/F^\times \\ |x|<1}} f(\alpha x)\,\omega(\alpha x)\,|\alpha x|^s\, d^\times x$$

$$= \int_{\substack{x\in A_F^\times/F^\times \\ |x|<1}} \Big(\sum_{\alpha\in F^\times} f(\alpha x)\Big)\,\omega(x)\cdot|x|^s\, d^\times x$$

Here, we need the key input of Fourier analysis: Possion Summation formula

$$\sum_{\alpha\in F} f(\alpha x) = |x|^{-1} \sum_{\alpha\in F} \hat{f}(\alpha x^{-1}) \qquad x\in \mathbb{A}_F^\times$$

then $z_0(s, w, f) = \int_{\substack{x\in A_F^\times/F^\times \\ |x|<1}} \Big(\sum_{\alpha\in F} f(\alpha x)\Big)\,\omega(x)\cdot|x|^s\, d^\times x - f(0)\cdot\int_{\substack{x\in A_F^\times/F^\times \\ |x|<1}} \omega(x)\cdot|x|^s\, d^\times x$

$$= \int_{\substack{x\in A_F^\times/F^\times \\ |x|<1}} \Big(\sum_{\alpha\in F} \hat{f}(\tfrac{\alpha}{x})\Big)\,\omega(x)\cdot|x|^{s-1}\, d^\times x - f(0)\int_0^1 \int_{\substack{x\in A_F^\times/F^\times \\ |x|=t}} \omega(x)\cdot|x|^s\, d^\times x\,\frac{dt}{t}$$

$$\overset{x\to\frac{1}{x}}{=} \int_{\substack{x\in A_F^\times/F^\times \\ |x|>1}} \Big(\sum_{\alpha\in F^\times} \hat{f}(\alpha x)\Big)\,\omega^{-1}(x)\,|x|^{1-s}\, d^\times x + \hat{f}(0)\int_0^1 \int_{\substack{x\in A_F^\times/F^\times \\ |x|=t}} \omega(x)\,|x|^{s-1}\, d^\times x\,\frac{dt}{t}$$

$$- f(0)\int_0^1 \int_{\substack{x\in A_F^\times/F^\times \\ |x|=t}} \omega(x)\cdot|x|^s\, d^\times x\,\frac{dt}{t}$$

$$\hat{f}(0)\int_0^1 \int_{\substack{x\in A_F^\times/F^\times\\ |x|=t}} \omega(x)\,|x|^{s-1}\, d^\times x\, \frac{dt}{t} \;-\; f(0)\int_0^1 \int_{\substack{x\in A_F^\times/F^\times\\ |x|=t}} \omega(x)\cdot |x|^{s}\, d^\times x\, \frac{dt}{t}$$

$$= \hat{f}(0)\int_0^1 t^{s-1}\int_{\substack{x\in A_F^\times/F^\times\\ |x|=t}} \omega(x)\, d^\times x\, \frac{dt}{t} - f(0)\int_0^1 t^{s}\int_{\substack{x\in A_F^\times/F^\times\\ |x|=t}} \omega(x)\, d^\times x\, \frac{dt}{t}$$

if $\omega|(A_F^\times)^1$ is non-trivial, then both $=0$

if $\omega|(A_F^\times)^1$ is trivial $\Rightarrow$ $\omega(x)=|x|^\lambda$ for some $\lambda\in\mathbb{C}$

$$\int_0^1 t^{s-1}\int_{\substack{x\in A_F^\times/F^\times\\ |x|=t}} \omega(x)\, d^\times x\, \frac{dt}{t} = \int_0^1 t^{s+\lambda-1}\,\frac{dt}{t}\cdot Vol\left((A_F^\times)^1/F^\times\right) = \frac{t^{s+\lambda-1}}{s+\lambda-1}\Big|_0^1\cdot V = \frac{V}{s+\lambda-1}$$

$$\int_0^1 t^{s}\int_{\substack{x\in A_F^\times/F^\times\\ |x|=t}} \omega(x)\, d^\times x\, \frac{dt}{t} = \int_0^1 t^{\lambda+s}\,\frac{dt}{t}\cdot Vol\left((A_F^\times)^1/F^\times\right) = \frac{t^{\lambda+s}}{\lambda+s}\Big|_0^1\cdot V = \frac{V}{s+\lambda}$$

i.e. $$Z_0(s,w,f) = Z_1(1-s,w^{-1},\hat{f}) + \frac{\hat{f}(0)V}{\lambda+s-1} - \frac{f(0)V}{\lambda+s}$$

$$\Rightarrow\quad Z(s,w,f) = Z_1(s,w,f) + Z_1(1-s,w^{-1},\hat{f}) + \begin{cases} 0 & \text{if } w|_{A_F^{\times 1}} \text{ is non-trivial} \\ -V\left(\dfrac{\hat{f}(0)}{1-s-\lambda} + \dfrac{f(0)}{s+\lambda}\right), & \omega(x)=|x|^\lambda \end{cases}$$

i.e. although $z(s,w,f)$ originally converges for $\mathrm{Re}(s)>1$

it can be analytically continued to $\forall s\in\mathbb{C}$

moreover:

$$z(1-s,\omega^{-1},\hat{f}) = z(s,\omega,f),\quad \forall s\in\mathbb{C}$$

Now $L(s,w) = \dfrac{z(s,w,f)}{z_0(s,w,f)}$ for $\mathrm{Re}(s)>1$ $\Rightarrow$ $L(s,w)$ has analytic continuation

moreover $\quad Z_0(1-s,w_v^{-1},\hat{f}_v) = \epsilon(s,w_v,\psi_v)\,Z_0(s,w_v,f_v)$

$$\Rightarrow\quad Z_0(1-s,w^{-1},\hat{f}) = \prod_v Z_0(1-s,w_v^{-1},\hat{f}_v) = \prod_v \epsilon(s,w_v,\psi_v)\cdot Z_0(s,w,f) = \epsilon(s,\omega,\psi)\,Z_0(s,w,f)$$

$$L(1-s,w^{-1}) = \frac{z(1-s,w^{-1},\hat{f})}{z_0(1-s,w^{-1},\hat{f})} = \frac{z(s,w,f)}{\epsilon(s,w,\psi)\,z_0(s,w,f)} = \epsilon(s,w,\psi)^{-1}L(s,w)$$

· About zeta integral

In the case of GL_1, for each place v, we consider the following integral

$$Z_v(s, w_v, f_v) = \int_{F_v} f_v(x)\, w_v(x)\, |x|_v^s\, d^\times x \quad , \text{where } f_v \in S(F_v)$$

We proved that it is a distribution satisfying certain conditions. now we want to generalize it to the case GL_2. To do this, we should find good replacement of f_v, w_v, $|\cdot|^s$

It seems that we should consider $S(M_2(F_v))$, $\beta_{v,w}(g)$, $|\det g|^s$

$$\begin{array}{ccc} S(M_2(F_v)) & \beta_{v,w}(g) & |\det g|^s \\ \uparrow & \uparrow & \uparrow \\ S(F_v) & \text{matrix coefficient} & |\cdot|^s \end{array}$$

$\beta_{v,w}(g) = (\pi(g)v, w)$
$w \in \check{\pi}$

We consider

$$Z_v(s, \beta_v, f_v) = \int_{M_2(F_v)} f_v(g)\, \beta(g) \cdot |\det g|^s\, dg$$

Actually, it may seems to be more naturally to consider

$$S(M_2(F_v)) \longrightarrow \text{operator on } \pi$$

$$f \longmapsto Z_v(s, f) = \int_{M_2(F_v)} f_v(g)\, \pi(g) \cdot |\det(g)|^s\, dg$$

$$v \to \int_{M_2(F_v)} f_v(g)\, \pi(g) v \cdot |\det g|^s\, dg$$

$\Rightarrow$ $Z_v(s, \beta_v, f_v)$ is just the pairing between
$\langle Z_v(s, f) v, w \rangle = Z_v(s, \beta_{v,w}, f_v)$

$$\pi(h) \cdot Z_v(s, f) = \int_{M_2(F_v)} f_v(g)\, \pi(hg) \cdot |\det g|^s\, dg = \int_{GL_2(F_v)} f_v(g)\, \pi(hg)\, |\det g|^s\, dg$$

$$= \int_{GL_2(F_v)} f_v(h^{-1}g)\, \pi(g)\, |\det h|^{-s} |\det g|^s\, dg$$

$$= |\det h|^{-s} \cdot Z_v(s, \rho_\ell(h) f_v)$$

$$Z_v(s, f) \cdot \pi(h) = \int_{GL_2(F_v)} f_v(g)\, \pi(\overset{g' = gh}{gh})\, |\det g|^s\, dg = |\det h|^{-s} \cdot Z_v(s, \rho(h^{-1}) f_v)$$

$$Z(s, f_v) = \int_{GL_2(F_v)} f_v(g)\,\pi(g)\,|\det g|^s\,dg \qquad W\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix} g\right) = \psi(x)\,W(g)$$

$$= \int_{BwN} f_v(g)\,\pi(g)\,|\det g|^s\,dg$$

$$\Big(Z(s, f_v)(W)\Big)(h) = \int_{BwN} \Big(f_v(g)\big(\pi(g)W\big)(h)\Big)\,|\det g|^s\,dg$$

$$\Big(Z(s, f_v)(W)\Big)(y) = \int_{BwN} \Big(f_v(g)\big(\pi(g)W\big)(y)\Big)\,|\det g|^s\,dg$$

$$= \int_B \int_N f_v(bwn)\big(\pi(bwn)W\big)(y)\,|\det b|^s\,db\,dn$$

$$W\begin{pmatrix} y & \\ x & 1\end{pmatrix}$$

$$\big(\pi(bwn)W\big)(y) = \big(\pi(wn)W\big)\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\begin{pmatrix} b_1 & x_1\\ & b_2\end{pmatrix}\right)$$

$$= \big(\pi(wn)W\big)\left(\begin{pmatrix} b_1y & x_1y\\ & b_2\end{pmatrix}\right)$$

$$= \psi(x_1y)\,\omega(b_2)\,\big(\pi(wn)W\big)\left(\begin{pmatrix} b_2^{-1}b_1y & \\ & 1\end{pmatrix}\right)$$

$$\big(\pi(n)W\big)(y) = W\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & x\\ & 1\end{pmatrix}\right)$$

$$= W\left(\begin{pmatrix} y & xy\\ & 1\end{pmatrix}\right) = W\left(\begin{pmatrix} 1 & xy\\ & 1\end{pmatrix}\begin{pmatrix} a & \\ & 1\end{pmatrix}\right)$$

$$= \psi(xy)\,W(y)$$

$\Rightarrow$ difficult to determine.

so we consider

① π is spherical, then

$$\Big(Z(s, f_v)(W)\Big)(y) = \int_G f_v(g)\big(\pi(g)W\big)(y)\,|\det g|^s\,dg$$

$$= \int_B \int_K f_v(bk)\big(\pi(b)W\big)(y)\,|\det b|^s\,db\,dk$$

π is spherical, then

$$\big(Z(s, f_v)(W)\big)(y) = \int_G f_v(g)\big(\pi(g)W\big)(y)\ |\det g|^s\, dg$$

$$= \int_B \int_K f_v(bk)\big(\pi(b)W\big)(y)\ |\det b|^s\, db\, dk$$

$$= \int_B \tilde{f}(b)\big(\pi(b)W\big)(y)\ |\det b|^s\, db$$

$$= \int_{F_v^\times} \int_{F_v^\times} \int_{F_v} \tilde{f}(y_1, y_2, x)\, W\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix}\right) |y_1 y_2|^s\, d^\times y_1\, d^\times y_2\, dx$$

$$W\left(\begin{pmatrix} y_1 y & xy \\ & y_2 \end{pmatrix}\right) = \omega(y_2)\, W\begin{pmatrix} y_1 y_2^{-1} y & x y_2^{-1} y \\ & 1 \end{pmatrix}$$

$$= \omega(y_2)\, \psi(x y_2^{-1} y)\, W(y_1 y_2^{-1} y)$$

$$= \int_{F_v^\times} \int_{F_v^\times} \int_{F_v} \tilde{f}(y_1, y_2, x)\, \omega(y_2)\, \psi(x y_2^{-1} y)\, W(y_1 y_2^{-1} y)\ |y_1 y_2|^s\, d^\times y_1\, d^\times y_2\, dx$$

$$= \int_{F_v^\times} \int_{F_v^\times} \tilde{\tilde{f}}(y_1, y_2, y)\, \omega(y_2)\ W(y_1 y_2^{-1} y)\, |y_1 y_2|^s\, d^\times y_1\, d^\times y_2$$

· $\pi(h)\circ z(s,f) = |\det h|^{-s} z(s, \rho_\ell(h) f)$

$z(s,f)\circ \pi(h) = |\det h|^{-s}\cdot z(s, \rho(h^{-1}) f)$

so actually we obtained:

$$S(M_2(F)) \longrightarrow \mathrm{End}(\pi) \qquad z(s,f): V \to V$$

$$f \longmapsto z(s,f) \qquad z(s,f)(v) = \int_G f(g)\pi(g)v\, |\det g|^s dg$$

both sides admit $G\times G$-action,

on LHS: $((g,h)f)(m) = f(g^{-1} m h)$

on RHS: $((g,h)\varphi)(v) = |\det gh^{-1}|^s \pi(g)\circ\varphi\circ\pi(h^{-1})$

$$\big(z(s,(g,h)f)\big)(v) = \big(z(s, \rho_\ell(g)\circ\rho(h)f)\big)(v)$$
$$= |\det g|^s \pi(g)\circ z(s,\rho(h)f)\,(v)$$
$$= |\det gh^{-1}|^s\, \pi(g)\circ z(s,f)\circ\pi(h^{-1})(v)$$

thus actually, we get $\quad \mathrm{Hom}_{G\times G}\big(S(M_2(F)),\ \mathrm{End}(\pi)\big)$

$\mathrm{End}(\pi) = \pi^*\otimes\pi$, we want to show, actually $z(s,f)\in\check{\pi}\otimes\pi$

i.e. $z(s,f)$ should be a smooth vector, hence we check the $G\times G$-action:

$(g,h)z(s,f) = z(s,(g,h)f) \Rightarrow$ since $G\times G$ acts on $S(M_2(F))$ smoothly

then $z(s,f)$ must also be smooth

$$\Rightarrow \mathrm{Hom}_{G\times G}\big(S(M_2(F)),\ \check{\pi}\otimes\pi\big) \simeq \mathrm{Hom}_{G\times G}\big(S(M_2(F))\otimes\pi\otimes\check{\pi},\ \mathbb{C}\big) \qquad (*)$$

$f\otimes v\otimes\varphi \longmapsto z(s,\beta_{v,\varphi}, f)$

$$(g,h)\big(f\otimes v\otimes\varphi\big) = (g,h)f \otimes \pi(h)v\otimes\check{\pi}(g)\varphi$$
$$\mapsto z(s,\ \beta_{\pi(h)v,\check{\pi}(g)\varphi},\ (g,h)f)$$
$$= (\check{\pi}(g)\varphi)\big(z(s,(g,h)f)(\pi(h)v)\big)$$
$$= \varphi\big(\pi(g)^{-1}\circ\pi(g)\circ z(s,f)(v)\big)\cdot \det|gh^{-1}|^s \sim \big(\det|gh^{-1}|^s\big) f\otimes v\otimes\varphi$$

$$\begin{pmatrix} y^{-1} & \\ & 1\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} y & \\ & 1\end{pmatrix} = \begin{pmatrix} ay^{-1} & by^{-1}\\ c & d\end{pmatrix}\begin{pmatrix} y & \\ & 1\end{pmatrix} = \begin{pmatrix} a & by^{-1}\\ cy & d\end{pmatrix}$$

i.e. $\mathbb{C}$ carries $G\times G$-repn: $|\det gh^{-1}|^s$

$W_v \in$ Whittaker model

$$Z(s,\chi,W) = \int_{F_v^\times} W_v\begin{pmatrix} y & \\ & 1 \end{pmatrix} \chi(y)\,|y|^s\, d^\times y \quad \rightsquigarrow \text{Bump}$$

$\dfrac{Z(s,\chi,W)}{L(s,\chi\cdot\pi)}$ will be entire in s

$$Z(s,\beta,f) = \int_{GL_2(F_v)} f(g)\,\beta(g)\,|\det g|^s\, dg \qquad \beta \in \pi \otimes \check{\pi}$$

$\dfrac{Z(s,\beta,f)}{L(s,\pi)}$ is entire in s

$=$

$$\beta(g) = \langle \pi(g)u, v\rangle, \quad u \in \pi,\ v \in \check{\pi}$$

$$u = W_1, \quad v = W_2$$

$$= \langle \pi(g)W_1, W_2\rangle = \int_{GL_2(\mathcal{O}_v)} (\pi(g)W_1)(k)\, W_2(k)\, dk$$

$$\pi(f)u = \int_{GL_2(F_v)} f(g)\,\pi(g)u\,|\det g|^s\, dg \quad \text{operator on } \pi$$

$$Z(s,\beta,f) = \langle \pi_s(f)u, v\rangle \qquad \beta = \langle \pi(g)u, v\rangle$$

<u>Claim 1</u>: $\forall\ \varphi \in (*)$ is determined by its restriction on $C_c^\infty(G) \otimes \pi \otimes \check{\pi}$

pf: $\varphi(f, v, w)$

Theory of L-functions

We will first define the local L-factors at <u>spherical place</u>, since by tensor product theorem, these kinds of representations appear in almost all places, hence functional equation will essentially rely on this type

Characterization of Spherical representation:

π is an irr admissible repn of $GL_2(F_v)$, then

π is spherical $\Leftrightarrow$ $\pi \simeq \pi(\chi_1, \chi_2)$, χ_i are both unramified

π shouldn't be Steinberg

Therefore we define $\alpha_1 = \chi_1(\varpi_v)$, $\alpha_2 = \chi_2(\varpi_v)$

We only focus on unitary repns (because automorphic repn is unitary), then we have more information:

complex pairing: $B(\chi_1,\chi_2) \times B(\bar{\chi}_1, \bar{\chi}_2) \to \mathbb{C}$ $\mathbb{C}$-bilinear

$\Rightarrow B(\chi_1, \chi_2) \simeq B(\bar{\chi}_1^{-1}, \bar{\chi}_2^{-1}) \Rightarrow$ $\chi_1 = \bar{\chi}_1^{-1}$ & $\chi_2 = \bar{\chi}_2^{-1}$ or $\chi_1 = \bar{\chi}_2^{-1}$ $\alpha_1 = \bar{\alpha}_2^{-1}$

$$\pi(\chi_1, \chi_2) \text{ unitary} \Leftrightarrow \begin{cases} |\alpha_1| = |\alpha_2| = 1 \\ \alpha_1 = \alpha q^{\sigma},\ \alpha_2 = \alpha q^{-\sigma}, \text{ where } |\alpha| = 1,\ -\frac{1}{2} < \sigma < \frac{1}{2} \end{cases}$$

是不是大行分对是 in 书上 $\sigma = 0$??

then we define the local L-factor:

$$L_v(s, \pi) = (1 - \alpha_1 q^{-s})^{-1}(1 - \alpha_2 q^{-s})^{-1}, \quad \text{here } q = |\mathcal{O}/\varpi_v|$$

we also define twist of L_v by an unramified character of $F^\times$ by:

$$L_v(s, \pi, \xi) = (1 - \alpha_1 \xi(\varpi_v) q^{-s})^{-1}(1 - \alpha_2 \xi(\varpi_v) q^{-s})^{-1} = L_v(s, \xi \otimes \pi)$$

Now suppose we have an irr admissible repn π of $GL(2, \mathbb{A})$ (actually an irr module of $(\mathfrak{g}, K) \times GL(2, \mathbb{A}_f)$) then by tensor product theorem,

$$\pi \simeq \bigotimes_v \pi_v$$

almost all π_v are spherical

then we could define:

$$L_S(s, \pi) = \prod_{v \notin S} L_v(s, \pi_v), \quad Re(s) \gg 0$$

here $\forall v \notin S$, π_v is spherical, S is a finite set.

Main goal & Idea

Our main goal in this section is to show that L_S has an analytic continuation to all $s \in \mathbb{C}$ and satisfies a functional equation.

The idea is that we relate L_S/L_v with the "zeta integral" of some privileged Whittaker function in the Whittaker model of π/π_v, the get the functional equation by analyzing the zeta integral

Prop: $L_S(s,\pi)$ is absolutely convergent for $re(s) > \frac{3}{2}$

pf: $\log|1-\alpha_1 q^{-s}||1-\alpha_2 q^{-s}| \le |\alpha_1 q^{-s}| + |\alpha_2 q^{-s}| = \begin{cases} 2q^{-Re(s)} \\ q^{\frac{1}{2}-Re(s)} + q^{-\frac{1}{2}-Re(s)} \end{cases} \le 2q^{\frac{1}{2}-Re(s)}$

hence when $Re(s) > \frac{3}{2}$, we must have absolute convergence

Zeta integral

We now focus on the case that (π, V) is a cuspidal automorphic representation, and take $\phi \in V$
(note: here we actually take V to be the K-finite subspace of an closed irr subrepn of $L_0^2(G_F \backslash G_{\mathbb{A}}, \omega)$)

Prop 1: 1. For $\forall N$, $\exists C_N > 0$, s.t.

$$\left|\phi\begin{pmatrix} y & \\ & 1\end{pmatrix}\right| \le C_N \cdot |y|^{-N}, \text{ when } |y| \to +\infty$$

$$\left|\phi\begin{pmatrix} y & \\ & 1\end{pmatrix}\right| \le C_N |y|^{N}, \text{ when } |y| \to 0$$

2. The following integral is absolutely convergent for $\forall s \in \mathbb{C}$

$$Z(s,\phi) = \int_{\mathbb{A}^\times / F^\times} \phi\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\right) |y|^{s-\frac{1}{2}} d^\times y$$

pf: Since ϕ is a cuspidal automorphic form, the rapid decay at ∞ is just [Gelbart]$' \Leftrightarrow$ [Gelbart]
now consider rapid decay at 0

$$\left|\phi\begin{pmatrix} y & \\ & 1\end{pmatrix}\right| = \left|\phi\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} y & \\ & 1\end{pmatrix}\right)\right| = \left|\phi\left(\begin{pmatrix} 1 & \\ & y\end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right)\right| = \left|\pi(w)\phi\left(\begin{pmatrix} 1 & \\ & y\end{pmatrix}\right)\right|$$

$$= \left|\omega(y)\cdot\left(\pi(w)\phi\right)\begin{pmatrix} y^{-1} & \\ & 1\end{pmatrix}\right| \le C_N' \cdot |y^{-1}|^{-N} = C_N' \cdot |y|^N \text{ when } |y| \to 0$$

Rmk: the restriction on the norm of $y \in \mathbb{A}^\times$ is essentially a restriction on norm of y_∞!
because we have seen by finiteness of class number,

$$\mathbb{A}^\times = \coprod F^\times(\mathbb{A}_\infty \times t_i \prod_v \mathcal{O}_{F,v}^\times) \Rightarrow \mathbb{A}^\times / F^\times = \coprod_{\text{some } i} \left(\mathbb{A}_\infty \times t_i \prod_v \mathcal{O}_{F,v}^\times\right)$$

Now when we do the integral, we are actually do integral over some $\mathbb{A}_\infty \times t_i \prod_v \mathcal{O}_{F,v}^\times$
then $|y| = a_i \cdot |y_\infty|$ for some $a_i > 0$ on each such piece, on each piece.

$$|y_\infty| = |y_1| \cdots |y_r|, \ r \text{ means infinite places of } F$$

then the integrand function $\phi(y_1, \dots, y_r)$ is rapid decay at each coordinate, it must be integrable!

We recall Whittaker function W_ϕ associated to ϕ.

$$\phi(g) = \sum_{\alpha \in F^\times} W_\phi\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\right)$$

then at least formally,

$$\int_{A^\times/F^\times} \phi\begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{s-\frac{1}{2}} d^\times y = \int_{A^\times} W_\phi\begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{s-\frac{1}{2}} d^\times y \qquad (1)$$

this equality holds if the integral on the RHS absolutely converge. now we analyze W_ϕ

we assume $\phi = \otimes_v \phi_v$, then we know

$$W_\phi(g) = \prod_v W_v(g_v) \quad \text{. here } W_v \text{ corresponds } \phi_v$$

$$\Rightarrow \quad W_\phi\begin{pmatrix} y & \\ & 1 \end{pmatrix} = \prod_v W_v\begin{pmatrix} y_v & \\ & 1 \end{pmatrix}$$

By the definition of integral theory over adeles, we have factorization if both sides converge absolutely

$$\int_{A^\times} W_\phi\begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{s-\frac{1}{2}} d^\times y = \prod_v \int_{F_v^\times} W_v\begin{pmatrix} y_v & \\ & 1 \end{pmatrix} |y_v|^{s-\frac{1}{2}} d^\times y_v$$

Therefore we define local zeta integral w.r.t a Whittaker function:

$$Z_v(s, W_v) = \int_{F_v^\times} W_v\begin{pmatrix} y_v & \\ & 1 \end{pmatrix} |y_v|^{s-\frac{1}{2}} d^\times y_v \qquad (2)$$

<u>Prop 2</u>: The local zeta integral (2) absolutely converges for $Re(s) > \frac{1}{2}$

- For almost all v, we have

$$Z_v(s, W_v) = L_v(s, \pi_v)$$

therefore, the global zeta integral (1) absolutely converges for $Re(s) > \frac{3}{2}$

<u>pf</u>: the convergence of (2) depends on the analysis of Killilov model of the local repn π_v

We first look at NA case

- π_v is cuspidal, $W_\phi\begin{pmatrix} y & \\ & 1 \end{pmatrix} \in C_c^\infty(F_v^\times)$, the integral converges for $\forall s \in \mathbb{C}$
- $\pi_v \simeq \pi(\chi_1, \chi_2)$, then $W_\phi\begin{pmatrix} y & \\ & 1 \end{pmatrix}$ satisfies

  when $|y| \to +\infty$, it is zero

  when $|y| \to 0$, then it is a linear combination of $\chi_1(y)|y|^{\frac{1}{2}}$ & $\chi_2(y)|y|^{\frac{1}{2}}$
- $\pi \simeq \chi \cdot St_G$ (subrepn of $\pi(\chi, \chi)$)

  when $|y| \to +\infty$, it is zero

  when $|y| \to 0$, it is a multiple of $\chi(y)|y|^{\frac{1}{2}}$

Now with these results in hand, we calculate (2)

$$F_v^\times = \coprod_{n \in \mathbb{Z}} \{x \mid v(x) = n\} \text{, we focus on } n \gg 0,$$

$$\int_{v(y) \geq n} \chi(y)|y|^s \frac{dy}{|y|} = \lim_{N \to +\infty} \sum_{k=n}^{N} \int_{v(y)=k} \chi(y)|y|^s \frac{dy}{|y|}, \quad \int_{v(y)=k} |\chi(y)| \cdot |y|^s \frac{dy}{|y|} = \int_{v(y)=k} |y|^{\sigma+s} \frac{dy}{|y|} = q^{-k(\sigma+s)}, \quad -\frac{1}{2} < \sigma < \frac{1}{2}$$

therefore to make sure the integral converges, we must have $Re(s) > \frac{1}{2}$ $\left(Re(s) + \sigma > 0 \right)$

Now we consider the Archimedean place,

By Prop 1, we have $W_v\begin{pmatrix} y & \\ & 1\end{pmatrix}$ must decay rapidly when $|y| \to +\infty$ and 0

Finally we turn to the global case:

we consider the place of the following type:

- $\nu \simeq \pi(\chi_1, \chi_2)$ with χ_1, χ_2 unramified $\Leftrightarrow$ Spherical principal series
- ψ_v has conductor $= \mathcal{O}_v$
- $\phi_v \in \pi_v$ is the unique spherical element, s.t. $W_v(k_v) = 1$, $\forall k_v \in K_v$

Claim: $Z_v(s, \phi_v) = L_v(s, \pi_v)$

pf: under these assumptions, we have

$$W_v\begin{pmatrix} y & \\ & 1\end{pmatrix} = \begin{cases} q^{-\frac{m}{2}} \dfrac{\alpha_1^{m+1} - \alpha_2^{m+1}}{\alpha_1 - \alpha_2} & , \text{if } m \geq 0 \\ 0 & , \text{if } m < 0 \end{cases}$$

here $\alpha_1 = \chi_1(p_v)$, $\alpha_2 = \chi_2(p_v)$, $m = \nu(y)$, then

$$\int_{\mathbb{Q}_v^*} W_v\begin{pmatrix} y & \\ & 1\end{pmatrix} |y|^{s-\frac{1}{2}} \frac{dy}{|y|} = \sum_{m=0}^{\infty} \int_{v(y)=m} q^{-\frac{m}{2}} \frac{\alpha_1^{m+1} - \alpha_2^{m+1}}{\alpha_1 - \alpha_2} q^{-m(s-\frac{1}{2})} \frac{dy}{|y|}$$

$$= \sum_{m=0}^{\infty} q^{-\frac{m}{2}} \frac{\alpha_1^{m+1} - \alpha_2^{m+1}}{\alpha_1 - \alpha_2} q^{m(\frac{1}{2}-s)} = L_v(s, \pi_v)$$

Then by the convergence condition for L_s we have already shown, (1) converges for $Re(s) > \frac{3}{2}$

Rmk: 1. By the analysis above, we see that W_ϕ is very similar to L_s ,

2. We can twist W_ϕ & Z_v by unitary characters, then the convergence analysis still works

Functional equations:

For zeta integral: We have global & local functional equations:

globally: $Z(s, \phi, \xi) = Z(1-s, \pi(w_1)\phi, w^{-1}\xi^{-1})$, $w_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

locally: $Z_v(s, W_v, \xi_v)$ has meromorphic continuation to all $s \in \mathbb{C}$, and there exists meromorphic function $\gamma_v(s, \pi_v, \xi_v, \psi_v)$, s.t.

$$Z_v(1-s, \pi_v(w_1)W_v, \xi_v^{-1}w_v^{-1}) = \underbrace{\gamma_v(s, \pi_v, \xi_v, \psi_v)}_{\text{independent of choice of } W_v \in \mathcal{W}_v} Z_v(s, W_v, \xi_v)$$

pf: the global version is rather easy

$$Z(s, \phi, \xi) = \int_{\mathbb{A}^\times/F^\times} \phi\begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{s-\frac12} \xi(y)\, d^\times y = \int_{\mathbb{A}^\times/F^\times} \phi\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) |y|^{s-\frac12} \xi(y)\, d^\times y$$

$$= \int_{\mathbb{A}^\times/F^\times} \phi\left(\begin{pmatrix} 1 & \\ & -y \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\right) |y|^{s-\frac12} \xi(y)\, d^\times y = \int_{\mathbb{A}^\times/F^\times} (\pi(w_1)\phi)\begin{pmatrix} -y^{-1} & \\ & 1 \end{pmatrix} \cdot w(-y)\, \xi(y)\, |y|^{s-\frac12} d^\times y$$

$$= \int_{\mathbb{A}^\times/F^\times} (\pi(w_1)\phi)\begin{pmatrix} y & \\ & 1 \end{pmatrix} |y|^{\frac12 - s} \xi^{-1}w^{-1}(y)\, d^\times y = Z(1-s, \pi(w_1)\phi, \xi^{-1}w^{-1})$$

the local case is much more difficult, we deal with NA case first:

the local zeta integral is essentially an integral of elements in Kirillov model. Let's consider

$$L_1: K(\pi) \longrightarrow \mathbb{C} \qquad \phi \mapsto Z_v(s, \phi, \xi_v)$$
$$L_2: K(\pi) \longrightarrow \mathbb{C} \qquad \phi \mapsto Z_v(1-s, \pi(w_1)\phi, w_v^{-1}\xi_v^{-1})$$

Claim. $L_i\left(\pi\begin{pmatrix} y & \\ & 1 \end{pmatrix}\phi\right) = \xi_v(y)^{-1}|y|^{\frac12 - s} L_i(\phi)$ $\qquad x' = xy$

pf: $L_1\left(\pi\begin{pmatrix} y & \\ & 1 \end{pmatrix}\phi\right) = \int_{F_v^\times} \phi(yx)|x|^{s-\frac12}\xi_v(x)\, d^\times x = \xi_v(y)^{-1}|y|^{\frac12 - s}\int_{F_v^\times} \phi(x)|x|^{s-\frac12}\xi_v(x)\, d^\times x = \xi_v(y)^{-1}|y|^{\frac12 - s} L(\phi)$

now consider L_2, we know

$$Z_v\left(s, \pi(w_1)\pi\begin{pmatrix} y & \\ & 1 \end{pmatrix}\phi, w_v^{-1}\xi_v^{-1}\right) = Z_v\left(s, \pi\begin{pmatrix} 1 & \\ & y \end{pmatrix}\pi(w_1)\phi, w_v^{-1}\xi_v^{-1}\right) = w_v(y)^{-1} Z_v\left(s, \pi\begin{pmatrix} y^{-1} & \\ & 1 \end{pmatrix}\pi(w_1)\phi, w_v^{-1}\xi_v^{-1}\right)$$

$$= w_v(y)\, w_v^{-1}(y)\, \xi_v^{-1}(y)\, |y|^{s-\frac12} Z_v(s, \pi(w_1)\phi, w_v^{-1}\xi_v^{-1})$$

hence the Claim is proved as soon as we prove the analytic continuation of Z_v

Now due to my Bachelor's thesis $\mathrm{Hom}_{GL_1}(\pi|_{GL_1}, \chi) \simeq \mathbb{C}$ is alway 1-dimensional !! i.e the existence of γ_v

Let's continue the proof, we must show that $Z_v(s, \varphi, \xi_v)$ which is originally defined for $\mathrm{Re}(s) > \frac{1}{2}$, can be meromorphically continued to $\mathbb{C}$, here we should use the asympotic properties of Kirillov models:

We only need to show that, for $\forall\, \chi : F_v^\times \to \mathbb{C}^\times$, the following expression:

$$\int_{|y| \le C} \chi(y)\, |y|^s\, \xi(y)\, d^\times y \;, \quad \int_{|y| \le C} \chi(y)\, v(y)\, |y|^s\, \xi(y)\, d^\times y \quad \text{has analytic continuation:}$$

- $$\int_{|y|\le C} \chi(y)|y|^s d^\times y = \lim_{N\to+\infty} \int_{M \le v(y) \le N} \chi(y)|y|^s d^\times y = \lim_{N\to+\infty} \sum_{k=M}^{N} \int_{v(y)=k} \chi(y) q^{-ks} d^\times y = \begin{cases} \dfrac{q^{-M(\alpha+s)}}{1-q^{-(\alpha+s)}}, & \chi \text{ unramified, } \chi(p_v) = q^{-\alpha} \\ 0, & \chi \text{ ramified} \end{cases}$$
- $$\int_{|y|\le C} \chi(y) v(y) |y|^s d^\times y = \lim_{N\to\infty} \sum_{k=M}^{N} \int_{v(y)=k} \chi(y)\, k\, q^{-ks} d^\times y = \begin{cases} (M-1)\, q^{-M(s+\alpha)} + \dfrac{q^{-M(s+\alpha)}}{1-q^{-(s+\alpha)}}, & \chi \text{ unramified, } \chi(p_v) = q^{-\alpha} \\ 0, & \chi \text{ ramified} \end{cases}$$

We can also show that these factors $\gamma(s, \pi_v, \xi_v, \psi_v)$ determines π

<u>Theorem</u>: If π_1, π_2 has the same central character & $\gamma(s, \pi_1, \xi_v, \psi_v) = \gamma(s, \pi_2, \xi_v, \psi_v)$ for $\forall\, \xi_v$, then $\pi_1 \simeq \pi_2$

<u>pf</u>:

<u>For L-functions</u>: Assume π is an unitary irr repn of $GL(2, \mathbb{A})$ (hence must be admissible), such as automorphic repns
then $\pi \simeq \bigotimes_v \pi_v$, with each π_v is an irr G_v-repn, then almost all π_v are spherical.
ξ is a Grossencharacter of $\mathbb{A}^\times$ (i.e. unitary, continuous, trivial on $F^\times$)
define a finite set S:

$$v \in S \text{ if any one of these is satisfied: } \begin{cases} v \text{ is Archimedean} \\ \pi_v \text{ is not spherical} \\ \xi_v \text{ is not unramified} \\ \text{conductor of } \psi_v \text{ is not } \mathcal{O}_v \end{cases}$$

i.e. if $v \notin S$, then v is NA, spherical, ξ_v unramified, ψ_v has conductor $= \mathcal{O}_v$,
then we can apply the results for zeta integral to obtain:

$$L_S(s, \pi, \xi) = \prod_{v \notin S} L_v(s, \pi_v, \xi_v)$$ converges for $Re(s) > \frac{3}{2}$, it has a meromorphic continuation to $\forall s \in \mathbb{C}$, it also satisfies a functional equation:

$$L_S(s, \pi, \xi) = \left(\prod_{v \in S} \gamma_v(s, \pi_v, \xi_v, \psi_v) \right) L_S(1-s, \hat{\pi}, \xi^{-1})$$

pf: We know that, for $\forall v \notin S$, $\exists \phi_v \in \pi_v$, $W_v \in \mathcal{W}_v$, s.t. $L_v(s, \pi_v, \xi_v) = Z_v(s, W_v, \xi_v)$
now for other places $v \in S$, choose $0 \neq \phi_v \in \pi_v$ and associated $W_v \in \mathcal{W}_v$, consider $W = \otimes W_v \in$ global Whittaker model
for $Re(s) > \frac{3}{2}$, we have

$$Z(s, \phi, \xi) = \prod_v Z_v(s, W_v, \xi_v) = \prod_{v \in S} Z_v(s, W_v, \xi_v) \cdot \prod_{v \notin S} Z_v(s, W_v, \xi_v) = L_S(s, \pi, \xi) \cdot \prod_{v \in S} Z_v(s, W_v, \xi_v) \quad (*)$$

since $Z_v(s, W_v, \xi_v)$ can be meromorphically continued to $\forall s \in \mathbb{C}$, $L_S(s, \pi, \xi)$ can also be meromorphically continued
after meromorphic continuation, $(*)$ holds for $\forall s \in \mathbb{C}$. now when $Re(s) \ll 0$

$$Z(s, \phi, \xi) = Z(1-s, \pi(w_1)\phi, \xi^{-1}\omega^{-1}) = \prod_{v \in S} Z_v(1-s, \pi_v(\omega_1) W_v, \xi_v^{-1}\omega_v^{-1}) \cdot \prod_{v \notin S} Z_v(1-s, \pi_v(w_1) W_v, \xi_v^{-1}\omega_v^{-1})$$

$$= \prod_{v \in S} \gamma_v(s, \pi_v, \xi_v, \psi_v) Z_v(s, W_v, \xi_v) \cdot \underbrace{\prod_{v \notin S} L_v(1-s, \hat{\pi}_v, \xi_v^{-1})}_{= L_S(1-s, \hat{\pi}, \xi^{-1})}$$

because when $v \notin S$, $Z_v(s, \pi_v(w_1) W_v, \xi_v^{-1}\omega_v^{-1}) = Z_v(s, W_v, \xi_v^{-1}\omega_v^{-1}) = L_v(s, \pi_v, \xi_v^{-1}\omega_v^{-1}) = L_v(s, \hat{\pi}_v, \xi_v^{-1})$

$\pi_v \simeq \pi(\chi_1, \chi_2)$, Satake parameter $\alpha_1, \alpha_2 \Rightarrow \hat{\pi}_v \simeq \pi(\chi_1^{-1}, \chi_2^{-1})$
$\alpha_1^{-1}, \alpha_2^{-1}$
LHS $= (1 - \alpha_1 \xi_v^{-1}\omega_v^{-1}(p_v) q^{-s})(1 - \alpha_2 \xi_v^{-1}\omega_v^{-1}(p_v) q^{-s})$
RHS $= (1 - \alpha_1^{-1} \xi_v^{-1}(p_v) q^{-s})(1 - \alpha_2^{-1} \xi_v^{-1}(p_v) q^{-s})$
since ω_v is central character:

$$\omega_v(p_v) = \chi_1(p_v) \cdot \chi_2(p_v) = \alpha_1 \cdot \alpha_2$$

Remaining problem: How to define L-factor at every place ??
Before answering this question, let's see the meaning of <u>L-factor</u>
and finally we will see this L-factor can also be obtained by zeta integral of some privileged Whittaker function!

More on the L-factors

[Gelbart 6.12]

Prop: For every irr admissible repn π_v of $GL_2(F_v)$, v a NA place. $\exists$ Euler factor $L(s, \pi_v)$, s.t.

$$\frac{Z_v(s, W_v, \xi_v)}{L_v(s, \xi_v \otimes \pi_v)} \quad \text{is entire for } \forall W_v \in \mathcal{W}_v,\ \forall \text{ character } \xi_v,\ \forall s \in \mathbb{C}$$

Moreover, for $\forall \xi_v$, there exists $W^\circ \in \mathcal{W}_v$, s.t.

$$Z_v(s, W^\circ, \xi_v) = L_v(s, \xi_v \otimes \pi_v).$$

pf: We divide the proof into three cases:

- π_v cuspidal, then $\xi_v \otimes \pi_v$ is also cuspidal, we know $Z_v(s, W_v, \xi_v)$ is always entire, we set $L_v(s, \pi_v) = 1$ and now we know the Kirillov model for cuspidal repn is just $C_c^\infty(F_v^\times)$, hence we could choose a suitable $W^\circ \in C_c^\infty(F_v^\times)$, s.t. $Z_v(s, W^\circ, \xi_v) = 1$

- $\pi_v \simeq \chi \cdot St$, then we know the Kirillov model is:
  $f: F_v^\times \to \mathbb{C}$ locally constant, vanish when $|y| \to +\infty$, and when $|y| \to 0$, $f(y) = C|y|^{\frac{1}{2}}\chi(y)$
  then
  $$Z_v(s, f, \xi_v) = \int_{F_v^\times} f(y)\, \xi_v(y) \cdot |y|^{s-\frac{1}{2}}\, d^\times y = \int_{-N \le v(y) \le N} f(y)\, \xi_v(y) \cdot |y|^{s-\frac{1}{2}}\, d^\times y + C \int_{v(y) \ge N} \chi\xi_v(y) \cdot |y|^s\, d^\times y$$
  $$= \text{linear combination of } \{q^{-Ns}, q^{-(N-1)s}, \dots, q^{(N-1)s}, q^{Ns}\} + C \cdot \underbrace{\sum_{k=N}^{\infty} \int_{v(y)=k} \chi\xi_v(y) \cdot q^{-ks}\, d^\times y}$$
  if $\chi\xi_v$ is ramified, this is 0
  if $\chi\xi_v$ is unramified, this equals to
  $$C \cdot \sum_{k=N}^{\infty} \alpha^k \cdot q^{-ks} = C \cdot \frac{\alpha^N q^{-Ns}}{1 - \alpha q^{-s}}$$
  here $\alpha = \chi\xi_v(P_v)$

  hence we define
  $$L_v(s, \pi_v) = \begin{cases} 1, & \text{if } \chi_v \text{ ramified}. \\ (1 - \chi(P_v)q^{-s})^{-1}, & \text{if } \chi_v \text{ unramified}. \end{cases}$$
  and the function f is easy to chose, s.t. $Z_v(s, f, \xi_v) = L_v(s, \xi_v \otimes \pi_v)$

- $\pi_v \simeq \pi(\chi_1, \chi_2)$ irr, then the Kirillov model is:
  $f: F_v^\times \to \mathbb{C}$, locally constant, vanish when $|y| \to +\infty$, and when $|y| \to 0$, $f(y) = C_1|y|^{\frac{1}{2}}\chi_1(y) + C_2|y|^{\frac{1}{2}}\chi_2(y)$
  then the previous calculation works well and tells us that
  $$L_v(s, \pi_v) = (1 - \chi_1(P_v)q^{-s})^{-1}(1 - \chi_2(P_v)q^{-s})^{-1} \text{ is the factor we want}$$

Rmk: 1. We have a list of L-factors

π	L
$\pi(\chi_1, \chi_2)$	$L_v(s, \chi_1)\, L_v(s, \chi_2)$
$\chi \cdot St$	$L(s, \chi)$
cuspidal	1

2. By the above computations, we obtain:
$$\frac{Z_v(s, W_v, \xi_v)}{L_v(s, \xi_v \otimes \pi_v)} \in \mathbb{C}[q^{-s}, q^s]$$

We see that the local Euler factor can be interpreted as the g.c.d of all the zeta integral therefore it is more reasonable to consider the following "normalized" zeta integral

$$\frac{Z_v(s, W_v, \xi_v)}{L_v(s, \xi_v \otimes \pi_v)} \in \mathbb{C}[q^{-s}, q^s]$$

We also have a functional equation for this kind of zeta integral

$$\frac{Z_v(s, W_v, \xi_v)}{L_v(s, \xi_v \otimes \pi_v)} \epsilon(s, \pi_v, \xi_v, \psi_v) = \frac{Z_v(1-s, \pi_v(w_1)W_v, \omega_v^{-1}\xi_v^{-1})}{L_v(1-s, \xi_v^{-1} \otimes \hat{\pi}_v)} \qquad (*)$$

here

$$\epsilon(s, \pi_v, \xi_v, \psi_v) = \frac{L_v(s, \xi_v \otimes \pi_v)}{L_v(1-s, \xi_v^{-1} \otimes \hat{\pi}_v)} \gamma(s, \pi_v, \xi_v, \psi_v) \in \mathbb{C}(q^s)$$

<u>Prop</u>: For an irr admissible repn π_v of $GL_2(F_v)$, we have the following equality:

$$\epsilon(s, \pi_v, \xi_v, \psi_v)\,\epsilon(1-s, \hat{\pi}_v, \xi_v^{-1}, \psi_v) = \omega_v(-1)$$

Moreover, $\epsilon(s, \pi_v, \xi_v, \psi_v) \in \mathbb{C}[q^s, q^{-s}]$, and $\exists\, a \in \mathbb{C}^\times$, $b \in \mathbb{Z}$, s.t.

$$\epsilon(s, \pi_v, \xi_v, \psi_v) = a \cdot q^{bs}$$

and we also have $\epsilon(s, \pi_v, \xi_v, \psi_v) = \epsilon(s, \xi_v \otimes \pi_v, \psi_v)$

pf:

$$\frac{Z_v(s, W_v, \xi_v)}{L_v(s, \xi_v \otimes \pi_v)} \epsilon(s, \pi_v, \xi_v, \psi_v) = \frac{Z_v(1-s, \pi_v(w_1)W_v, \omega_v^{-1}\xi_v^{-1})}{L_v(1-s, \xi_v^{-1} \otimes \hat{\pi}_v)}$$

$$\frac{Z_v(s, W_v', \xi_v^{-1}\omega_v)}{L_v(s, \xi_v^{-1} \otimes \hat{\pi}_v)} \epsilon(s, \hat{\pi}_v, \xi_v^{-1}, \psi_v) = \frac{Z_v(1-s, \hat{\pi}_v(w_1)W_v', \xi_v)}{L_v(1-s, \xi_v \otimes \pi_v)}$$

now we suppose $W_v' \in C_c^\infty(F_v^\times)$, then $\hat{\pi}_v(w_1)W_v' \in C_c^\infty(F_v^\times)$ also lies in the Whittaker space of $\hat{\pi}_v$, then obviously

$$\left(\pi_v(w_1) \circ \hat{\pi}_v(w_1)\, W_v'\right)(y) = W_v'\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} w_1^2\right) = W_v'\left(\begin{pmatrix} -y & \\ & -1 \end{pmatrix}\right) = (\omega_v(-1)\, W_v)(y)$$

then:

$$\frac{\omega_v(-1)\, Z_v(s, W_v', \omega_v^{-1}\xi_v^{-1})}{L_v(1-s, \xi_v^{-1} \otimes \hat{\pi}_v)} = \frac{Z_v(s, \hat{\pi}_v(w_1)W_v', \xi_v)}{L_v(s, \xi_v \otimes \pi_v)} \epsilon(s, \pi_v, \xi_v, \psi_v)$$

$$= \frac{Z_v(1-s, W_v', \xi_v^{-1}\omega_v)}{L_v(1-s, \xi_v^{-1} \otimes \hat{\pi}_v)} \epsilon(1-s, \hat{\pi}_v, \xi_v^{-1}, \psi_v)\,\epsilon(s, \pi_v, \xi_v, \psi_v)$$

i.e. $\epsilon(s, \pi_v, \xi_v, \psi_v)\,\epsilon(1-s, \hat{\pi}_v, \xi_v^{-1}, \psi_v) = \omega_v(-1)$

now by the definition of ϵ, there exists W_v, s.t. $Z_v(s, W_v, \xi_v) = L_v(s, \xi_v \otimes \pi_v)$, therefore

$$\epsilon(s, \pi_v, \xi_v, \psi_v) = \frac{Z_v(1-s, \pi_v(w_1)W_v, \omega_v^{-1}\xi_v^{-1})}{L_v(1-s, \xi_v^{-1} \otimes \hat{\pi}_v)} \in \mathbb{C}[q^{-s}, q^s]$$

and by the above relation, we get it belongs to $\mathbb{C}[q^{-s}, q^s]^\times$. i.e. $\exists\, a \in \mathbb{C}^\times$, $b \in \mathbb{Z}$, s.t.

$$\epsilon(s, \pi_v, \xi_v, \psi_v) = a \cdot q^{bs}$$

the last equality follows from take $\xi_v = 1$ in $(*)$, and compare with original definition, then take $W_v \in C_c^\infty(F_v^\times)$

Computation of $\epsilon(s, \pi_v, \psi_v)$

$$\frac{Z_v(s, W_v, \xi_v)}{L_v(s, \xi_v \otimes \pi_v)}\, \epsilon(s, \pi_v, \xi_v, \psi_v) = \frac{Z_v(1-s, \pi_v(w_1)W_v, \omega_v^{-1}\xi_v^{-1})}{L_v(1-s, \xi_v^{-1}\otimes \hat{\pi}_v)}$$

1. We first consider the simplest case: π_v is spherical $\simeq \pi(\chi_1, \chi_2)$ & ξ_v is unramified

In this case, we could take $W_v = W_v^0$, which is the normalized K-fixed function, then $\pi_v(w_1)W_v = W_v$.
then suppose $\chi_1(p_v) = \alpha_1$, $\chi_2(p_v) = \alpha_2$, then $Z_v(s, W_v) = L_v(s, \pi_v) = (1-\alpha_1 q^{-s})^{-1}(1-\alpha_2 q^{-s})^{-1}$

$$\epsilon(s, \pi_v, \xi_v, \psi_v) = \frac{Z_v(1-s, W_v, \omega_v^{-1}\xi_v^{-1})}{L_v(1-s, \xi_v^{-1}\otimes\hat{\pi}_v)} \cdot \left(\frac{Z_v(s, W_v, \xi_v)}{L_v(s, \xi_v\otimes\pi_v)}\right)^{-1}$$

Recall that

$$W_v\begin{pmatrix} y & \\ & 1\end{pmatrix} = \begin{cases} q^{-\frac{m}{2}}\dfrac{\alpha_1^{m+1}-\alpha_2^{m+1}}{\alpha_1-\alpha_2} & , \text{if } m \geq 0 \\ 0 & , \text{if } m < 0\end{cases}$$

then

$$Z_v(s, W_v, \chi_v) = \int_{\mathbb{Q}_v^\times} W_v\begin{pmatrix} y & \\ & 1\end{pmatrix} \chi_v(y)\, |y|^{s-\frac{1}{2}}\, d^\times y$$

zero if χ is ramified
therefore we assume χ is unramified, $\chi(p_v) = \alpha$

$$= \sum_{k=0}^{\infty} \int_{v(y)=k} q^{-\frac{k}{2}} \frac{\alpha_1^{k+1}-\alpha_2^{k+1}}{\alpha_1-\alpha_2} \cdot \alpha^k \cdot q^{k(\frac{1}{2}-s)}\, d^\times y$$

$$= \sum_{k=0}^{\infty} q^{-k(s+1)}\alpha^k \cdot \frac{\alpha_1^{k+1}-\alpha_2^{k+1}}{\alpha_1-\alpha_2} = (1-q^{-s}\alpha\alpha_1)^{-1}(1-q^{-s}\alpha\alpha_2)^{-1}$$

suppose $\xi_v(p_v) = \alpha$, then $\xi_v^{-1}\omega_v^{-1}(p_v) = \alpha_1^{-1}\alpha_2^{-1}\alpha^{-1}$, then

$$\frac{Z_v(1-s, W_v, \omega_v^{-1}\xi_v^{-1})}{L_v(1-s, \xi_v^{-1}\otimes\hat{\pi}_v)} = (1-q^{s-1}\alpha_1^{-1}\alpha^{-1})^{-1}(1-q^{s-1}\alpha_2^{-1}\alpha^{-1})^{-1}(1-q^{s-1}\alpha^{-1}\alpha_1^{-1})(1-q^{s-1}\alpha^{-1}\alpha_2^{-1}) = 1$$

$$\frac{Z_v(s, W_v, \xi_v)}{L_v(s, \xi_v\otimes\pi_v)} = (1-q^{-s}\alpha\alpha_1)^{-1}(1-q^{-s}\alpha\alpha_2)^{-1}(1-q^{-s}\alpha\alpha_1)(1-q^{-s}\alpha\alpha_2) = 1$$

i.e. for unramified ξ_v, we have $\epsilon(s, \pi_v, \xi_v, \psi_v) = \epsilon(s, \xi_v\otimes\pi_v, \psi_v) = 1$

2 Let's try for $\pi \simeq \pi(\chi_1, \chi_2)$, here $\chi_i(x) = \xi_i(x)\cdot|x|^{s_i}$, $\mathrm{Re}(s_1-s_2) > 0$

$$f\left(\begin{pmatrix} y_1 & z \\ & y_2\end{pmatrix} w_0 \begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right) = \chi_1(y_1)\chi_2(y_2)\left|\frac{y_1}{y_2}\right|^{\frac{1}{2}}, \text{ for } y_1, y_2 \in F_v^\times,\ z \in F_v,\ x \in \mathcal{O}_v$$

then
$$W_f(y) = \int_{F_v} f\left(w_0\begin{pmatrix} y & x \\ & 1\end{pmatrix}\right)\psi(-x)\,dx = \int_{F_v} \chi_2(y)|y|^{-\frac{1}{2}} f\left(w_0\begin{pmatrix} 1 & xy^{-1} \\ & 1\end{pmatrix}\right)\psi(-x)\,dx = \chi_2(y)|y|^{-\frac{1}{2}}\int_{y\mathcal{O}_v}\psi(-x)\,dx = \begin{cases}\chi_2(y)|y|^{\frac{1}{2}} & y \in c(\psi) \\ 0, & y \notin c(\psi)\end{cases}$$

and
$$(\pi(w_1)W_f)(y) = \int_{F_v} f\left(w_0\begin{pmatrix} y & x \\ & 1\end{pmatrix}w_1\right)\psi(-x)\,dx = \int_{F_v}\chi_1(-x^{-1})\chi_2(-x)\chi_1\chi_2(y)\, f\left(w_0\begin{pmatrix} 1 & -x^{-1} \\ & 1\end{pmatrix}\begin{pmatrix} y^{-1} & \\ & 1\end{pmatrix}\right)\psi(-x)\,dx$$

$$= \int_{F_v}\chi_1(-x^{-1})\chi_2(-x)\chi_1(y)|y|^{\frac{1}{2}} f\left(w_0\begin{pmatrix} 1 & -x^{-1}y \\ & 1\end{pmatrix}\right)\psi(-x)\,dx$$

$f\left(\begin{pmatrix} y & \\ & y\end{pmatrix}g\right) = \chi_1 \times$

$=\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & \\ & y\end{pmatrix}$

$=\begin{pmatrix} & -1 \\ 1 & x\end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & \\ & y\end{pmatrix} = \begin{pmatrix} 1 & \\ -x & 1\end{pmatrix}\begin{pmatrix} 1 & \\ & y\end{pmatrix} = \begin{pmatrix} 1 & \\ -x & 1\end{pmatrix} = \begin{pmatrix} -x^{-1} & 1 \\ & -x\end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & -x^{-1} \\ & 1\end{pmatrix}\begin{pmatrix} 1 & \\ & y\end{pmatrix}$

$$= \int_{y\mathcal{O}_v} \chi_1(-x^{-1})\chi_2(-x)\chi_1(y)|y|^{\frac12}\psi(-x)\,dx = \chi_1(y)|y|^{\frac12}\chi_1\chi_2(-1)\int_{y\mathcal{O}_v}(\chi_1^{-1}\chi_2)(x)\psi(-x)\,dx$$

With these W by hand, we compute the local zeta integral.

$$Z_v(s, W_v, \xi_v) = \int_{F_v^\times} W_v(y)|y|^{s-\frac12}\xi_v(y)\,d^\times y = \int_{v(y)\geq c}\chi_2\xi(y)\cdot|y|^s\,d^\times y = \begin{cases} \dfrac{(\chi_2\xi(p)q^{-s})^c}{1-\chi_2\xi(p)q^{-s}}, & \chi_2\xi \text{ unramified} \\ 0, & \chi_2\xi \text{ ramified}\end{cases}$$

Let's try another f:

$$f\left(\begin{pmatrix} y_1 & z \\ & y_2\end{pmatrix} w_0 \begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right) = \chi_1(y_1)\chi_2(y_2)\left|\frac{y_1}{y_2}\right|^{\frac12}|x|, \text{ for } y_1, y_2 \in F_v^\times,\ z\in F_v,\ x\in\mathcal{O}_v$$

then $W_f(y) = \int_{F_v} f\left(w_0\begin{pmatrix} y & x \\ & 1\end{pmatrix}\right)\psi(-x)\,dx = \int_{F_v}\chi_2(y)|y|^{-\frac12} f\left(w_0\begin{pmatrix} 1 & xy^{-1} \\ & 1\end{pmatrix}\right)\psi(-x)\,dx = \chi_2(y)|y|^{-\frac12}\int_{y\mathcal{O}_v}|xy^{-1}|\psi(x)\,dx =$

$$\chi_2(y)|y|^{-\frac32}\int_{y\mathcal{O}_v}|x|\psi(x)\,dx$$

$y\in\mathcal{O}_v$, $v(y)=k$

$$\int_{p^k}|x|\,dx$$

$$= \sum_{i=k}^{\infty}\int_{p^k\mathcal{O}^\times} q^{-k}\,dx$$

$$= \sum_{i=k}^{\infty} q^{-2k}\int_{p^k\mathcal{O}^\times}\frac{dx}{|x|}$$

$$= \sum_{i=1}^{\infty} q^{-2i}\left(1-\frac1q\right)$$

$$= \left(1-\frac1q\right)q^{-2k}\left(1+q^{-1}+\ldots\right.$$

$$= \frac{q-1}{q}\cdot q^{-2k}\cdot\frac{1}{1-q^{-2}}$$

$$= \frac{q-1}{q^{2k+1}}\cdot\frac{q^2}{q-1\cdot q+1} = \frac{1}{(q+1)q^{2k-1}}$$

$$\int_{p^k}|x|\psi(-x)\,dx$$

$$\int_{p^{-1}-\mathcal{O}}|x|\psi(x)\,dx$$

$$= q\int_{p^{-1}\mathcal{O}^\times}\psi(-x)\,dx = q\cdot\sum_{x\in(\mathcal{O}/p)^\times}\psi(-w^{-1}x)\cdot\frac1q$$

$$= 1-\psi$$

$$\psi: p^{-1}/\mathcal{O}\to\mathbb{C}^\times$$

$$\mathcal{O}/p\to\mathbb{C}^\times$$

$$e^{2\pi i\frac1p}\quad e^{2\pi i\frac2p}\quad\ldots\quad e^{2\pi i\frac{p-1}{p}}$$

$$\int_{p^{-k}}|x|\psi(-x)\,dx$$

$$p^{-k}/\mathcal{O}$$

$$= \sum_{a\in\mathcal{O}/p^k-0}|w^{-k}a|\cdot\psi(w^{-k}a)$$

Intertwining Operators & Eisenstein series

Local Intertwining operator

In the theory of Whittaker model, we have seen that. for suitable $\chi_1, \chi_2 : F^\times \to \mathbb{C}^\times$, the following perator explicitly gives the Whittaker model: ($\chi_1 = \xi_1 |\cdot|^{s_1}$, $\chi_2 = \xi_2 \cdot |\cdot|^{s_2}$, ψ nontrivial : $F_v \to \mathbb{C}^*$, $Re(s_1 - s_2) > 0$)

$$\Lambda_\psi f(g) = \int_{F_v} f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g) \psi(-x) dx$$

ξ_i can be taken to be finite order

We are curious about the case that ψ is trivial. we will see that, under the same condition, we get an operator which gives an isomorphism between $\pi(\chi_1, \chi_2)$ & $\pi(\chi_2, \chi_1)$

Prop: When $Re(s_1 - s_2) > 0$, then the following operator converges absolutely:

$$(M(s)f)(g) = \int_{F_v} f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g) dx$$

it also gives a non-zero intertwining map between $B(\chi_1, \chi_2)$ & $B(\chi_2, \chi_1)$, so when $B(\chi_1, \chi_2)$ is irreducible i.e. when $\chi_1 \chi_2^{-1} \neq |\cdot|$, then $M(s)$ is an isomorphism

$$\pi(\chi_1, \chi_2) \xrightarrow{M(s)} \pi(\chi_2, \chi_1)$$

pf: when $|x| \to +\infty$, by the following equality:

$$f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g) = f(\begin{pmatrix} x^{-1} & -1 \\ & x \end{pmatrix}\begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix} g) = \chi_1^{-1}\chi_2(x)|x|^{-1} f(\begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix} g) = \chi_1^{-1}\chi_2(x)|x|^{-1} f(g)$$

hence

$$\int_{|x| \geq q^N} f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g) dx = \int_{|x| \geq q^N} \chi_1^{-1}\chi_2(x)|x|^{-1} f(g) dx = f(g) \sum_{k=N}^{\infty} \int_{v(x)=-k} \chi_1^{-1}\chi_2(x) \, d^\times x$$

the absolute value is bounded by:

$$\sum_{k=N}^{\infty} \int_{v(x)=-k} q^{k(s_2 - s_1)} d^\times x = \sum_{k=N}^{\infty} q^{k(s_2 - s_1)}$$

, it converges absolutely iff $Re(s_1 - s_2) > 0$

it also converges absolutely if $\chi_1^{-1}\chi_2$ is ramified i.e. $\xi_1^{-1}\xi_2$ is ramified

Now we try to give an analytic continuation of this operator $M(s)$
So let's fix a flat section f, then

Prop: For $\forall g \in G$, the function $M(s) f_{s_1, s_2}$ originally defined for $Re(s_1 - s_2) > 0$, has analytic continuation to $\forall (s_1, s_2) \in \mathbb{C}^2$ s.t. $\chi_1 \neq \chi_2$

Hence in conclusion, we got a non-zero intertwining morphism when $\chi_1 \neq \chi_2$

$$M(s): \; B(\chi_1, \chi_2) \xrightarrow{\sim} B(\chi_2, \chi_1)$$

which can be written explicity when $Re(s_1 - s_2) > 0$

Here is a natural question, we have $M: \pi(\chi_1, \chi_2) \xrightarrow{\sim} \pi(\chi_2, \chi_1)$, $M': \pi(\chi_2, \chi_1) \xrightarrow{\sim} \pi(\chi_1, \chi_2)$
hence $M' \circ M$ is a non-zero intertwing endomorphism of $\pi(\chi_1, \chi_2)$ itself, therefore it must be a scalar multiple
Q1: What is this scalar? $M' \circ M$
Moreover, we know that when χ_1 & χ_2 are unramified, then $\pi(\chi_1, \chi_2)$ are spherical, then M also maps the K-invariant vector of $\pi(\chi_1, \chi_2)$ to the K-invariant vector of $\pi(\chi_2, \chi_1)$, since $\dim_{\mathbb{C}} \pi(\chi_1, \chi_2)^K = \dim_{\mathbb{C}} \pi(\chi_2, \chi_1)^K = 1$, there is also a scalar relating them!
Q2: What is $M\phi_\chi^K = ?\ \phi_{\chi^w}^K$
These questions will be solved via local functional equations

· Unitaricity

In this section, we consider the following problem:
Q: When does $B(\chi_1, \chi_2)$ is an unitary repn?

Firstly, when χ_1 & χ_2 are both unitary, then the induced repn $B(\chi_1, \chi_2)$ is also unitary
i.e. Case 1: $|\chi_1| = |\chi_2| = 1$.

Now if we have a Hermitian bilinear pairing

$$\underset{\mathbb{C}\text{-linear}}{B(\chi_1, \chi_2)} \times \underset{\mathbb{C}\text{-anti-linear}}{B(\chi_1, \chi_2)} \longrightarrow \mathbb{C} \qquad (f_1, f_2) \mapsto \int_K \varphi(f_1)(k) \overline{f_2}(k)\, dk$$

$$\Rightarrow \text{we have } \underset{\mathbb{C}\text{-linear}}{B(\chi_1, \chi_2)} \times \underset{\mathbb{C}\text{-linear}}{B(\overline{\chi}_1, \overline{\chi}_2)} \to \mathbb{C} \qquad (f_1, f_2) \mapsto \varphi(f_1)(f_2) = \int_K \varphi(f_1)(k) f_2(k)\, dk$$

then we get intertwing homos

$$B(\chi_1, \chi_2) \longrightarrow \underbrace{B(\overline{\chi}_1, \overline{\chi}_2)^\vee}_{\text{smooth dual}} \simeq B(\overline{\chi}_1^{-1}, \overline{\chi}_2^{-1}) \qquad f \mapsto \varphi(f)$$

then we must have

Case 1 $\quad \chi_1 = \overline{\chi}_1^{-1}$ & $\chi_2 = \overline{\chi}_2^{-1}$

Case 2: $\quad \chi_1 = \overline{\chi}_2^{-1} \Leftrightarrow \chi_2 = \overline{\chi}_1^{-1}$

Now we focus on case 2. $B(\chi, \overline{\chi}^{-1})$, Suppose $\chi = \xi \cdot |\cdot|^s$, then $\overline{\chi}^{-1} = \overline{\xi}^{-1} \cdot |\cdot|^{-\overline{s}} = \xi \cdot |\cdot|^{-\overline{s}}$ (ξ is finite order)

$$B(\chi, \overline{\chi}^{-1}) \simeq \underset{\uparrow\ \text{unitary}}{\xi} \cdot B(|\cdot|^s, |\cdot|^{-\overline{s}}) \simeq \underset{\uparrow\ \text{unitary}}{\underbrace{\xi \cdot |\cdot|^{ib}}} \cdot B(|\cdot|^a, |\cdot|^{-a})$$

$$|\cdot|^s = |\cdot|^{a+ib}, \quad |\cdot|^{-\overline{s}} = |\cdot|^{-a+ib}$$

So our essential problem is: When does $B(|\cdot|^a, |\,|^{-a})$ become unitary? For $a \in \mathbb{R}$

And we have see before that a Hermitian pairing is equivalent to an intertwing homomorphism

$$B(|\cdot|^a, |\ |^{-a}) \xrightarrow{M} B(|\cdot|^{-a}, |\cdot|^a)$$

the pairing is given by

$$(f, g) = \int_K (Mf)(k)\, \overline{g(k)}\, dk$$

we want this to be Hermitian, i.e. $(f, g) = \overline{(g, f)}$

i.e.

$$\int_K (Mf)(k)\,\overline{g(k)}\, dk \overset{?}{=} \int_K \overline{Mg(k)}\, f(k)\, dk \qquad (*)$$

both sides of $(*)$ defines a pairing.

$$B(|\cdot|^a, |\cdot|^{-a}) \times B(|\cdot|^a, |\cdot|^{-a}) \to \mathbb{C}$$

but we know this pairing is essentially unique, up to a constant. i.e.

$$\int_K (Mf)(k)\,\overline{g(k)}\, dk = C \cdot \int_K \overline{Mg(k)}\, f(k)\, dk$$

take a special f:

Iwahori decomposition

$$G = B K_0(p) \sqcup B w_0 K_0(p)$$

define
$$f_0\left(\begin{pmatrix} b_1 & x \\ & b_2 \end{pmatrix} k\right) = \left|\frac{b_1}{b_2}\right|^{a+\frac{1}{2}} \qquad (f_0(1) = 1)$$

$$f_0(w_0) = 0$$

assume $a > 0$, then

$$(Mf_0)(k) = \int_F f_0\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) dx \qquad k \in K_0(p)$$

$$\Rightarrow \langle f_0, f_0 \rangle = \mathrm{Vol}(K_0(p)) \int_F f_0\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) dx$$

$$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} x^{-1} & -1 \\ & x \end{pmatrix}\begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix} \Rightarrow f_0\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) = |x^{-1}|^{a+\frac{1}{2}} |x|^{-a-\frac{1}{2}}\, \mathbb{1}_{v(x) \le -1} = |x|^{-1-2a} \cdot \mathbb{1}_{v(x) \le -1}$$

hence $\langle f_0, f_0\rangle = Vol(K_0(p)) \int_{|x|\geq q} |x|^{-1-2a}\, dx$

$$= Vol(K_0(p)) \sum_{k=1}^{\infty} \int_{v(x)=-k} q^{-k(1+2a)}\, dx$$

$$= V \cdot \sum_{k=1}^{\infty} q^{-k(1+2a)} \int_{v(x)=-k} dx$$

$$= V \cdot \sum_{k=1}^{\infty} q^{-2ak} = V \cdot \frac{q^{-2a}}{1-q^{-2a}} \in \mathbb{R}$$

now $(f,g) = C\overline{(g,f)} \Rightarrow (f_0, f_0) = C \cdot \overline{(f_0, f_0)}$, since $(f_0, f_0) \in \mathbb{R} \Rightarrow C = 1$

therefore $(f,g) = \overline{(g,f)}$, $\Rightarrow$ we get a Hermitian form

<u>Next job</u>: find the region, s.t. $\langle \cdot, \cdot \rangle$ is positive-definite!

First: we look back on the intertwining operator $M(a)$

$$M(a): B(|\cdot|^a, |\cdot|^{-a}) \longrightarrow B(|\cdot|^{-a}, |\cdot|^a)$$

$$f \longmapsto (Mf)(g) = \int_F f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix} g\right) dx$$

we have seen that

$$f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix} g\right) = f\left(\begin{pmatrix} x^{-1} & -1 \\ & x\end{pmatrix}\begin{pmatrix} 1 & \\ x^{-1} & 1\end{pmatrix} g\right) = |x^{-1}|^{a+\frac{1}{2}} |x|^{-a-\frac{1}{2}} f\left(\begin{pmatrix} 1 & \\ x^{-1} & 1\end{pmatrix} g\right)$$

$$= |x|^{-1-2a} f\left(\begin{pmatrix} 1 & \\ x^{-1} & 1\end{pmatrix} g\right)$$

$\Rightarrow$ when $|x| \gg 0 \Rightarrow f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix} g\right) = |x|^{-1-2a} f(g)$

the integration becomes

$$(Mf)(g) = \text{something holomorphic} + \int_{|x|\geq q^N} |x|^{-1-2a} f(g)\, dx$$

$$= \cdots + f(g) \frac{q^{-2Na}}{1-q^{-2a}} \Rightarrow$$

we consider the operator $M^*(a) = (1-q^{-2a}) M(a)$ is the endomorphism of $B(1,1) \simeq St_G$

and easily $M^*(0) = 1 - q^{-1}$

Input:

Proposition 4.6.7 *We have*

$$M\phi_{K,\chi} = \frac{1-q^{-1}\alpha_1\alpha_2^{-1}}{1-\alpha_1\alpha_2^{-1}}\phi_{K,\chi'}. \tag{6.7}$$

now $\quad \alpha_1 = q^{-a}, \quad \alpha_2 = q^{a} \Rightarrow \quad M\phi_K = \dfrac{1-q^{-1-2a}}{1-q^{-2a}}\phi'_K$

$$\Rightarrow \quad \langle \phi_K, \phi_K \rangle = V \cdot \frac{1-q^{-1-2a}}{1-q^{-2a}}$$

when $a < -\frac{1}{2} \Rightarrow \langle \phi_K, \phi_K \rangle < 0 \Rightarrow a > -\frac{1}{2}$, 同 h 对称性. $-\frac{1}{2} < a < \frac{1}{2}$

the next term stops an elegant:

$$B(|\cdot|^{a}, |\cdot|^{-a}) \xrightarrow[\text{inclusion}]{\text{return}} C^{\infty}(K)$$

$$f \longmapsto f|_K$$

for all $-\frac{1}{2} < a < \frac{1}{2}$, this is an isomorphism!

hence we have is: $\quad \langle \cdot, \cdot \rangle_a^* \quad$ for $-\frac{1}{2} < a < \frac{1}{2}$

positive definite at $a = 0$

now consider an irr subrep $\quad C^{\infty}(K)(\rho) = L^2(K)(\rho)$

it is finite-dimensional, then

$\langle \cdot, \cdot \rangle_a^*(\rho)$ is a Hermitian form on it

we consider the matrix of this pairing! it is always non-degenerate.

then if it has negative eigenvalue $\Rightarrow$ it must be degenerate at some point

$-\frac{1}{2} < a < \frac{1}{2}$

which is impossible!

Theorem 4.6.7 *The representation $B(\chi_1, \chi_2)$ is unitary if and only if either χ_1 and χ_2 are unitary, or else there exists a unitary character χ_0 and a real number $-\frac{1}{2} < s < \frac{1}{2}$ such that $\chi_1(y) = \chi_0(y)|y|^s$ and $\chi_2(y) = \chi_0(y)|y|^{-s}$.*

Eisenstein series

Our goal in this section is constructing automorphic forms which are not cuspidal, these kinds of automorphic forms will orthogonal to cusp forms, and help us understand the non-cuspidal part of $L^2(G_{\mathbb{Q}}\backslash G_{\mathbb{A}}, \omega)$

Our starting point is a function space:

Def: Fix ξ_1, ξ_2 unitary Hecke character $F^\times\backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$, $s_1, s_2 \in \mathbb{C}$, define $\chi_i = \xi_i \cdot |\cdot|^{s_i}$, define $B(\chi_1, \chi_2)$ to be a function space consisting of Smooth, K-finite functions satisfying:

$$f(bg) = \left|\frac{b_1}{b_2}\right|^{\frac{1}{2}} \chi_1(b_1)\chi_2(b_2)\cdot f(g), \quad \forall b = \begin{pmatrix} b_1 & * \\ & b_2 \end{pmatrix} \in B_{\mathbb{A}}$$

Rmk: By the condition Smooth, K-finite, we immediately got that $B(\chi_1, \chi_2)$ is $GL(2, \mathbb{A})$-representation, or to be more precise, $(\mathfrak{g}, K_\infty)\times GL_2(\mathbb{A}_f)$-repn, then we can compare it with another similar space $\bigotimes_v B(\chi_{1,v}, \chi_{2,v})$

Claim: $B(\chi_1, \chi_2) = \bigotimes_v B(\chi_{1,v}, \chi_{2,v})$

pf: Obviously, $\bigotimes_v B(\chi_{1,v}, \chi_{2,v})$ makes sense, because for almost all v, $\chi_{1,v}$ & $\chi_{2,v}$ are unramified, hence they are irreducible & Spherical at almost all places, hence the restricted tensor product makes sense, and obviously,

$$\bigotimes_v B(\chi_{1,v}, \chi_{2,v}) \subseteq B(\chi_1, \chi_2)$$

Now we know Both sides are K-finite vectors of some Hilbert space, we compare the ρ-part, for $\rho \in \hat{K}$

We will show that

$$\left(\bigotimes_v B(\chi_{1,v}, \chi_{2,v})\right)(\rho) = B(\chi_1, \chi_2)(\rho) \quad \text{are both finite-dimensional}$$

Since K is compact, and $K = \prod_v K_v$, we get $\rho \simeq \bigotimes_v \rho_v$ (almost all ρ_v is trivial)

Define S a finite subset, s.t. $\forall v \notin S$, $B(\chi_{1,v}, \chi_{2,v})$ is irr spherical & ρ_v is trivial, then we get

$$\left(\bigotimes_v B(\chi_{1,v}, \chi_{2,v})\right)(\rho) = \left(\bigotimes_{v\in S} B(\chi_{1,v}, \chi_{2,v})\right)\left(\bigotimes_{v\in S} \rho_v\right)$$

let's now identify $B(\chi_1, \chi_2)(\rho)$, $\forall \phi \in B(\chi_1, \chi_2)(\rho)$, we know

$$\phi(\mathring{g}^v \cdot g_v) \in B(\chi_{1,v}, \chi_{2,v})^{K_v}, \text{ which is one-dimensional if } v \notin S$$

$$\mathring{g}^v = (g_\infty, g_{v'}, \ldots, \overset{v}{1}, \ldots)$$

therefore $\phi(\mathring{g}^v \cdot g_v) = \tilde{\phi}(\mathring{g}^v)\cdot f_v(g_v)$, where $f_v \in B(\chi_{1,v}, \chi_{2,v})^{K_v}$, then since for $\forall g \in GL(2, \mathbb{A})$, $g_v \in K_v$ for almost $\forall v$, we get

$$\phi(g) = \phi_S(g_S)\cdot \bigotimes_{v\notin S} f_v(g_v), \text{ where obviously } \phi_S \in B(\chi_{1,S}, \chi_{2,S})(\rho_S)$$

therefore we only need to show $B(\bigotimes_{v\in S}\chi_{1,v}, \bigotimes_{v\in S}\chi_{2,v})(\bigotimes_{v\in S}\rho_v) = \left(\bigotimes_{v\in S} B(\chi_{1,v}, \chi_{2,v})\right)\left(\bigotimes_{v\in S}\rho_v\right)$

$\forall \phi \in$ LHS, $\phi(\mathring{g}^v\cdot g_v) \in \underbrace{B(\chi_{1,v}, \chi_{2,v})(\rho_v)}_{\substack{\text{finite-dimensional} \\ \text{has a basis } \phi_1^v, \ldots, \phi_{n_v}^v}} \Rightarrow \phi(\mathring{g}^v \cdot g_v) = \sum_{i=1}^{n_v} \tilde{\phi}_i(\mathring{g}^v)\cdot \phi_{n_v}^v(g_v)$

doing this successively, we get: $\phi(g) = \sum C_{\vec{i}} \bigotimes_{v\in S} \phi_i^v(g_v) \in \left(\bigotimes_{v\in S} B(\chi_{1,v}, \chi_{2,v})\right)\left(\bigotimes_{v\in S}\rho_v\right)$

$$B\left(\bigotimes_{v\in S}\chi_{1,v}, \bigotimes_{v\in S}\chi_{2,v}\right)\left(\bigotimes_{v\in S}\rho_v\right) = \left(\bigotimes_{v\in S} B(\chi_{1,v}, \chi_{2,v})\right)\left(\bigotimes_{v\in S}\rho_v\right) = \bigotimes_{v\in S}\left(B(\chi_{1,v}, \chi_{2,v})(\rho_v)\right)$$

Next, we will define a $GL(2,\mathbb{A})$-equivariant map

$$B(\chi_1, \chi_2) \longrightarrow \mathcal{A}(G_F \backslash G_{\mathbb{A}}, \omega)$$

We will first define this when $Re(s_1 - s_2) > 1$ for convergence consideration, then we will show this map can be meromorphically continued

<u>Def</u>: (Eisenstein series map)

$$f \longmapsto E(g,f) = \sum_{\underbrace{\gamma \in B_F \backslash G_F}} f(\gamma g)$$

different choice of B_F give different map

Prop: 1. The summation is absolutely convergent when $Re(s_1 - s_2) > 1$

2. When $Re(s_1 - s_2) > 1$, $E(g,f) \in \mathcal{A}(G_F \backslash G_{\mathbb{A}}, \chi_1\chi_2)$, and this map is $GL(2,\mathbb{A})$-equivariant

pf: ($F = \mathbb{Q}$). We first identify $B_{\mathbb{Q}} \backslash G_{\mathbb{Q}}$, since $G = B \amalg B\omega_0 N$, we have $\forall g \in G$, $\exists \gamma \in B$, $\gamma g = \omega_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}$

Suppose $x = \frac{m}{n}$, where $(m,n) = 1$, let $\gamma' = \begin{pmatrix} n^{-1} & c \\ & n \end{pmatrix}$, where $cm \equiv 1 \mod n$, then $\gamma'\gamma g \in SL_2(\mathbb{Z}) = \Gamma$

therefore $B_{\mathbb{Q}} \backslash G_{\mathbb{Q}} = \Gamma_\infty \backslash \Gamma$, $\Gamma_\infty = \Gamma \cap B_{\mathbb{Q}}$

Now by the identication: $B(\chi_1, \chi_2) = \bigotimes_v B(\chi_{1,v}, \chi_{2,v})$, WLOG, suppose $f = \prod_v f_v$, here $f_v \in B(\chi_{1,v}, \chi_{2,v})$, and for almost all places, $f_v = f_v^\circ$ is the unique K_v-fixed vector, s.t. $f_v(K_v) = 1$. Suppose $g = (g_v)$, then $g_v \in K_v$ for almost $\forall v$

$$\gamma \cdot g = (\gamma g_\infty, \gamma g_v)$$

since K_v is compact, $\exists B_v > 0$, and almost all $B_v = 1$, s.t. $|f(K_v g_v)| \le B_v$, then

$$|f(\gamma g)| = |f_\infty(\gamma g_\infty) \cdot \prod_{v<\infty} f_v(\gamma g_v)| \le \left(\prod_{v<\infty} B_v\right) \cdot |f_\infty(\gamma g_\infty)| = B \cdot |f_\infty(\gamma g_\infty)|$$

$f_\infty \in \pi_\infty(\chi_{1,\infty}, \chi_{2,\infty})$ and K-finite, WLOG, we assume $f_\infty(g\kappa_\theta) = e^{ik\theta} f_\infty(g)$.

Now suppose
$$g_\infty = \begin{pmatrix} u & \\ & u \end{pmatrix}\begin{pmatrix} y & x \\ & 1 \end{pmatrix} \kappa_\theta, \quad u > 0,\ y \neq 0,\ x \in \mathbb{R}$$

if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = SL_2(\mathbb{Z})$, then

$$\gamma g_\infty = \begin{pmatrix} u' & \\ & u' \end{pmatrix}\begin{pmatrix} y' & x' \\ & 1 \end{pmatrix} \kappa_{\theta'}, \quad u' > 0,\ y' \neq 0,\ x' \in \mathbb{R}$$

$$|f_\infty(\gamma g_\infty)| = |\chi_{1,\infty}(u'y') \cdot \chi_{2,\infty}(u')| \cdot |y'|^{\frac{1}{2}} \cdot |f_\infty(\kappa_{\theta'})| \le B_\infty \cdot |u'|^{Re(s_1+s_2)} \cdot |y'|^{Re(s_1)+\frac{1}{2}}$$

where $\frac{az+b}{cz+d} = \gamma g_\infty(i) = \gamma z$, hence $y' = Im(\gamma z) = Im\left(\frac{az+b}{cz+d}\right) = \frac{y}{|cz+d|^2}$

and $u^2 \cdot y = u'^2 \cdot y' \Rightarrow u' = u \cdot (y/y')^{\frac{1}{2}} = u \cdot |cz+d|$, hence

$$|f_\infty(\gamma g_\infty)| \le B_\infty \cdot u^{Re(s_1+s_2)} \cdot |cz+d|^{Re(s_1+s_2)} \cdot \frac{|y|^{Re(s_1)+\frac{1}{2}}}{|cz+d|^{2Re(s_1)+1}} = B_\infty \cdot u^{Re(s_1+s_2)} \cdot |y|^{Re(s_2)} \frac{|y|^r}{|cz+d|^{2r}} = B_\infty(g) \frac{|y|^r}{|cz+d|^{2r}}$$

here $r = \frac{1}{2}(Re(s_1 - s_2) + 1)$, therefore,

$$|E(g,f)| = \left|\sum_{\gamma \in B_F \backslash G_F} f(\gamma g)\right| \le \sum_{\Gamma_\infty \backslash \Gamma} |f_\infty(\gamma g_\infty)| \le B_\infty(g) \cdot \sum_{\Gamma_\infty \backslash \Gamma} \frac{|y|^r}{|cz+d|^{2r}},$$ it converges iff $r > 1$

Now let's show $E(g,f) \in \mathcal{A}(G_F \backslash G_{\mathbb{A}}, \chi_1\chi_2)$

- invariance under G_F is easily verified
- K-finite: follow from f is K-finite
- Z-finite: only need to show f is Z-finite, this is because $B(\chi_{1,\infty}, \chi_{2,\infty})$ is an irr $(\mathfrak{g}, K)$-mod, hence Δ acts by scalar
- moderate growth: easy to see that, over any compact subset $\omega \subseteq G(\mathbb{A})$, we have a universe bound $B_\omega(g)$, i.e.

$$|E(g,f)| \le B_\omega \cdot |y|^r, \ \forall g \in \omega$$

this is condition [Gelbart]', which is equivalent to moderate growth condition □

Now we have defined Eisenstein series for $Re(s_1 - s_2) > 1$, we would like to extend (meromorphically continue) this functions to the whole $\mathbb{C}^2$. We will prove this fact by finding Fourier expansions.
As usual, since $E(g,f) \in \mathcal{A}(G_F \backslash G_{\mathbb{A}}, \omega)$ is invariant under G_F, then we consider the following function over $\mathbb{A}/F$

$$x \in \mathbb{A}/F \longmapsto E\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g, f\right)$$

then $$E\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g, f\right) = \sum_{\alpha \in F} c_\alpha(g,f)\,\psi(\alpha x), \text{ where } c_\alpha(g,f) = \int_{\mathbb{A}/F} E\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g, f\right)\psi(-\alpha x)\,dx$$

We now analysis those Fourier coefficients $c_\alpha(g,f)$

$$c_\alpha(g,f) = \int_{\mathbb{A}/F} E\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g, f\right)\psi(-\alpha x)\,dx = \int_{\mathbb{A}/F} \sum_{\gamma \in B_F\backslash G_F} f\left(\gamma \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)\psi(-\alpha x)\,dx$$

$$\overset{\text{abs conver}}{=\!=\!=} \sum_{\gamma \in B_F \backslash G_F} \int_{\mathbb{A}/F} f\left(\gamma \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)\psi(-\alpha x)\,dx \underset{G_F = B_F \sqcup B_F w_0 N_F}{=} \int_{\mathbb{A}/F} f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)\psi(-\alpha x)\,dx + \sum_{\lambda \in F} \int_{\mathbb{A}/F} f\left(w_0 \begin{pmatrix} 1 & x+\lambda \\ & 1 \end{pmatrix} g\right)\psi(-\alpha x)\,dx$$

$$= \delta_\alpha^0 \cdot f(g) + \int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)\psi(-\alpha x)\,dx$$

this suggests that we can only focus on the analytic continuation of the following form: $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \alpha^{-1} & \\ & 1 \end{pmatrix} = \begin{pmatrix} \alpha^{-1} & x \\ & 1 \end{pmatrix}$

$$\int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)\psi(-\alpha x)\,dx = \begin{cases} \displaystyle\int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & \alpha^{-1}x \\ & 1 \end{pmatrix} g\right)\psi(-x)\,dx = \int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\right)\psi(-x)\,dx, & \alpha \ne 0, \\ \displaystyle\int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)dx & \alpha = 0. \end{cases}$$

We define the Whittaker model for $B(\chi_1, \chi_2)$ as the following:

$$W(g,f) = \int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)\psi(-x)\,dx$$

then we know $$E(g,f) = \sum_{\alpha \in F} c_\alpha(g,f) = f(g) + \overset{①}{\int_{\mathbb{A}} f\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)dx} + \overset{②}{\sum_{\alpha \in F^\times} W\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g, f\right)}$$

Since f is a linear combination of pure tensors, WLOG, we assume $f = \otimes_v f_v$, where almost all f_v is K_v-invariant, $f_v(K_v) = 1$

Now let's consider the "constant term" ①, we only focus on the NA place in this page

$$\int_{\mathbb{A}} f(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g)\,dx = \int_{\mathbb{A}} \prod_v f_v(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}g_v)\,dx$$

$$= \prod_v \int_{F_v} f_v(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}g_v)\,dx_v = \prod_v \int_{F_v} (\rho_v(g_v)f_v)(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix})\,dx_v$$

here f_v, $\rho_v(g_v)f_v \in B(\chi_{1,v}, \chi_{2,v})$, then when $|x_v|\to+\infty$, we have $\rho_v(g_v)f_v(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}) = \xi_{1,v}^{-1}\xi_{2,v}(x_v)\cdot|x_v|^{s_2-s_1-1} f_v(g_v)$

so there is a uniform N, s.t. when $v(x_v)\le -N$, then

$$\int_{F_v} (\rho(g_v)f_v)(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix})\,dx = f(g_v)\int_{v(x_v)\le -N} \xi_{1,v}^{-1}\xi_{2,v}(x_v)\cdot|x_v|^{s_2-s_1}\,d^\times x_v + \int_{v(x_v)\ge -N} f_v(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}g_v)\,dx_v$$

$$= \begin{cases} \displaystyle\int_{v(x)\ge -N} f_v(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}g_v)\,dx_v, & \text{when } \xi_{1,v}^{-1}\xi_{2,v} \text{ is ramified} \\ \displaystyle\frac{f(g_v)\,(\alpha^{-1}q^{s_2-s_1})^N}{1-\alpha^{-1}q^{s_2-s_1}} + \int_{v(x)\ge -N} f_v(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}g_v)\,dx & \text{when } \xi_{1,v}^{-1}\xi_{2,v} \text{ is unramified} \\ & \alpha = \xi_{1,v}^{-1}\xi_{2,v}(\pi_v) \end{cases}$$

Seemingly, we should take this expression into ② and see what happened, but actually our life is almost much easier! since for almost every place v, we have $f_v \in B(\chi_{1,v}, \chi_{2,v})^{K_v}$, and $f_v(K_v)=1$, i.e. f_v is the normalized spherical vector, then

$$\int_{F_v} f_v(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}g_v)\,dx_v = M(s)f_v = \frac{L_v(2s-1, \xi_{1,v}\xi_{2,v}^{-1})}{L_v(2s, \xi_{1,v}\xi_{2,v}^{-1})}\tilde{f}_v, \quad \tilde{f}_v \text{ is the normalized spherical vector in } B(\chi_{2,v}, \chi_{1,v})$$

hence except a finite set S, we can identify ① by:

$$\int_{\mathbb{A}} f(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g)\,dx = \prod_{v\in S}\int_{F_v} f_v(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}g_v)\,dx_v \cdot \prod_{v\notin S}\int_{F_v} f_v(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}g_v)\,dx_v$$

$$= \prod_{v\in S}\int_{F_v} f_v(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}g_v)\,dx_v \cdot \prod_{v\notin S}\frac{L_v(2s-1, \xi_{1,v}\xi_{2,v}^{-1})}{L_v(2s, \xi_{1,v}\xi_{2,v}^{-1})}\tilde{f}_v(g_v)$$

this suggests us to "normalize" $C_0(g,f)$ by multiplying $L_S(2s, \xi_1\xi_2^{-1})$, so we define $C_0^*(g,f) = L_S(2s, \xi_1\xi_2^{-1})\,C_0(g,f)$, we got

$$②\text{-part of } C_0^*(g,f) = \prod_{v\in S}\int_{F_v} f_v(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}g_v)\,dx_v \cdot \prod_{v\notin S} L_v(2s-1, \xi_{1,v}\xi_{2,v}^{-1})\,\tilde{f}_v(g_v)$$

$$= L(2s-1, \xi_1\xi_2^{-1})\cdot \underbrace{\prod_{v\in S} L_v(2s-1, \xi_{1,v}\xi_{2,v}^{-1})^{-1}\int_{F_v} f_v(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}g_v)\,dx_v}_{\text{every single term is entire in } s_1, s_2} \cdot \prod_{v\notin S}\tilde{f}_v(g_v) \quad (*)$$

· possible poles for the constant term $C_0(g,f)$, suppose $\xi_1\xi_2^{-1} = |\cdot|^{\nu}$,

$$C_0^*(g,f) = \underbrace{L_S(2s, \xi_1\xi_2^{-1})}_{\substack{\text{possible pole at} \\ 2s+\nu=0,\,1}} f(g) + \underbrace{(*)}_{\substack{\text{possible pole at} \\ 2s-1+\nu=0,\,1}}$$

$$s+\frac{\nu}{2} = \begin{cases} 0 \\ \frac{1}{2} \end{cases} \qquad s+\frac{\nu}{2} = \begin{cases} 1 \\ \frac{1}{2} \end{cases} \qquad \text{we will show that the possible pole at } s+\frac{\nu}{2}=\frac{1}{2} \text{ cancels}$$

Now we analyze the last term ②, we only focus on NA place in this page, by definition

$$W(g,f) = \int_{A} f\left(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix} g\right)\psi(-x)\,dx = \int_{A}\prod_v f_v\left(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix} g_v\right)\psi_v(-x_v)\,dx$$

$$= \prod_v \int_{F_v} f_v\left(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix} g_v\right)\psi_v(-x_v)\,dx_v = \prod_v \int_{F_v}(\rho_v(g_v)f_v)\left(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}\right)\psi_v(-x_v)\,dx_v = \prod_v W_v(g_v, f_v)$$

and for $\forall\,\alpha\in F^\times$

$$W\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}g, f\right) = \prod_v\int_{F_v} f_v\left(w_0\begin{pmatrix}\alpha & x_v\\ & 1\end{pmatrix}g_v\right)\psi_v(-x_v)\,dx_v = \prod_v\int_{F_v} f_v\left(\begin{pmatrix}1 & \\ & \alpha\end{pmatrix}w_0\begin{pmatrix}1 & \alpha^{-1}x_v\\ & 1\end{pmatrix}g_v\right)\psi_v(-x_v)\,dx_v$$

$$= \prod_v \chi_{2,v}(\alpha)|\alpha|_v^{\frac12}\int_{F_v} f_v\left(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}g_v\right)\psi_v(-\alpha x_v)\,dx_v = \prod_v\int_{F_v}\left(\rho(g_v)f_v\right)\left(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}\right)\psi_v(-\alpha x_v)\,dx_v$$

here $f_v,\ \rho_v(g_v)f_v\in B(\chi_{1,v},\chi_{2,v})$, then when $|x_v|\to+\infty$, we have $\rho_v(g_v)f_v\left(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}\right) = \xi_{1,v}^{-1}\xi_{2,v}(x_v)\cdot|x_v|^{s_2-s_1-1}f_v(g_v)$

so there is a uniform N, s.t. when $v(x_v)\le -N$, then

$$\int_{F_v}\left(\rho(g_v)f_v\right)\left(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}\right)\psi_v(-\alpha x_v)\,dx = \int_{v(x_v)\le -N}\xi_{1,v}^{-1}\xi_{2,v}(x_v)\cdot|x_v|^{s_2-s_1}\psi_v(-\alpha x_v)\,d^\times x_v + \underbrace{\int_{v(x_v)\ge -N} f_v\left(w_0\begin{pmatrix}1 & x_v\\ & 1\end{pmatrix}g_v\right)\psi_v(-\alpha x_v)\,dx_v}_{\text{integral over compact subset, must be analytic}}$$

Now we focus on the first integration

Claim: When $N>>0$, $\displaystyle\int_{v(x_v)\le -N}\xi_{1,v}^{-1}\xi_{2,v}(x_v)\cdot|x_v|^{s_2-s_1}\psi_v(-\alpha x_v)\,d^\times x_v = 0$

pf: focus on $v(x_v)=-k$, when $k>>0$, and consider $\psi_v(\alpha\,\cdot\,): F_v\to\mathbb{C}^\times$ is nontrivial over $\mathfrak{a}$

then when $v(x)=v(y)=-k$, $x-y\in\mathfrak{a}\Rightarrow 1-x^{-1}y\in x^{-1}\mathfrak{a}\rightsquigarrow x^{-1}y\in 1+\mathfrak{p}^{k-m}$, for some m

when $k>>0$, $k-m>>0$, then $\xi_{1,v}^{-1}\xi_{2,v}(x)\cdot|x|^{s_2-s_1+1} = \xi_{1,v}^{-1}\xi_{2,v}(y)\cdot|y|^{s_2-s_1+1}$, i.e. on $x+\mathfrak{a}$, the multiplicative character part is constant. we only focus on $\int_{x+\mathfrak{a}}\psi_\alpha(-x_v)\,dx_v$, we claim this integration is 0

because $\displaystyle\int_{x+\mathfrak{a}}\psi_\alpha(-x_v)\,dx_v = \psi_\alpha(y)\int_{x+\mathfrak{a}}\psi_\alpha(-x_v)\,dx_v$, for $y\in\mathfrak{a}$. $\Rightarrow$ Since ψ_α is nontrivial at $\mathfrak{a}$, this is 0

Therefore, every single part in the product is holomorphic, but we almost can't say anything for the convergence of the product. Recall what we have done in constant term $C_0(g,f)$ case, we identify almost all terms appearing in the product as another privilliged function, hence the product automatically makes sense. Now we would like to do the same thing.

We state the extra result we need:

Lemma 1: When f_v is normalized spherical vector, then $L_v(2s,\xi_{1,v}\xi_{2,v}^{-1})W_v$ is the normalized spherical vector W_v°

With the help of this lemma, we immediately get:

$$L_S(2s,\xi_1\xi_2^{-1})\,W(g,f) = \underbrace{\prod_{v\in S}W_v(g_v,f_v)}\cdot\underbrace{\prod_{v\notin S}W_v^\circ(g_v,f_v)}$$

automatically analytic

Next, we shall show that the following summation

$$\sum_{\alpha \in F^\times} W\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g, f \right) \quad \text{is absolutely convergent for any choice of } s_1, s_2$$

then this will imply that $E^*(g, f) = L_S(2s, \xi_1 \xi_2^{-1}) E(g, f)$ has meromorphic continuation

---

We have shown that $E(g, f) \in \mathcal{A}(G_F \backslash G_{\mathbb{A}}, \omega)$, let's compute its "constant term"

$$\int_{F \backslash \mathbb{A}_F} E\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g, f \right) dx = \int_{F \backslash \mathbb{A}_F} \sum_{\alpha \in F} C_\alpha(g, f) \psi(\alpha x)\, dx = C_0(g, f)$$

i.e. $C_0(g, f)$ is the "constant term" for automorphic forms we have defined before, so we consider

$$\tilde{E}(g, f) = E(g, f) - C_0(g, f)$$

then $\tilde{E}$ satisfies:

- G_F-invariant
- K-finite
- Z-finite
- moderate growth
- cuspidal condition

$\Rightarrow \tilde{E}(g, f) \in \mathcal{A}_0(G_F \backslash G_{\mathbb{A}}, \omega)$ is a cuspidal automorphic form

and calculate the Fourier exponsion of $\tilde{E}$, and by rapid decay of cusp forms, the summation must be absolutely convergent
But this is just the heuristic, because up to now, we can say nothing about the analytic continuation of E

Previous tells us that $\exists N \in \mathbb{Z}$, the summation term is nonzero only if $N\alpha \in \mathcal{O}_F$. now we prove this is our case

Lemma 2 : 1 For NA place v, suppose $B(\chi_{1,v}, \chi_{2,v})$ is spherical irreducible. W_v to be the normalized Whittaker function, we have:

$$L_v(2s, \xi_{1,v} \xi_{2,v}^{-1})\, W_v\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix} k \right) = \begin{cases} \psi_v(x) (\alpha_1 \alpha_2)^{m_2} q^{-\frac{m}{2}} \dfrac{\alpha_1^{m+1} - \alpha_2^{m+1}}{\alpha_1 - \alpha_2}, & \text{if } m \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad \text{(Thm 4.6.5)}$$

here $m_i = \mathrm{ord}(y_i)$, $m = m_1 - m_2$, $\alpha_i = q^{-s_i} \xi_i(\pi)$

2. For $\forall$ NA place v, $\exists C_v > 0$, s.t. when $|\alpha|_v \geq C_v$, $W_v\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g_v, f_v \right) = 0$

now 1 tells us that we could chose C_v to be 1 for almost all v

The rapid decay property is guaranteed by Lemma 2 (NA) and the rapid decay property of Archimedean place

Let's now go to the Archimedean place, to complete our proof, it remains to show

- $\int_{F_\infty} f_\infty\left(w_0\begin{pmatrix}1&x\\&1\end{pmatrix}g_\infty\right)\underbrace{\psi_\infty(-\alpha x)}_{\text{we can omit it}}\,dx$ has analytic continuation w.r.t s_1 & s_2
- absolute convergence of $\sum_{\alpha\in F^*}$

suppose $F_\infty=\mathbb{R}$, then since f_∞ is K-finite, WLOG, we assume

$$f_\infty(ge^{i\theta}) = e^{ik\theta}f_\infty(g),\ \forall g\in GL_2(\mathbb{R})$$

now by Iwasawa decomposition, (WLOG, we assume $g_\infty\in GL_2(\mathbb{R})^+$, because we can modify f by $\begin{pmatrix}&1\\1&\end{pmatrix}$)

$$g_\infty=\begin{pmatrix}\zeta&\\&\zeta\end{pmatrix}\begin{pmatrix}\eta^{\frac12}&\eta^{-\frac12}\xi\\&\eta^{-\frac12}\end{pmatrix}\kappa_\phi$$

then

$$f_\infty\left(w_0\begin{pmatrix}1&x\\&1\end{pmatrix}g_\infty\right) = (\chi_1^{-1}\chi_2)(\eta^{\frac12})\,|\eta|^{-1}(\chi_1\chi_2)(\zeta)\,e^{ik\phi}f_\infty\begin{pmatrix}1&\eta^{-1}(x+\xi)\\&1\end{pmatrix}$$

now in the intergration, make the change of variable: $x\to\eta x-\xi$, we are reduced to prove these two results for:

$$\int_{\mathbb{R}} f_\infty\left(w_0\begin{pmatrix}1&x\\&1\end{pmatrix}\right)\overbrace{\psi_\infty(-\eta x)}^{\text{we can omit}}\,dx$$

so suppose $\psi_\infty(-\eta x)=e^{i\lambda x}$, for some $\lambda\in\mathbb{R}$ (ψ on $\mathbb{A}/F\to\mathbb{C}^\times$, must be unitary, since $\mathbb{A}/F$ is compact)

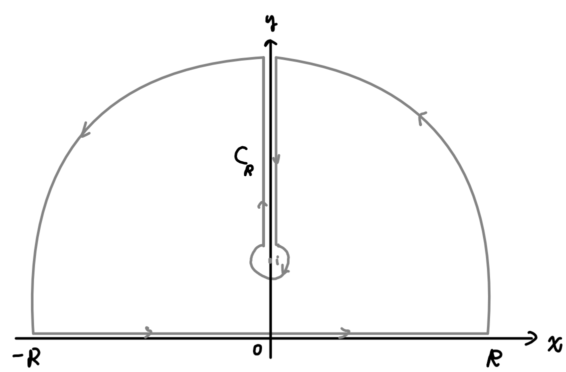
then using Iwasawa decomposition

$$w_0\begin{pmatrix}1&x\\&1\end{pmatrix}=\begin{pmatrix}1&\\&1\end{pmatrix}\begin{pmatrix}\Delta^{-1}&-x\Delta^{-1}\\&\Delta\end{pmatrix}\begin{pmatrix}x\Delta^{-1}&-\Delta^{-1}\\\Delta^{-1}&x\Delta^{-1}\end{pmatrix},\quad \Delta=\sqrt{1+x^2}$$

$$f_\infty\left(w_0\begin{pmatrix}1&x\\&1\end{pmatrix}\right) = \chi_2\chi_1^{-1}(\Delta)\cdot\Delta^{-1}\cdot\left(\frac{x-i}{\sqrt{x^2+1}}\right)^k = (x^2+1)^{-s-\frac{k}{2}}\cdot(x-i)^k$$

therefore the integration becomes

$$\int_{\mathbb{R}}(x^2+1)^{-s-\frac{k}{2}}\cdot(x-i)^k e^{i\lambda x}\,dx = \lim_{R\to+\infty}\int_{-R}^{R}(x^2+1)^{-s-\frac{k}{2}}(x-i)^k e^{i\lambda x}\,dx$$



assume $\lambda>0$, suppose $Re(s)>0$, then over the circle, $\left|(x^2+1)^{-s-\frac{k}{2}}(x-i)^k e^{i\lambda x}\right|\le(R^2+1)^{-Re(s)}$

then the integration over the half cirle $\le(R^2+1)^{Re(s)}\cdot\pi R\to0$, when $R\to+\infty$

therefore $\int_{\mathbb{R}}(x^2+1)^{-s-\frac{k}{2}}\cdot(x-i)^k e^{i\lambda x}\,dx = \lim_{R\to+\infty}\int_{C_R}(x^2+1)^{-s-\frac{k}{2}}\cdot(x-i)^k e^{i\lambda x}\,dx$

the limit at RHS always exitss, because $e^{i\lambda x}$ decay rapidly when $x\to i\infty$, Moreover, in this case, the integral on the RHS converges absolutely for $\forall s\in\mathbb{C}$, i.e. $\forall s_1,s_2\in\mathbb{C}$, this gives the analytic continuation of W_∞.

the same argument applies to $\lambda<0$, but we chould take the circle on the $\mathcal{H}^-=\{x+iy\mid y<0\}$

therefore we prove that W_∞ has analytic continuation to all $s_1,s_2\in\mathbb{C}$, it remains to show the rapid decay property

We have proved that

$$W^*(g,f) = L_S(2s, \xi_1\xi_2^{-1})\, W(g,f) = \prod_{v\in S} W_v(g_v, f_v)\cdot \prod_{v\notin S} W_v^\circ(g_v, f_v)$$

By the formula for W_v°:

$$W_v^\circ(g_v, f_v) = L_v(2s, \xi_{1,v}\xi_{2,v}^{-1})\, W_v\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}y_1 & \\ & y_2\end{pmatrix}k, f_v\right) = \begin{cases} \psi_v(x)\,(\alpha_1\alpha_v)^{m_2} q^{-\frac{m}{2}} \dfrac{\alpha_1^{m+1}-\alpha_2^{m+1}}{\alpha_1-\alpha_v}, & \text{if } m\geq 0 \\ 0 & \end{cases}$$

hence we get the following equality for almost every place, (since almost every place, $B(\chi_{1,v},\chi_{2,v})$ spherical, irreducible, $g_v\in K_v$)

$$\left| W_v^\circ\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}g_v, f_v\right)\right| = q^{-\frac{m}{2}}\cdot\left|\frac{\alpha_1^{m+1}-\alpha_2^{m+1}}{\alpha_1-\alpha_2}\right| \leq q^{-\frac{m}{2}}\max_{i=1,2}\{|\alpha_i|\}^m\cdot(m+1) \leq q^{\frac{m}{2}}\cdot q^{-m\cdot\min_{i=1,2}\{Re(s_i)\}} = |\alpha|_v^{\min_{i=1,2}\{Re(s_i)\}-\frac{1}{2}}$$

for those remaining NA places, since α only valued in a subset like $v(\alpha)\geq -N$, then we can try to find the upper bound for:

$$|\alpha|_v^{\frac{1}{2}-\min_{i=1,2}\{Re(s_i)\}}\cdot\left| W_v^\circ\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}g_v, f_v\right)\right|, \quad \text{for } v(\alpha)\geq -N$$

we know that when $|\alpha|\to 0$, $W_v^\circ\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}g_v, f_v\right) = C_1\xi_{1,v}(\alpha)|\alpha|^{s_1+\frac{1}{2}} + C_2\xi_{2,v}(\alpha)|\alpha|^{s_2+\frac{1}{2}}$, then LHS is bounded when $|\alpha|\to 0$

therefore we get a constant $C_v>0$, and almost all $C_v=1$. s.t. for $\forall$ NA place v,

$$\left| W_v^\circ\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}g_v, f_v\right)\right| \leq C_v\cdot|\alpha|_v^{\min_{i=1,2}\{Re(s_i)\}-\frac{1}{2}}$$

this also holds for those remaining $v\in S$, therefore we get:

$$\left|\prod_{v\in S} W_v\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}g_v, f_v\right)\cdot\prod_{v\notin S} W_v^\circ\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}g_v, f_v\right)\right| \leq \left|W_\infty\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}g_\infty, f_\infty\right)\right|\cdot\prod_{v<\infty} C_v\cdot|\alpha|_v^{\min_{i=1,2}\{Re(s_i)\}-\frac{1}{2}}$$

$$= C_f\, |\alpha|_\infty^{\frac{1}{2}-\min_{i=1,2}\{Re(s_i)\}}\cdot\left|W_\infty\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}g_\infty, f_\infty\right)\right|$$

now we know that

$$W_\infty\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}g_\infty, f_\infty\right) = \int_{F_\infty} f_\infty\left(w_0\begin{pmatrix}\alpha & x\\ & 1\end{pmatrix}g_\infty\right)\psi_\infty(-x)\,dx = \int_{F_\infty}\chi_{2,\infty}(\alpha)\cdot|\alpha|_\infty^{-\frac{1}{2}}\cdot f_\infty\left(w_0\begin{pmatrix}1 & \alpha^{-1}x\\ & 1\end{pmatrix}g_\infty\right)\psi_\infty(-x)\,dx$$

$$= \xi_{2,\infty}(\alpha)|\alpha|_\infty^{s_2+\frac{1}{2}}\int_{F_\infty} f_\infty\left(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g_\infty\right)\psi_\infty(-\alpha x)\,dx$$

this last formula equals to

$$\int_{\mathbb{R}}(x^2+1)^{-s-\frac{k}{2}}(x-i)^k e^{i\lambda\alpha x}\,dx = \lim_{R\to\infty}\int_{C_R}(x^2+1)^{-s-\frac{k}{2}}(x-i)^k e^{i\lambda\alpha x}\,dx \quad, C_R \text{ is depicted in the previous page}$$

$$\int_{C_R}\left|(x^2+1)^{-s-\frac{k}{2}}(x-i)^k e^{i\lambda\alpha x}\right|dx \leq \int_{C_R}\left|(x^2+1)^{-s-\frac{k}{2}}(x-i)^k e^{\frac{i\lambda\alpha x}{2}}\right|\cdot e^{-\frac{\lambda|\alpha|_\infty}{2}}\,dx = e^{-\frac{\lambda}{2}|\alpha|_\infty}\cdot\underbrace{\int_{C_R}\left|(x^2+1)^{-s-\frac{k}{2}}(x-i)^k e^{\frac{i\lambda\alpha x}{2}}\right|dx}$$

↳ when $|x|_\infty\to+\infty$, goes to zero, hence bounded uniformly by some constant C_∞

therefore $\left|W^*\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}g, f\right)\right| \leq C_f\cdot C_\infty\cdot|\alpha|_\infty^{\frac{1}{2}-\min_{i=1,2}\{Re(s_i)\}}\cdot e^{-\frac{\lambda}{2}|\alpha|_\infty}$, then the summation converges absolutely

Now, the meromorphic continuation of $E^*(g,f)$ has been proved with the help of Lemma 1 & Lemma 2 & local intertwining operator, Let's briefly review what we have proved, the starting point is the following Fourier expansion

$$E(g,f) = \sum_{\alpha\in F} C_\alpha(g,f) = f(g) + \int_A f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g)\,dx + \sum_{\alpha\in F^\times} W(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g, f)$$

What we have shown is, the normalized Eisenstein series:

$$E^*(g,f) = L_s(2s, \xi_1\xi_2^{-1})\,E(g,f) = \underset{①}{C_0^*(g,f)} + \underset{②}{\sum_{\alpha\in F^\times} W^*\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g, f\right)}$$

Both ① and each term in ③ has analytic continuation to the whole $\mathbb{C}$-plane, we normalize by $L_s(2s, \xi_1\xi_2^{-1})$ in order to disregard the convergence property of infinite product, this was done with the help of Lemma 1, the next step is to show that the summation ② is absolutely convergent, this was done by Lemma 2 in NA case and the analysis of the integral at the infinite place.

We also show that ③ is analytic w.r.t $\forall s_1, s_2 \in \mathbb{C}$, and the only possible poles appear in the $C_0^*(g,f)$ term and they comes from the poles of $L(2s, \xi_1\xi_2^{-1})$ and $L(2s-1, \xi_1\xi_2^{-1})$, Let's now look more closely at these poles

Recall our previous analysis:

possible poles for the constant term $C_0(g,f)$, suppose $\xi_1\xi_2^{-1} = |\cdot|^{v}$, v is purely imaginary

$$C_0^*(g,f) = \underbrace{L_s(2s, \xi_1\xi_2^{-1})}_{\alpha} f(g) + \underbrace{L_s(2s, \xi_1\xi_2^{-1}) \int_A f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g)\,\psi(-x)\,dx}_{\beta}$$

possible pole at $2s+v = 0, 1$, $\quad s+\frac{v}{2} = \begin{cases} 0 \\ \frac12 \end{cases}$

possible pole at $2s-1+v = 0, 1$, $\quad s+\frac{v}{2} = \begin{cases} 1 \\ \frac12 \end{cases}$

now we would like to show that the possible pole at $s+\frac{v}{2} = \frac12$ cancels, i.e. the residue at $s+\frac{v}{2} = \frac12$ cancells

It's easily seen that $\alpha \in \pi(\chi_1, \chi_2)$, $\beta \in \pi(\chi_2, \chi_1)$ (should combine the NA case & A case together)

when $s+\frac{v}{2} = \frac12$, $\chi_1 = \chi_2$, then α & β "should" live in $\pi(\chi_1, \chi_1)$, but they have possible poles, it is not themselves in $\pi(\chi_1, \chi_2)$ rather their "residue" live in there.

<u>Claim</u>: $R(g)$ = residue of $E^*(g,f)$ at $s+\frac{v}{2} = \frac12$, then $R(g)$ is an automorphic form

pf: Obviously $R(\gamma g) = R(g)$, $\forall \gamma \in G_F$, K-finiteness is trivial to verify, Let's consider Z-finiteness & moderate growth

By definition $R(g) = \lim_{s\to\frac{1-v}{2}} (s - \frac{1-v}{2})\, E^*(g,f) = \tilde{E}(g,f)(s_1,s_2)$

↳ meromorphic function evaluated at s_1, s_2, s.t. $s_1 - s_2 + 1 = 2(1-v)$

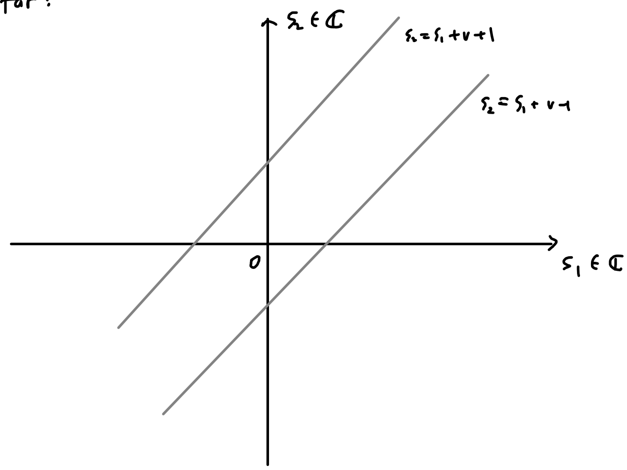
then obviously $R(g)$ is Z-finite, since E^*, hence $\tilde{E}$ is

the moderate growth property also follows from the corresponding property of E^*, $\tilde{E}$

What's wrong with $R(g) \in \pi(\chi_1, \chi_1)$? The simple calculation will tells us $\pi(\chi_1, \chi_1) = 0$, why?

$$\left.\begin{aligned} R\left(w_0 \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix} g\right) &= R\left(\begin{pmatrix} y & \\ & y^{-1} \end{pmatrix} g\right) = |y|\, R(g) \\ R\left(\begin{pmatrix} y^{-1} & \\ & y \end{pmatrix} w_0\, g\right) &= |y|^{-1} R(w_0 g) \;= |y|^{-1} R(g) \end{aligned}\right\} \Rightarrow R(g) = 0 \text{ since } y \in \mathbb{A}_F^\times \text{ is arbitrary}$$

therefore the possible pole of both sides at $s + \frac{v}{2} = \frac{1}{2}$ must cancell. Now the calculation above gives us the belief the "Residues are automorphic", so what's the residue at $s + \frac{v}{2} = 0, 1$? We will answer this question later, and summarize results we have obtained so far:



$$S = \frac{1}{2}(s_1 - s_2 + 1)$$

$$2s + v = s_1 - s_2 + 1 + v = 0,\ 2$$

except these two locus, $E(g, f) \in \mathcal{A}(G_F \backslash G_{\mathbb{A}}, \omega)$, $\omega = \chi_1 \cdot \chi_2$

Now let's consider the decomposition of $L^2(G_F \backslash G_{\mathbb{A}_F}, \omega)$. $\omega : \mathbb{A}_F^\times \to \mathbb{C}^*$

$$L^2(G_F \backslash G_{\mathbb{A}_F}, \omega) = L_0^2(G_F \backslash G_{\mathbb{A}_F}, \omega) \oplus \bigoplus_{\chi^2 = \omega} \mathbb{C} \cdot \chi \circ \det \oplus \int_{\chi_1 \chi_2 = \omega} \pi(\chi_1, \chi_2)\, dy$$

$$\mathcal{L}^2\left(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2\right) = \mathbb{C} \oplus \mathcal{L}^2_{\text{cusp}}\left(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2\right) \oplus \mathcal{L}^2_{\text{cont}}\left(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2\right).$$

Theorem 3.16.1 (Selberg spectral decomposition) *Let $f \in \mathcal{L}^2(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2)$. Then we have*

$$f(z) = \sum_{j=0}^{\infty} \langle f, \eta_j \rangle\, \eta_j(z) + \frac{1}{4\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle f, E(*,s) \rangle\, E(z,s)\, ds,$$

where

$$\langle f, g \rangle = \iint_{SL(2,\mathbb{Z})\backslash\mathfrak{h}^2} f(z)\overline{g(z)}\, \frac{dx\,dy}{y^2}$$

denotes the Petersson inner product on $\mathcal{L}^2\left(SL(2,\mathbb{Z})\backslash\mathfrak{h}^2\right)$.

Iwahori-fixed vector & Lemma 2

In this section, we aim to give a sketch of the proof of Lemma 2 which plays a central role in proving the analytic continuation of Eisenstein series. Let's state this lemma, it gives an explicit description of spherical Whittaker function

<u>Lemma 2</u>: For NA place v, suppose $B(\chi_1, \chi_2)$ is spherical irreducible. W_0 to be the normalized Whittaker function, we have:

$$(1-q^{-1}\alpha_1\alpha_2^{-1})^{-1} W_0\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}y_1 & \\ & y_2\end{pmatrix}k\right) = \begin{cases} \psi_v(x)(\alpha_1\alpha_2)^{m_2} q^{-\frac{m}{2}} \dfrac{\alpha_1^{m+1}-\alpha_2^{m+1}}{\alpha_1-\alpha_2}, & \text{if } m \geq 0 \\ 0, & \text{otherwise}\end{cases}$$

here $m_i = \mathrm{ord}(y_i)$, $m = m_1 - m_2$, $\alpha_i = \chi_i(\pi)$

<u>Rmk</u>: find a coset representative of $Z(F)N(F)\backslash GL_2(F)/K$, we only need to prove this lemma 2 for $a_m = \begin{pmatrix}\pi^m & \\ & 1\end{pmatrix}$

i.e. our goal is to evaluate $W_0(a_m)$

naive try: Recall that we have an explicit description for Whittaker model, i.e.

$$W_0(g) = \lim_{k\to+\infty}\int_{p^{-k}} f_0\left(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g\right)\psi(-x)\,dx$$

this integral is essentially a finite summation, we can assume ψ has conductor $\mathcal{O}$, and fix our choice of f_0:

$$f_0\left(\begin{pmatrix}y_1 & x\\ & y_2\end{pmatrix}k\right) = \chi_1(y_1)\chi_2(y_2)\left|\frac{y_1}{y_2}\right|^{\frac{1}{2}}, \quad y_i\in F^\times, x\in F, k\in K$$

and all we need to evaluate is the following integral.

$$W_0\left(\begin{pmatrix}\pi^m & \\ & 1\end{pmatrix}\right) = \lim_{k\to+\infty}\int_{p^{-k}} f_0\left(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\pi^m & \\ & 1\end{pmatrix}\right)\psi(-x)\,dx$$

- $m=0$, over $v(x)\geq 0$, $\displaystyle\int_{v(x)\geq 0} f_0\left(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right)\psi(-x)\,dx = \int_{v(x)\geq 0} 1\,dx = 1$

  over $v(x) = -k$, $k>0$,

  $$f_0\left(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right) = f_0\left(\begin{pmatrix}x^{-1} & -1\\ & x\end{pmatrix}\begin{pmatrix}1 & \\ x^{-1} & 1\end{pmatrix}\right) = \overset{\text{constant, since } \chi_1, \chi_2 \text{ are unramified}}{\chi_1^{-1}\chi_2(x)}\cdot|x|^{-1}$$

  hence the integral over $v(x)=-k$ is 0, since ψ is non-trivial

  therefore $W_0(a_0) = W_0(1) = 1$

- $m\geq 1$. $f_0\left(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\pi^m & \\ & 1\end{pmatrix}\right) = f_0\left(w_0\begin{pmatrix}\pi^m & \\ & 1\end{pmatrix}\begin{pmatrix}1 & \pi^{-m}x\\ & 1\end{pmatrix}\right) = \chi_2(\pi)^m\cdot q^{-\frac{m}{2}} f\left(w_0\begin{pmatrix}1 & \pi^{-m}x\\ & 1\end{pmatrix}\right)$

  $$\int_{p^{-k}} f_0\left(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\pi^m & \\ & 1\end{pmatrix}\right)\psi(-x)\,dx = \alpha_2^m\cdot q^{\frac{m}{2}}\int_{p^{-k}} f\left(w_0\begin{pmatrix}1 & \pi^{-m}x\\ & 1\end{pmatrix}\right)\psi(-x)\,dx$$

  $$= \alpha_2^m\cdot q^{-\frac{m}{2}}\int_{p^{-m-k}} f\left(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right)\psi(-\pi^m x)\,dx$$

  when $k\leq -m$, this integral equals $\alpha_2^m\cdot q^{-\frac{m}{2}}\cdot q^{m+k} = \alpha_2^m\cdot q^{\frac{m}{2}+k}$

  when $k>-m$, then $-m-k<0$, over $v(x) = -m-k<0$

  $$f\left(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right) = f\left(\begin{pmatrix}x^{-1} & -1\\ & x\end{pmatrix}\begin{pmatrix}1 & \\ x^{-1} & 1\end{pmatrix}\right) = \chi_2\chi_1^{-1}(x)\cdot|x|^{-1} = (\alpha_1\alpha_2^{-1})^{m+k}\cdot q^{-(m+k)}$$

therefore $$\int_{v(x)=-k-m} f\left(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right)\psi(-\pi^m x)\,dx = (\alpha_1\alpha_2^{-1})^{m+k} q^{-(m+k)}\int_{v(x)=-k-m}\psi(-\pi^m x)\,dx = (\alpha_1\alpha_2^{-1})^{m+k} q^{-k}\int_{v(x)=-k}\psi(-x)\,dx$$

here $k>-m$, i.e. $-k<m$, when $-k=0,1,2,\dots,m-1$, the final integral equals $(\alpha_1\alpha_2^{-1})^{m+k}\cdot q^{-k}\cdot\int_{v(x)=-k} dx = (\alpha_1\alpha_2^{-1})^{m+k}\cdot(1-q^{-1})$

when $-k<0$, the final integral is zero, therefore

$$\lim_{k\to+\infty}\int_{p^{-k}} f_0\left(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\pi^m & \\ & 1\end{pmatrix}\right)\psi(-x)\,dx = \alpha_2^m\cdot q^{-\frac{m}{2}} + \lim_{k\to+\infty}\int_{-k\le v(x)\le m-1} f_0\left(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\pi^m & \\ & 1\end{pmatrix}\right)\psi(-x)\,dx$$

$$= \alpha_2^m q^{-\frac{m}{2}} + \int_{0\le v(x)\le m-1} f_0\left(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\pi^m & \\ & 1\end{pmatrix}\right)\psi(-x)\,dx + \underbrace{\lim_{k\to+\infty}\int_{-k\le v(x)<0} f_0\left(w_0\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\pi^m & \\ & 1\end{pmatrix}\right)\psi(-x)\,dx}_{=0}$$

$$= \alpha_2^m\cdot q^{-\frac{m}{2}} + \sum_{k=0}^{m-1}\alpha_1^{m+k}\alpha_2^{-k}\cdot q^{-\frac{m}{2}}(1-q^{-1})$$

local & global, irr + unitary $\Rightarrow$ admissible

Smooth irr repn of profinite is finite-dimensional (PWT)

Steinberg repn: $C^\infty(\mathbb{P}^1(\mathbb{Q}_p)) \curvearrowleft G$ mobius $\quad St = C^\infty(\mathbb{P}^1(\mathbb{Q}_p))/\mathbb{C}$

contragredient:

$$V = \bigoplus_{\rho \in \hat{K}} V(\rho)$$

$$\Longrightarrow \hat{V} = \bigoplus_{\rho\in\hat{K}} V(\rho)^* \Rightarrow \hat{\hat{V}} = \bigoplus_{\rho\in\hat{K}} V(\rho)^{**}$$

Thm:
Irr smooth repn is admissible

Th: $K \subset G$ open compact, $\exists N = N(K)$, s.t. $\forall V$ irr smooth repn,

$$\dim V^K \le N$$

why in induced repn, there is a δ_B?

unitary induction: if μ is a unitary induction (take value in S^1)

we can define $\|f\|^2 = \int_K |f(k)|^2 dk$, G-invariant product

Jacquet module: $\mathrm{Ind}_B^G : \mathrm{Rep}(T) \to \mathrm{Rep}(G)$

(π, V) sm repn of G, $V_N = V/V(N)$ $\quad V(N) = \mathrm{Span}\{\pi(n)v - v \mid n \in N, v \in V\}$

is a B-module

V_N is T-module

$$J : \mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(T)$$
$$V \longmapsto V_N$$

$$\mathrm{Hom}_G(V, \mathrm{Ind}_B^G W) \simeq \mathrm{Hom}_T(V_N, W)$$

$f(1) = 0$

① $\mu \neq \mu^w$ $\quad (\mathcal{L}_B^G \mu)_N = \delta_B^{\frac12}\mu \oplus \delta_B^{\frac12}\mu^w$,

$0 \to U \to \mathcal{L}_B^G(\mu) \to \delta_B^{\frac12}\mu \to 0$

② $\mu = \mu^w$ $\quad (\mathcal{L}_B^G \mu)_N = \begin{pmatrix} \mu\delta_B^{\frac12} & * \\ & \mu^w\delta_B^{\frac12} \end{pmatrix}$

$\Rightarrow 0 \to U_N \to (\mathcal{L}_B^G(\mu))_N \to \delta_B^{\frac12}\mu \to 0$

Claim: $U_N = \delta_B^{\frac12}\mu^w$

Supercuspidal representation

$C(\pi)$: Space of matrix coefficients. $g \mapsto \langle \hat{v}, \pi(g)v \rangle$ is compactly supported modulo Z

Unramified representation

$\mathcal{H}(G, K) = e_K \mathcal{H}(G) e_K$ consists of compactly support bi K-invariant locally constant function on G
$= \{ f \in C_c^\infty(G) \mid f(kgk') = f(g) \}$

Prop: $\mathcal{H}(G, K_0)$ is commutative, e_{K_0} is a unit in this ring

Hecke algebra of T: $\mathcal{H}(T, T_0)$: locally constant on T/T_0 $\simeq \mathbb{C}[X, X^{-1}, Y, Y^{-1}]$

Prop: $\{$Irr unramified representation$\} \xleftrightarrow{1-1} \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathcal{H}(G, K_0), \mathbb{C})$

Thm: Satake isomorphism: S: $\mathcal{H}(G, K_0) \to \mathcal{H}(T, T_0)^W$ isomorphism

$$\phi \longmapsto S(\phi)(t) = \delta_B(t)^{\frac{1}{2}} \int_N \phi(tn)\,dn$$

then $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(G, K_0), \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}(T, T_0)^W, \mathbb{C}) = (\mathbb{C}^\times)^2/\sim$, $(\alpha, \beta) \sim (\beta, \alpha)$

then $(\alpha, \beta) \in (\mathbb{C}^\times)^2/\sim \longrightarrow \mathrm{Ind}(\delta_B^{\frac{1}{2}} \cdot (|\cdot|^\alpha \otimes |\cdot|^\beta))$

Classification theorem:
- (π, V) cuspidal, then $V^I = 0$, $I = \begin{pmatrix} o^\times & o \\ p & o^\times \end{pmatrix}$, $\dim_{\mathbb{C}} St_G^I = 1$
- $\begin{cases} \chi \circ \det & : \chi \text{ unramified} \leadsto I \text{ fixed is } 1 \qquad |\alpha - \beta| = 1 \\ \mathrm{Ind}_B^G \delta^{\frac{1}{2}} \mu & : \mu = \mu_1 \otimes \mu_2,\ \mu_1, \mu_2 \text{ unramified},\ \frac{\mu_1}{\mu_2} \neq |\cdot|^{\pm 1} \qquad |\alpha - \beta| \neq 1 \end{cases}$

$\hookrightarrow$ I fixed is 2-dim

admissibility of $L^2_0(\Gamma\backslash G) \Rightarrow$
in $(\mathfrak{g}, K)$-mod of $SL_2(\mathbb{R})$
HS - operator

存在性与唯一性. 由于 Whittaker model 可定
moderate
growth

Tensor product thm
Kirillov model
spherical conductor

Maass

Definition 3.5.7 (Maass form) Let $N, k \in \mathbb{Z}$ with $N \geq 1$. Let $\nu \in \mathbb{C}$. Fix a character $\chi \pmod N$. A Maass form of type ν of weight k and character χ for $\Gamma_0(N)$ is a smooth function $f : \mathfrak{h} \to \mathbb{C}$ satisfying the following conditions:

- $\left(f\big|_k \gamma\right)(z) = \chi(d) f(z), \qquad$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),\ z \in \mathfrak{h}$;
- $\Delta_k f = \nu(1-\nu) f$, where Δ_k is the Laplace operator given in Definition 3.5.3;
- f is of moderate growth as in Definition 3.3.3;
- $\displaystyle\iint_{\Gamma_0(N)\backslash\mathfrak{h}} |f(z)|^2 \, \frac{dx\,dy}{y^2} < \infty.$

Eisenstein

Definition 3.8.2 (Eisenstein series at a cusp) Let $N, k \in \mathbb{Z}$ with $N \geq 1$. Fix a character $\chi \pmod N$. Let $\mathfrak{a}$ be a singular cusp of $\Gamma_0(N)$ as in (3.8.1) and let $\sigma_{\mathfrak{a}}$ be defined by (3.7.1). For $\Re(s) > 1$ and $z \in \mathfrak{h}$, the Eisenstein series at the cusp $\mathfrak{a}$, of weight k, and character χ, for the congruence subgroup $\Gamma_0(N)$ is defined by the absolutely convergent series

$$E_{\mathfrak{a}}(z, s, \chi) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma_0(N)} \overline{\chi(\gamma)} \left(J\left(\sigma_{\mathfrak{a}}^{-1}\gamma,\ z\right)\right)^{-k} \operatorname{Im}\left(\sigma_{\mathfrak{a}}^{-1}\gamma z\right)^s$$

where

$$J\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = \frac{cz+d}{|cz+d|}.$$