Automorphic forms

§1. Review on classical theory of modular forms

1. Congruence subgroups:

$$\Gamma \supset \Gamma(N) \qquad (\Gamma_1(N),\ \Gamma_0(N))$$

Rmk: · conjugate of Γ by $GL_2(\mathbb{Q})$ is still congruence
· $\exists$ conjugate of Γ, s.t. $\alpha\Gamma\alpha^{-1} \cap SL_2(\mathbb{Z}) \supset \Gamma_1(N')$

2. Cusps: equivalence class on $\mathbb{P}^1(\mathbb{Q})$ under Γ-action

geometric picture:

Fundamental domain: for $SL_2(\mathbb{Z})$, $Y_0(N)$, $Y_1(N)$

$$\begin{array}{ccc} & Y_0(N) & Y_1(N) \\ & \downarrow \text{compactification} & \\ \hat{\mathbb{C}} & X_0(N) & X_1(N) \end{array}$$

3. Modular forms: $f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau)\cdot(c\tau+d)^k$, $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$SL_2(\mathbb{Z})$-action on functions on $\mathcal{H}$

$$\left(f[\gamma]_k\right)(\tau) = f(\gamma\tau)\, j(\gamma,\tau)^{-k}, \quad j(\gamma,\tau) = (c\tau+d)\cdot \det(\gamma)^{-\frac{1}{2}}$$

· f holomorphic on $\mathcal{H}$

· expansion of f at each cusp is holomorphic

if the constant term $=0$ $\Rightarrow$ cusp forms

Rmk: geometric interpretation: global section of certain line bundles

cusp $\to$ holomorph.

modul. $\to$ with singularity control

4. Hecke operators

· Double coset operators:

$$[\Gamma_1\alpha\Gamma_2]_k : \mathcal{M}_k(\Gamma_1) \to \mathcal{M}_k(\Gamma_2)$$

geometric interpretation:

Petersson inner product

$$\int_{\Gamma\backslash\mathcal{H}} f\bar{g}\, y^k \frac{dx\,dy}{y^2}$$

Hecke operator on $\Gamma_1(N)$ $\qquad$ Diamond operator

$T_p = [\Gamma_1(N)\alpha_p\Gamma_1(N)]_k$, $\quad \alpha_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ $\qquad$ $\langle d \rangle$: $\begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \in \Gamma_0(N)$ $\quad \delta \equiv d$ mod N

↳ double coset decomposition

· commutativity, $\quad T_p T_q = T_q T_p$, $\quad T_p\langle d\rangle = \langle d\rangle T_p$ $\quad \forall p, q, d$ $\Big\} \Rightarrow T_p$ can be defined in $\mathcal{M}_k(\Gamma_0(N), \chi)$

→ representation 可交换

$$a_n(T_p f) = a_{np}(f) + 1_N(p)\, p^{k-1} a_{n/p}(\langle p\rangle f),$$

· normality (see Old & New)

对于 T_p $(p,N)=1$ & $\langle d\rangle$, 则 in $S_k(\Gamma_1(N))$ ($S_k(\Gamma_0(N),\chi)$) 存在由特征向量组成的正交基.

特别地, 对于 newforms newform is also eigenform for T_p, $p \mid N$

考虑相应的系数: $\qquad T_p f = \lambda_p f$, $\forall p \nmid N$ $\quad \langle p\rangle f = \chi(p) f$

$$\Rightarrow \quad a_n(T_p f) = a_{np}(f) + \chi(p)\, p^{k-1} a_{n/p}(f)$$

$$\lambda_p a_{p^n}(f) = a_{p^n}(T_p f) = a_{p^{n+1}}(f) + \chi(p)\, p^{k-1} a_{p^{n-1}}(f), \ \forall n \geq 1$$

especially, if f is a normalized newform

$$\lambda_p = a_p(f) \Rightarrow \quad a_p(f)\, a_{p^n}(f) = a_{p^{n+1}}(f) + \chi(p)\, p^{k-1} a_{p^{n-1}}(f), \ \forall n \geq 1, \ \forall p$$

· L-functions

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \int_0^{\infty} f(it)\, t^s \frac{dt}{t}$$

Sum factor

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Gelbart Chapter 2.

Our first goal is to re-intepret <u>holomorphic cusp form on $\mathcal{H}$</u> as a <u>function on $SL_2(\mathbb{R}) / GL_2(\mathbb{R})^+$</u>

Note that $SL_2(\mathbb{R}) / SO(2) \simeq \mathcal{H}$

$GL_2(\mathbb{R})^+ / Z^+ SO(2)$

- Iwasawa decomposition for $GL_2(\mathbb{R})^+$, $SL_2(\mathbb{R})$

$$GL_2(\mathbb{R})^+: \quad \begin{pmatrix} u & \\ & u \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \begin{array}{l} u, y > 0 \\ x \in \mathbb{R} \\ \theta \in [0, 2\pi) \end{array}$$

$$SL_2(\mathbb{R}): \quad \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \begin{array}{l} x \in \mathbb{R} \\ y > 0, \\ \theta \in [0, 2\pi) \end{array}$$

these decompositions must be unique, apply each element to i, get unique x & y
take det, get unique $u \in \mathbb{R}_{>0}$, then θ is necessarily unique up to 2π

- Haar measure on $GL_2(\mathbb{R})^+$ & $SL_2(\mathbb{R})$ (they are both unimodular)

on $GL_2(\mathbb{R})^+$: $\quad \dfrac{du}{u} \cdot \dfrac{dx\,dy}{y^2} \cdot d\theta$

on $SL_2(\mathbb{R})$: $\quad \dfrac{dx\,dy}{y^2} \cdot d\theta$

Now we associate a <u>cusp form</u> to a function on $G = SL_2(\mathbb{R}) / GL_2(\mathbb{R})_+$
take $f \in S_k(\Gamma)$, define:

$$\phi_f(g) = f(g(i))\, j(g, i)^{-k} \qquad \left(j(g, z) = (cz+d)\cdot \det(g)^{\frac{1}{2}} \right) \qquad j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z)\, j(\gamma_2, z)$$

- Properties of these functions ϕ_f (First insight for automorphic forms)

① $\phi(\gamma g) = \phi(g), \quad \forall \gamma \in \Gamma$

② $\phi(g k_\theta) = e^{-ik\theta}\phi(g), \quad \forall \theta \in \mathbb{R}$

③ $\phi(g)$ is bounded, in particular,

$$\int_{\Gamma\backslash G} |\phi(g)|^2\, dg < +\infty \quad \Big(\int$$

④ $\phi(g)$ is cuspidal, i.e. $\forall \sigma \in SL_2(\mathbb{Z})$, $h = \text{width}(\sigma(\infty))$, $\forall g \in G$

$$\int_0^1 \phi\left(\sigma \begin{pmatrix} 1 & xh \\ & 1 \end{pmatrix} g\right) dx = 0$$

① $\phi(\gamma g) = f(\gamma g(i))\, j(\gamma g, i)^{-k}$

$= f(g(i))\, j(\gamma, g(i))^k\, j(\gamma, g(i))^{-k} \cdot j(g, i)^{-k}$

$= \phi(g)$

② trivial

③ suppose $g = h\kappa_\theta$, $h = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix}$

$|\phi(g)| = |f(h(i))\, j(h, i)^{-k}| = \underline{|f(z)| \cdot \operatorname{Im}(z)^{\frac{k}{2}}}$

bounded since f is cusp form

$\Gamma\backslash G$ has finite volume

④ $g = h\kappa_\theta$, $h(i) = z$

$\phi\left(\sigma\begin{pmatrix} 1 & xh \\ & 1 \end{pmatrix} g\right) = \left(f[\sigma]_k\right)(z + xh)\, (\operatorname{Im} z)^{\frac{k}{2}} e^{-ik\theta}$ see the Fourier expansion

· Question: How to characterize the image of $f \mapsto \phi_f$?

the image lies in $L^2(\Gamma \backslash G)$, which has a natural G-action by right multiplication

Our characterization will use the decomposition of this representation.

Interlude: Lie theory

G a Lie group, a representation of G on finite-dimensional vector space V is a continuous homomorphism:

$$\pi : G \longrightarrow GL(V)$$

· continuous homomorphism between Lie group is at once <u>smooth</u>

A great tool to study representation of the whole group is by studying the induced representation of the Lie algebra of G.

· $\mathfrak{g} = \mathrm{Tgt}_e G$

· exponential map: $\forall X \in \mathfrak{g}$, consider the geodesic starting from e, with initial velocity X

suppose this geodesic is $\phi_X : (-\varepsilon, \varepsilon) \to G$, actually, it can be defined over $\mathbb{R}$

define $\exp(X) = \phi_X(1)$ $\left(\exp(tX) = \phi_{tX}(1) = \phi_X(t) \right)$

example: $G = GL_n(\mathbb{R})$, $\mathfrak{g} = M_n(\mathbb{R})$, $\exp(X) = e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$

the induced representation of $\mathfrak{g}$ on V is defined as:

$$dX \cdot v := \frac{d}{dt}\left(\pi(\exp(tX)) v \right) \Big|_{t=0} \quad \left(\text{the representation is smooth} \right)$$

elements of Lie algebra may be viewed as differential operators

We can also pass the action to a larger space: $U(\mathfrak{g})$ (universal enveloping algebra of $\mathfrak{g}$)

$$U(\mathfrak{g}) = T(\mathfrak{g}) \big/ \left([X,Y] - (X \otimes Y - Y \otimes X) \right) \rightsquigarrow \text{viewed as differential operators}$$

But here, what we have is an <u>infinite dimensional space</u> $L^2(\Gamma \backslash G)$. How to define the Lie algebra action?

Generally, we can't "reasonably" define it on the whole space, but we can "reasonably" define it on the dense subspace $C^\infty(\Gamma \backslash G)$

$\mathfrak{g} = \mathfrak{sl}_2$: $R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad [H,R] = 2R$, $[H,L] = -2L$, $[R,L] = H$

$$dR = e^{2i\theta}\left(iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{1}{2i} \frac{\partial}{\partial \theta} \right), \quad dL = e^{-2i\theta}\left(-iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \frac{1}{2i} \frac{\partial}{\partial \theta} \right), \quad dH = -i \frac{\partial}{\partial \theta}$$

Casimir element $\Delta = -\frac{1}{4}\left(H^2 + 2RL + 2LR \right) = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + y \frac{\partial^2}{\partial x \partial \theta} \in C(U(\mathfrak{g})_{\mathbb{C}}) = \mathbb{C}[\Delta]$

Calculation

$\mathfrak{g} = sl_2$: $R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$dR = e^{2i\theta}\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial \theta}\right), \quad dL = e^{-2i\theta}\left(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial\theta}\right), \quad dH = -i\frac{\partial}{\partial\theta}$$

Casimir element $\Delta = -\frac{1}{4}\left(H^2 + 2RL + 2LR\right) = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial^2}{\partial x \partial\theta} \in \underline{C}(U(\mathfrak{g})_{\mathbb{C}}) = \mathbb{C}[\Delta]$

$$[H, R] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = 2R$$

$$[H, L] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -2L$$

$$(Rf)\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{\frac12} & \\ & y^{-\frac12} \end{pmatrix} k_\theta\right) = \lim_{t\to 0}\frac{f(g\exp tR) - f(g)}{t}$$

$$= \begin{pmatrix} \sqrt{1+t\sin 2\theta} & t\cos 2\theta\,(1+t\sin 2\theta)^{-\frac12} \\ & (1+t\sin 2\theta)^{-\frac12} \end{pmatrix}\begin{pmatrix} \cos\theta' & \sin\theta' \\ -\sin\theta' & \cos\theta' \end{pmatrix}$$

$$=$$

$$\exp(tR) = \sum_{n\geq 0}\frac{t^n}{n!}R^n = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} i = \begin{pmatrix} 1 & t\cos 2\theta \\ & 1 \end{pmatrix}\begin{pmatrix} \sqrt{1+t\sin 2\theta} & \\ & (1+t\sin 2\theta)^{-\frac12} \end{pmatrix}$$

then $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{\frac12} & \\ & y^{-\frac12} \end{pmatrix} k_\theta \cdot \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$

$$\frac{\cos\theta(i+t) + \sin\theta}{-\sin\theta(i+t) + \cos\theta} = \frac{-t^2\sin\theta\cos\theta + t(\cos 2\theta) + i}{1 - 2t\sin\theta\cos\theta + t^2\sin^2\theta}$$

$$= \left(-t^2\sin\theta\cos\theta + t\cos 2\theta + i\right)\left(1 + t\sin 2\theta + o(t)\right)$$

$$\frac{\cos\theta(i+t) + \sin\theta}{-\sin\theta(i+t) + \cos\theta}\cdot y + x = x + y\cdot\frac{\left((\sin\theta + t\cos\theta) + i\cos\theta\right)\left(\cos\theta - t\sin\theta + i\sin\theta\right)}{(\cos\theta - t\sin\theta)^2 + (\sin\theta)^2}$$

$$t\cos 2\theta + i(1 + t\sin 2\theta) + o(t)$$

$$= x + y\,\frac{-t^2\sin\theta\cos\theta + t(\cos 2\theta) + i}{1 - 2t\sin\theta\cos\theta + t^2\sin^2\theta}$$

$$\sin\theta'(1+t\sin 2\theta)^{-\frac12} = \sin\theta$$

$$= \begin{pmatrix} 1 & x + y\,\frac{-t^2\sin\theta\cos\theta + t(\cos 2\theta)}{1 - 2t\sin\theta\cos\theta + t^2\sin^2\theta} \\ & 1 \end{pmatrix}\begin{pmatrix} \frac{y^{\frac12}}{\sqrt{1 - 2t\sin\theta\cos\theta + t^2\sin^2\theta}} & \\ & \frac{\sqrt{1 - 2t\sin\theta\cos\theta + t^2\sin^2\theta}}{y^{\frac12}} \end{pmatrix} k_{\theta'(t)}$$

$$\sin\theta' = \sin\theta\cdot(1 + t\sin 2\theta)^{-\frac12}$$

$$= \sin\theta\left(1 - \tfrac12 t\sin 2\theta\right)$$

$$\theta' = \arcsin\left(\sin\theta\cdot\left(1 - \tfrac12 t\sin 2\theta\right)\right)$$

$$= \begin{pmatrix} 1 & x + y\,t\cos 2\theta + o(t) \\ & 1 \end{pmatrix}\begin{pmatrix} y^{\frac12}\left(1 + \frac12 t\sin 2\theta + o(t)\right) & \\ & y^{-\frac12}\left(1 - \frac12 t\sin 2\theta + o(t)\right) \end{pmatrix} k_{\theta'(t)}$$

$$\frac{d\theta'}{dt} = \frac{1}{\sqrt{1 - \sin\theta^2\left(1 - \frac12 t\sin 2\theta\right)^2}}\cdot -\frac12\sin\theta\sin 2\theta$$

$$= \frac{1}{\cos\theta} - \sin^2\theta\cos\theta = -\sin^2\theta$$

$$\theta'(t) = \theta - \sin^2\theta\, t + o(t)$$

$$\Rightarrow \quad Rf = \frac{\partial f}{\partial x}\cdot y\cos 2\theta + \frac{\partial f}{\partial y}\,y\sin 2\theta - \frac{\partial f}{\partial\theta}\sin^2\theta$$

$$dR = e^{2i\theta}\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial\theta}\right)$$

$$= (\cos 2\theta + i\sin 2\theta)\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{i}{2}\frac{\partial}{\partial\theta}\right)$$

$$= \frac{\partial}{\partial y}y\cos 2\theta - (\sin 2\theta)\left(y\frac{\partial}{\partial x} - \frac12\frac{\partial}{\partial\theta}\right)$$

For $f \in S_k(\Gamma)$,

$$\Delta \phi_f = -\frac{k}{2}\left(\frac{k}{2}-1\right)\phi_f$$

pf: obviously, $\phi_f \in C^\infty(\Gamma \backslash G)$, hence the Δ action makes sense

$$\phi_f(g) = y^{\frac{k}{2}} f(z)\, e^{-ik\theta}, \text{ where } g = h\kappa_\theta,\ h(i) = z = x+iy$$

just computation, and note f holomorphic $\Rightarrow$ $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ is 0 on f

Now we can recognize the image of $S_k(\Gamma) \to L^2(\Gamma \backslash G)$, denote $A_k^2(\Gamma)$ as the following subspace of $L^2(\Gamma\backslash G)$

(a) $\phi(\gamma g) = \phi(g)$, $\forall \gamma \in \Gamma$,

(b) $\phi(g\kappa_\theta) = \phi(g)e^{-ik\theta}$, $\forall \theta \in \mathbb{R}$ ~ we will see later that this imply ϕ is smooth

(c) $\Delta\phi = -\frac{k}{2}\left(\frac{k}{2}-1\right)\phi$

(d) ϕ is bounded & cuspidal

then there is an isomorphism:

$$\begin{aligned} S_k(\Gamma) &\xrightarrow{\sim} A_k^2(\Gamma) \\ f &\longmapsto \phi_f \\ f_\phi(z) = \phi(g)j(g,i)^k &\longleftarrow \phi \end{aligned}$$

pf: only need to show the reverse direction: $\phi \mapsto f_\phi$

- f_ϕ is well defined: $\forall g'$ s.t. $g'(i) = g(i) \Rightarrow g' = g\kappa_\theta$

$$\begin{aligned} \text{then}\quad \phi(g')j(g',i)^k &= \phi(g\kappa_\theta)j(g\kappa_\theta,i)^k \\ &= \phi(g)e^{-ik\theta}j(g,i)^k \cdot j(\kappa_\theta,i)^k = \phi(g)j(g,i)^k \end{aligned}$$

- (a) $\Rightarrow$ $f_\phi[\gamma]_k = f_\phi$
- (c) $\Rightarrow$ f_ϕ is holomorphic 事实上，这里还用到了 (b), wt 为 k!
- (d) $\Rightarrow$ $|\phi(g)| = |f(z)|\,|\mathrm{Im} z|^{\frac{k}{2}}$ is bounded, $\forall g \in G$ $\Leftrightarrow$ f is a cusp form

hence $f_\phi \in S_k(\Gamma)$

To explain (b). We introduce the notion of admissible.

We consider a reductive Lie group G ($GL_2(\mathbb{R})$, $GL_2(\mathbb{R})^+$, $SL_2(\mathbb{R})$) and its maximal compact subgroup K ($O(2,\mathbb{R})$, $SO(2,\mathbb{R})$)

a representation of G, say (π, V), V is a separable Hilbert space, has a decomposition when restricted to K

$$V \simeq \widehat{\bigoplus_{\sigma \in \mathrm{Irr}(K)}} V(\sigma) \quad \text{as Hilbert space decomposition}$$

Here σ ranges over all irreducible representation of K, and Peter-Weyl theorem also tells us σ must be finite dimensional

then π is admissible if each $V(\sigma)$ is finite dimensional, i.e. every irr repn of K appears with finite multiplicity

we also define $V_{fin} = \bigoplus_{\sigma \in \mathrm{Irr}(K)} V(\sigma)$ as an algebraic direct sum. called K-finite vectors, they are dense in V

this space can also be characterized as: $\{v \in V \mid \{k \cdot v\} \text{ spans a finite dimensional space}\}$

and hence G acts on V_{fin}

Holomorphicity

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial^2}{\partial\theta\partial x} \qquad L =$$

$$\phi_f(g) = y^{\frac{k}{2}} f(z)\cdot e^{-ik\theta}$$

$$\Delta\phi_f = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(y^{\frac{k}{2}}f(z)\right)\cdot e^{-ik\theta} + y\frac{\partial^2}{\partial\theta\partial x}\left(y^{\frac{k}{2}}f(z)e^{-ik\theta}\right)$$

$$= -y^2\left(y^{\frac{k}{2}}\frac{\partial^2 f}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{k}{2}y^{\frac{k}{2}-1}f(z) + y^{\frac{k}{2}}\frac{\partial f}{\partial y}\right)\right)e^{-ik\theta} + y\frac{\partial}{\partial x}\left(-ik\,y^{\frac{k}{2}}f(z)e^{-ik\theta}\right)$$

$$= \underline{-y^{\frac{k}{2}+2}}\frac{\partial^2 f}{\partial x^2} - y^2\left(\underline{\frac{k}{2}\left(\frac{k}{2}-1\right)y^{\frac{k}{2}-2}f(z)} + \frac{k}{2}y^{\frac{k}{2}-1}\frac{\partial f}{\partial y} + \frac{k}{2}y^{\frac{k}{2}-1}\frac{\partial f}{\partial y} + \underline{y^{\frac{k}{2}}}\frac{\partial^2 f}{\partial y^2}\right)e^{-ik\theta}$$

$$-ik\,y^{\frac{k}{2}+1}e^{-ik\theta}\frac{\partial f}{\partial x}$$

$$= \frac{k}{2}\left(1-\frac{k}{2}\right)\phi_f - y^{\frac{k}{2}+2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)e^{-ik\theta} - k\,y^{\frac{k}{2}+1}\frac{\partial f}{\partial y}e^{-ik\theta} - ik\,y^{\frac{k}{2}+1}e^{-ik\theta}\frac{\partial f}{\partial x}$$

$$= \frac{k}{2}\left(1-\frac{k}{2}\right)\phi_f - 4y^{\frac{k}{2}+2}\frac{\partial^2 f}{\partial z\partial\bar z}e^{-ik\theta} + 2ik\,y^{\frac{k}{2}+1}e^{-ik\theta}\frac{\partial f}{\partial z}$$

$$\Delta\phi_f = -RL\phi_f + \frac{k(1-k)}{2}\phi_f \quad \text{(almost)}$$

$\Rightarrow$ $RL\phi_f = 0$, but

$$\langle R\phi_1, \phi_2\rangle = \langle\phi_1, -L\phi_2\rangle$$

$$\Rightarrow \langle RL\phi_f, \phi_f\rangle = \langle L\phi_f, -L\phi_f\rangle = 0$$

$$\Rightarrow L\phi_f = 0$$

$$dL = e^{-2i\theta}\left(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial\theta}\right)$$

$$L\phi_f = 0 \Rightarrow \frac{\partial}{\partial\bar z}f = 0 \qquad L\left(f(z)\,y^{\frac{k}{2}}e^{-ik\theta}\right)$$

$$= e^{-2i\theta}\left(-iy^{\frac{k}{2}+1}\frac{\partial f}{\partial x}e^{-ik\theta} + y^{\frac{k}{2}+1}\frac{\partial f}{\partial y}e^{-ik\theta} + \frac{k}{2}f(z)\,y^{\frac{k}{2}}e^{-ik\theta}\right.$$
$$\left. - \frac{k}{2}f(z)\,y^{\frac{k}{2}}e^{-ik\theta}\right)$$

$$-i\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = -i\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = -i\frac{\partial f}{\partial\bar z}$$

$\tilde z \Leftarrow (\theta)$ in $\mathbb{C}$

Gelbart & notation

$$K_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

We also define the maximal space where $U(\mathfrak{g})$ can act:

$$V^\infty = \{ v \in V \mid d\pi(X_1) \circ d\pi(X_2) \circ \cdots \circ d\pi(X_r) v \text{ exists for } \forall X_1, \ldots, X_r \in \mathfrak{g} \}$$

Here, we say $d\pi(X)v$ exists if the following limit exists in the Hilbert space.

$$\lim_{t \to 0} \frac{\pi(\exp(tX))v - v}{t}$$

Thm: (π, V) is an admissible repn of G on a Hilbert space V, then

- $V_{fin} \subseteq V^\infty \rightsquigarrow V^\infty$ is dense in V
- V_{fin} is invariant under $\mathfrak{g}$ action

pf: convolution trick, see Bump 2.4.5

Apply to our situation, $G = SL_2(\mathbb{R})$, $K = SO(2, \mathbb{R})$. $V = L^2(\Gamma \backslash G)$, then $V^\infty = C^\infty(\Gamma \backslash G)$

now (b) implies ϕ is K-finite $\Rightarrow \phi$ is smooth

but we need to verify that G-representation $L^2_0(\Gamma \backslash G)$ is admissible $\rightsquigarrow$ 后面的讨论会讲 (luckily, this is true)

这里不对的. 下面的 argument 是 $L^2_0(\Gamma\backslash G)$ 是不可以约是 admissible 的, 而 $A^2_k(\Gamma)$ 它不在某个不可约表示中

因为 $A^2_k(\Gamma)$ invariant under G-action

Let's now introduce $(\mathfrak{g}, K)$-module (K: compact Lie group $\subseteq G$)

a $(\mathfrak{g}, K)$-module is a vector space V equipped with both $\mathfrak{g}$ & K-representation, s.t.

- $V = V_{fin}$, i.e. $V = \bigoplus_\sigma V(\sigma)$, and $\dim_{\mathbb{C}} V(\sigma) < +\infty$ $\quad (\Rightarrow V = V_{fin} = V^\infty)$
- $\mathrm{Lie}(K) = \mathfrak{t}$. $\left.\begin{array}{l} \mathfrak{t} \text{ acts on } V \text{ via classical notions} \\ \mathfrak{t} \text{ acts on } V \text{ via } \mathfrak{t} \hookrightarrow \mathfrak{g} \end{array}\right\} \Rightarrow$ these two actions agree
- $\forall k \in K$, $X \in \mathfrak{g}$

$$k \cdot (X \cdot v) = (Ad(k)X) \cdot (k \cdot v) \quad (\text{When } K \text{ is connected, this follows from 2})$$

For every admissible Hilbert space representation (π, V) of G, we can associate

V_{fin}, with usual $(\mathfrak{g}, K)$-module structure

this is a $(\mathfrak{g}, K)$-module (For general representation, define it to be $V^\infty \cap V_{fin}$), Moreover, it is also a G-repn (wrong!)

this associated $(\mathfrak{g}, K)$-module reflects some properties of original representation: (V admissible, π unitary)

V_{fin} as $(\mathfrak{g}, K)$-mod is irreducible $\Leftrightarrow$ V as G-repn is irreducible (no proper invariant closed subspace)

Sketch: $\Rightarrow$ if not irr, $V = W \oplus W^\perp \Rightarrow V_{fin} = W_{fin} \oplus W^\perp_{fin}$. $\Leftarrow$ if V_{fin} not irr, then $\exists W \subseteq V_{fin}$

then $\exists \sigma$, s.t. $V(\sigma) \subsetneq W \Rightarrow W^\perp$ not $0 \Rightarrow G$ acts on $\hat{W} \subsetneq V$

then to get irr repn of $SL_2(\mathbb{R}) = G$, we first classify $(\mathfrak{g}, K)$-mod, then construct corresponding representation

- Schur's lemma holds for $(\mathfrak{g}, K)$-mod

i.e. for $\forall (\pi_1, V_1)$ & (π_2, V_2) are two irreducible (no proper invariant subspace) $(\mathfrak{g}, K)$-mods, then

$$\mathrm{End}_{(\mathfrak{g}, K)}((\pi_1, V_1), (\pi_2, V_2)) \simeq \begin{cases} \mathbb{C}, & \text{if they are isomorphic} \\ 0, & \text{if not} \end{cases}$$

Motivated by the definition of $A_k^2(\Gamma)$, we define Γ-automorphic forms:

Def: Γ-automorphic form ϕ on G is a (smooth) function s.t.

(a) $\phi(\gamma g) = \phi(g)$, $\forall \gamma \in \Gamma$
(b) ϕ is right K-finite ($\rightsquigarrow$ automatic smooth)
(c) ϕ is an eigenfunction for Δ
(d) ϕ satisfies growth condition: $\exists\, C, N > 0$, s.t.

$$|\phi(z, \theta)| \le C \cdot |y|^N \text{, when } y \to +\infty$$

if ϕ also satisfy the cuspidal condition $\oplus \Rightarrow \phi$ is a cusp form $\sim L_0^2(\Gamma\backslash G)$

Examples are given later, let's now come to the question of decomposing $L^2(\Gamma\backslash G)$ as G-repn
We will see that this decomposition is closely related to the Γ-automorphic forms we have defined

Why? because $\Delta \in C(U(\mathfrak{g}))$, hence Δ "commutes" with G-action, thus on every G-irr subspace Δ "should" acts as a scalar by heuristic from Schur's lemma, but rigorously speaking, Δ is not defined on the whole space $L^2(\Gamma\backslash G)$, and we don't know Schur's lemma in this case

But we can make it rigorous in the following sense, remember that Schur's lemma holds for $(\mathfrak{g}, K)$-mod and in our cases π is unitary, $L^2(\Gamma\backslash G)$ as G-repn is admissible, hence if

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus_{\sigma}} L^2(\Gamma\backslash G)(\sigma) \rightsquigarrow \text{Wrong: } L_0^2(\Gamma\backslash G) = \cdots$$

then Δ will acts as a scalar on each finite-dimensional subspace $L^2(\Gamma\backslash G)(\sigma)$
therefore each subspace $L^2(\Gamma\backslash G)(\sigma)$ consists of right K-finite, eigenfunctions for Δ, essentially (a), (b), (c)

need edit: L^2 先分解成 G-irr subspace, each subspace 再分解成 K-finite!

Why cusp forms are square-integrable?

consider the case $\phi(g\kappa_\theta) = \phi(g)e^{-ik\theta} \Rightarrow$ let's consider the function $f_\phi(z) = \phi(g)\, j(g, i)^k$

then $|\phi(g)| = |\underbrace{f_\phi(z) \cdot \operatorname{Im}(z)^{\frac{k}{2}}}|$ is invariant under Γ

$\hookrightarrow$ function on $\Gamma\backslash\mathcal{H}$, $\le C y^{N+\frac{k}{2}}$ when $y \to \infty$

Example for Γ-automorphic forms

- non-holomorphic cusp forms $W_s(\Gamma)$, s purely imaginary or $-1 < s < 1$, $s \neq 0$

a) $\phi \in C^\infty(\Gamma \backslash G)$
b) $\phi(g k_\theta) = \phi(g)$
c) $\Delta \phi = \frac{1-s^2}{4}\phi$
d) ϕ is bounded

$$\xrightarrow{f(z) = \phi(g)}$$

a) $f \in C^\infty(\Gamma \backslash \mathcal{H})$
b) $\Delta^* f = \frac{1-s^2}{4} f$
c) f is bounded

because L^2 is unitary

Bump:

• Irreducible unitary representations of $SL_2(\mathbb{R})$ (general strategy: classify irr $(\mathfrak{g},K)$-mod, construct certain representation, then test which one is unitary)

automatic admissible

1. Class 1 principal series π_s^+

they are induced from unitary characters:

$$\begin{pmatrix} a & * \\ & a^{-1} \end{pmatrix} \longrightarrow \mathbb{C}^*$$

$$a \longmapsto |a|^s, \quad Re(s) = 0 \qquad \Delta: \frac{1-s^2}{4} \geq \frac{1}{4}$$

i.e. they are measurable functions on $SL_2(\mathbb{R})$ satisfying:

$H^+(s)$

• $f\left(\begin{pmatrix} a & * \\ & a^{-1} \end{pmatrix} g \right) = |a|^{s+1} f(g)$ ~ f is determined by K

• f is square-integrable: $\int_K |f|^2 dg < +\infty$

Rmk: $\pi_s^+ \simeq \pi_{-s}^+$

2. non-class 1 principal series π_s^-

they are induced from unitary characters:

$$\begin{pmatrix} a & * \\ & a^{-1} \end{pmatrix} \longmapsto |a|^s \operatorname{sgn}(a), \quad Re(s) = 0 \qquad \Delta = \frac{1-s^2}{4} \geq \frac{1}{4}$$

i.e. they are measurable functions on $SL_2(\mathbb{R})$ satisfying:

• $f\left(\begin{pmatrix} a & * \\ & a^{-1} \end{pmatrix} g \right) = |a|^{s+1} \operatorname{sgn}(a) f(g)$

• $\int_K |f|^2 dg < +\infty$

Rmk: π_s^- is irr $\Leftrightarrow s \neq 0$

3. complementary series π_s^c

they are induced from non-unitary characters:

$$\begin{pmatrix} a & * \\ & a^{-1} \end{pmatrix} \longmapsto |a|^s, \quad s \neq 0, -1 < s < 1 \qquad \Delta = \frac{1-s^2}{4} \in \left(0, \frac{1}{4}\right)$$

4. Discrete series $\pi_k^{\pm}$, $k > 1$

they are induced from:

$$\pi_k^+ : \begin{pmatrix} a & * \\ & a^{-1} \end{pmatrix} \mapsto |a|^k$$

$$\Delta = -\frac{k}{2}\left(\frac{k}{2} - 1\right) \leq 0$$

$$\pi_k^- : \begin{pmatrix} a & * \\ & a^{-1} \end{pmatrix} \longmapsto |a|^k \operatorname{sgn}(a)$$

another description:

$H(k)$: f is holomorphic on $\mathcal{H}$, and $f(z)\, \mathrm{Im}(z)^{\frac{k}{2}} \in L^2(\mathcal{H})$

$$\left(\pi_k^+(g) f\right)(z) = (bz+d)^{-k} f\left(\frac{az+c}{bz+d}\right), \text{ i.e. } \pi_k^+ f = f[g^T]_k$$

5. Trivial representation

Decomposition theorem for $L^2(\Gamma \backslash G)$:

We expect $L^2(\Gamma\backslash G)$ decomposes into Hilbert direct sum of G irr repns, but this doesn't hold in general (holds for $\Gamma\backslash G$ compact), what we can get is the following:

$L^2_d(\Gamma\backslash G)$ is the <u>Hilbert direct sum</u> of subspaces invariant & irreducible as G-repns, then

- $L^2_d(\Gamma\backslash G) = L^2_0(\Gamma\backslash G) \oplus \mathbb{C}$ ← constant functions
- By definition of L^2_0, we have

$$L^2_0(\Gamma\backslash G) = \hat{\bigoplus_{\sigma}} L^2_0(\Gamma\backslash G)(\sigma)$$ ← 表示可能多次出现在其中!

$\forall \sigma$ irr repn of G,

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\sigma, L^2_0) = \dim_{\mathbb{C}} \mathrm{Hom}_G(\sigma, L^2_0(\Gamma\backslash G)(\sigma)) < +\infty$$

i.e.

$$L^2_0(\Gamma\backslash G)(\sigma) \simeq \sigma^{\oplus n}$$

- (What's the remaining?)

Denote $R_c = L^2_d(\Gamma\backslash G)^{\perp}$, then

$$R_c = \bigoplus \int_0^{\infty} \pi_s^+ \, ds$$

Rmk: When $\Gamma = \Gamma_0(N)$, $m = \#\{\text{cusps}\}$ then this part is m-dimensional represented by Eisenstein series

For cusps??

$$\sum_{(c,d) \equiv \nu \bmod N} \frac{1}{(c\tau+d)^k}$$

Question: Which one appears in $L_0^2(\Gamma\backslash G)$? and multiplicity?

Multiplicity one lemma: (π, V) irr unitary repn of G (automatic admissible)

- π contains the trivial repn K at most <u>once</u>

i.e. $\dim V^K \leq 1$

- $\dim V^K = 1 \iff \pi \simeq \pi_s^{\pm}$ or π_s^C
- $\pi_k^{\pm}$ contains $\sigma_{\pm k}: k_\theta \longmapsto e^{\mp ik\theta}$, doesn't contain σ_0

Theorem: multiplicity of $\pi_k^+, \pi_s^+, \pi_s^C$ in the decomposition of $L_0^2(\Gamma\backslash G)$

- π_k^+: multiplicity $= \dim_{\mathbb{C}} S_k(\Gamma)$
- π_s^+ & π_s^C: multiplicity $= \dim_{\mathbb{C}} W_s(\Gamma)$

pf: Consider $H = L_0^2(\Gamma\backslash G)(\pi_k^+) \simeq (\pi_k^+)^{\oplus n}$ then restricted to $K = SO(2)$

$$H \simeq \hat{\bigoplus_{\ell}} H(\ell)$$

We also know that

$$\pi_k^+ \simeq \hat{\bigoplus_{\ell}} \pi_k^+(\ell), \text{ and each } \pi_k^+(\ell) \text{ is 1-dim or } 0$$

moreover, $\pi_k^+(\pm k) \neq 0 \Rightarrow \dim \pi_k^+(\pm k) = 1 \Rightarrow$ multiplicity of $\pi_k^+ = \dim_{\mathbb{C}} H(k)$

$H(k)$ consists of functions ϕ:

a) $\phi \in L_0^2(\Gamma\backslash G) \cap C^\infty(\Gamma\backslash G)$

b) $\phi(g k_\theta) = \phi(g) e^{-ik\theta}$

c) $\Delta\phi = -\frac{k}{2}(\frac{k}{2}-1)\phi$

d) growth condition: $|\phi(z,\theta)| \leq C \cdot |y|^N$ when $y \to +\infty$ for some C & N

define $f(z) = \phi(g(i)) j(g,i)^k$, b) guarantee well-defined, a) $\Rightarrow$ smooth, cuspidal, $f[\gamma]_k = f, \forall \gamma \in \Gamma$

c) $\Rightarrow$ holomorphic on $\mathcal{H}$ lowest weight $\Rightarrow$ holomorphic   constant term

d) $\Rightarrow$ holomorphic at cusps! $\rightsquigarrow$ meromorphic term $= 0$ $\Big] \Rightarrow$ cusp form

i.e.

$$H(k) \simeq S_k(\Gamma) \Rightarrow \text{multiplicity of } \pi_k^+ \text{ in } L_0^2(\Gamma\backslash G) = \dim_{\mathbb{C}} H(k) = \dim_{\mathbb{C}} S_k(\Gamma)$$

Let's consider the case $\pi_s^{\pm}$ or π_s^C we know

$$\pi_s^+ \simeq \hat{\bigoplus_{\ell}} \pi_s^+(\ell), \text{ and } \pi_s^+(0) \neq 0, \text{ has } \dim = 1$$

then <u>multiplicity of π_s^+ in $L_0^2(\Gamma\backslash G)$</u> $= \dim_{\mathbb{C}} L_0^2(\Gamma\backslash G)(\pi_s^+)(0)$

these space consists of functions ϕ:

a) $\phi \in L_0^2(\Gamma\backslash G) \cap C^\infty(\Gamma\backslash G)$
b) $\phi(g k_\theta) = \phi(g)$
c) $\Delta \phi = \frac{1-s^2}{4}\phi$
d) growth condition $|\phi(z,\theta)| \leq C \cdot |y|^N$

$$\xRightarrow{f(z) = \phi(g)}$$

a) $f(\gamma z) = f(z)$, $\forall \gamma \in \Gamma$, $f \in C^\infty(\mathcal{H}) \cap L^2(\Gamma\backslash\mathcal{H})$
b) well-defined of f
c) $\Delta^* f = \frac{1-s^2}{4} f$, $\Delta^* = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$
d) $|f(z)| \leq C\, |y|^N$ for $y \to +\infty$

why this imple f is bounded ?

↳ actually, cusp form $\Rightarrow$ decrease rapidly

Miscellaneous results about the decomposition $L^2(\Gamma \backslash SL_2(\mathbb{R}))$

Thm: In the decomposition of $L_0^2(\Gamma\backslash G)$, infinitely many class 1 repns occur
i.e. $W_s(\Gamma) \neq 0$ for infinitely many s

pf:

$$L_0^2(\Gamma\backslash G) = \hat{\bigoplus_\ell} L_0^2(\Gamma\backslash G, \sigma_\ell) \quad \text{as } K\text{-repns}$$

We know $L^0(\Gamma\backslash G, \sigma_0) = L_0^2(\Gamma\backslash G)^K = L_0^2(\Gamma\backslash G/K)$ is made up of $\hat{\bigoplus_\pi} \pi^K$, here π ranges over irr G-repn appearing in L_0
each π^K is 0 or 1-dim, but $\pi^K \neq 0 \iff \pi \simeq \pi_s^+$ or π_s^C

- Hilbert-Schmidt operators & Admissibility & irr decomposition theorem

Our goal in the section is to prove that:

Suppose $G = SL_2(\mathbb{R})$ or $GL_2(\mathbb{R})^+$, then

1. irr unitary $\Rightarrow$ admissible
2. in admissible repn, K-finite $\Rightarrow$ smoothness
3. $L_0^2(\Gamma\backslash G)$ decomposes into Hilbert direct sum of irr repns   3'. $L_0^2(G_{\mathbb{Q}}\backslash G_{\mathbb{A}}, \psi)$ decomposes ..
4. every irr repn appears in $L_0^2(\Gamma\backslash G)$ with finite multiplicity   4'. every irr repn appears in $L_0^2(G_{\mathbb{Q}}\backslash G_{\mathbb{A}}, \psi)$ with ...
5. Rapid decreasing property of automorphic functions $\Rightarrow$ Whittaker function summation is absolutely convergent

These results will be consequences of good properties of Hilbert-Schmidt operators

<u>Def</u>: (Smooth functions representation)

强弱

Suppose (π, V) is a repn of G on a ~~separable~~ Hilbert Space V (continuous, not necessarily unitary) → don't need this condition, any Hilbert space, we can directly define integral by approximating by locally constant function

take $\phi \in C_c^\infty(G)$, define $\pi(\phi) \in End(V)$ by:

$$\pi(\phi)v = \int_G \phi(g)\,\pi(g)v\,dg$$

take a basis for V, this becomes integral on coordinates

here the integral makes sense because $g \longmapsto \phi(g)\pi(g)v$ is compactly supported, continuous function

<u>Def</u>. (Hilbert-Schmidt operator)

X locally compact topological space endowed with positive Borel measure, assume $L^2(X)$ separable,

Let $K \in L^2(X\times X)$, then

$$(Tf)(x) = \int_X f(y)\,K(x,y)\,dy \quad \text{is compact operator on } L^2(X)$$

<u>Prop</u>:
- $\pi(\phi)$ is bounded, hence continuous
- if π is unitary, and $\psi(g) = \overline{\phi(g^{-1})}$, then $\pi(\psi)$ is the adjoint of $\pi(\phi)$

pf: Suppose ϕ is supported on compact subset K, and $|\pi(g)| \le B$, $\forall g \in K$

then $$|\pi(\phi)v| \le B\cdot \int_K |\phi(g)|\,dg \cdot |v| \Rightarrow \text{bounded} \Leftrightarrow \text{continuous}$$

now we consider adjoint:

$$\langle \pi(\phi)v, w\rangle = \int_G \phi(g)\langle \pi(g)v, w\rangle\,dg = \int_G \phi(g)\,\langle v, \pi(g^{-1})w\rangle\,dg = \langle v, \pi(\psi)w\rangle$$

here we omit the fact that $\int_G \phi(g)\,dg = \int_G \phi(g^{-1})\,dg$, this follows from the fact that $I\circ inv$ is also Haar measure, and on compact subset K, two integrals agree! thus they must equal to each other

$\rightsquigarrow Z^+$-invariant

We focus on the most interested case to us: $V = L^2(\Gamma\backslash G, \chi)$, π is right regular representation ρ

then $\rho(\phi)$ has even better properties as follows

But Notice that, for congruence subgroups, $\Gamma\backslash G/Z^+$ is not compact, then V is not necessarily separable,
So here We first focus on the case that $\Gamma\backslash G/Z^+$ is compact $\Leftrightarrow \Gamma\backslash\mathcal{H}$ compact, then V is separable

Prop 1: Suppose $\Gamma\backslash G$ is compact, then

① $\rho(\phi)$ is compact & $\forall f \in L^2(\Gamma\backslash G, \chi) \Rightarrow \rho(\phi) f \in C^\infty(\Gamma\backslash G, \chi)$

② If $\phi(g) = \overline{\phi(g^{-1})}$, then $\rho(\phi)$ is self-adjoint

③ If $\phi(e^{i\theta} g) = e^{-ik\theta}\phi(g)$, then $\rho(\phi)$ maps $L^2(\Gamma\backslash G, \chi)$ to $C^\infty(\Gamma\backslash G, \chi, k)$

pf: $$(\rho(\phi) f)(g) = \int_G \phi(h) f(gh)\, dh = \int_G f(h)\phi(g^{-1}h)\, dh$$
$$= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} \chi(\gamma) f(h) \phi(g^{-1}\gamma h)\, dh$$
$$= \int_{\Gamma\backslash G/Z^+}\int_{Z^+} \sum_{\gamma\in\Gamma} \chi(\gamma) f(h)\ \phi(g^{-1}\gamma h u)\, du\, dh$$
$$= \int_F f(h) \underbrace{\left(\sum_{\gamma\in\Gamma} \chi(\gamma) \int_{Z^+} \phi(g^{-1}\gamma h u)\, du\right)}_{= K(g,h)} dh$$ $\rightsquigarrow$ Hilbert-Schmidt operator

ϕ smooth $\Rightarrow K(g,h)$ is smooth (For details, see my note in Automorphic representation)
then since $F\times F$ compact $\Rightarrow K(g,h)$ is square-integrable, then by Thm 2.3.2 in [Bump] $\Rightarrow \rho(\phi)$ compact
What's more, since $K(g,h)$ is smooth w.r.t $g\in G \Rightarrow \rho(\phi) f$ must be smooth

②, ③ follows from simple computation

Prop 2. (π, H) unitary representation of G on a Hilbert space H. take $0\neq f\in H$ can we generalize this to adelic representation?

① $\forall \varepsilon > 0$, $\exists \phi \in C_c^\infty(G)$, s.t.
$\pi(\phi)$ is self-adjoint & $|\pi(\phi) f - f| < \varepsilon$ Smooth vectors are dense

② if $\pi(e^{i\theta}) f = e^{ik\theta} f$, then we can choose ϕ in ① s.t. $\phi(\kappa_\theta g) = \phi(g\kappa_\theta) = e^{-ik\theta}\phi(g)$

③ if $\dim_{\mathbb{C}} H(k) < +\infty$, then we can find ϕ, s.t. $\pi(\phi) f = f$ $(f\in H(k))$

pf: ① the idea is to find a smooth function compactly supported in a small nbd U of $1\in G$
U has the property that $|\pi(g) f - f| < \varepsilon$, $\forall g\in U$, then choose $\phi_0 \geq 0$, $\int_U \phi_0(g)\, dg = 1$, $\phi_0(g) = \phi_0(g^{-1})$

② too complicated, see [Bump 2.3.2]. ③, ① tells us $\{\pi(\phi) f\}_{\phi\in C_c^\infty(K\backslash G/K, \sigma_k)}$ is dense in $H(k)$, must be $H(k)$

Now let's see some applications of Prop 1 & Prop 2

Admissibility Theorem: Every irr unitary repn (π, V) of G is admissible

pf: We should prove that $\dim_{\mathbb{C}} V(k) < +\infty$ $\qquad \pi(K_\theta)V = e^{ik\theta}V \qquad \pi(\phi)v = \int_G \phi(g)\pi(g)v\,dg \qquad g' = K_\theta g K_\theta^{-1}$

$\pi(K_\theta)(\pi(\phi)v) = \int \phi(g)\,\pi(K_\theta g)v\,dg \sim \int \phi(K_\theta^{-1} g K_\theta)\pi(gK_\theta)v\,dg$

The idea is: $C_c^\infty(K\backslash G/K, \sigma_{-k})$ acts on $V(k)$, if we choose $\phi \in C_c^\infty(K\backslash G/K, \sigma_{-k})$ carefully, s.t. $\pi(\phi)$ is self-adjoint $\quad e^{ik\theta}\int \phi(g)\pi(g)v\,dg$

then $\exists \lambda \neq 0$, s.t. $\dim_{\mathbb{C}} V(k)_\lambda < +\infty$, $V(k)_\lambda \neq 0$

Then by the fact that $C_c^\infty(K\backslash G/K, \sigma_{-k})$ is commutative under convolution, its action will preserve the eigenspace $V(k)_\lambda$, then by another result that $V(k)$ is irr $C_c^\infty(K\backslash G/K, \sigma_{-k})$-mod $\Rightarrow V(k) = V(k)_\lambda$ finite dimensional

So we will prove: ① $C_c^\infty(K\backslash G/K, \sigma_{-k})$ is commutative under convolution $\quad \mathcal{H}_K$ is commutative

② $V(k)$ is an irr $C_c^\infty(K\backslash G/K, \sigma_{-k})$-mod $\quad \Leftrightarrow V^K$ is irr $\mathcal{H}_K$-mod

$\Downarrow$

考虑可能k型

1 证明!

① Cartan decomposition

$$K\backslash G/K = \left\{ \begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix} \,\middle|\, d_1 \geq d_2 > 0 \right\} \qquad g \rightsquigarrow \begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix}. \text{ then } g^{-1} \sim \begin{pmatrix} d_2^{-1} & \\ & d_1^{-1} \end{pmatrix}$$

then we define a convolution on $C_c^\infty(K\backslash G/K, \sigma_{-k})$

$$\hat{\phi}(g) = \phi\left(\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} g^T \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \right)$$

$$\hat{\phi}(K_\theta g) = \phi\left(\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} g^T K_{-\theta} \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \right) = \phi\left(\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} g^T \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} K_\theta \right) = \phi\left(\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} g^T \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \right) e^{ik\theta}$$

easy to verify that $(\phi_1 * \phi_2)^\wedge = \hat{\phi}_2 * \hat{\phi}_1$, and $\phi = \hat{\phi}$

$$\hat{\phi}(gK_\theta) = \phi\left(\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} K_{-\theta} g^T \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \right) = \phi\left(K_\theta \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} g^T \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \right) = e^{ik\theta}\hat{\phi}(g)$$

② $V(k)$ is an irr $C_c^\infty(K\backslash G/K, \sigma_{-k})$-mod, i.e. the only closed subspace invariant under $\pi(\phi)$ is $V(k)$ or 0

Now suppose we have such a subspace $H \neq 0$, we will prove $H = V(k)$, If $H \neq V(k)$, $H^\perp \neq 0$ (in $V(k)$), take $0 \neq v \in H^\perp$

$$X = \text{closure of subspace spanned by } \pi(\phi)v,\ \phi \in C_c^\infty(G)$$

Claim: $X \perp H$

pf: we only need to show $\pi(\phi)v \in H^\perp$ (in V)

$$\langle \pi(\phi)v, \underset{H}{\underset{\cap}{w}} \rangle = \int_G \phi(h)\langle \pi(h)v, w\rangle\,dh$$

$$= \int_G \phi(h) \int_{S^1\times S^1} \langle \pi(h)\pi(K_\theta)v, \pi(K_{-\sigma})w\rangle\, \sigma_k(K_\theta \cdot K_\sigma)\,d\theta\,d\sigma\,dh$$

$$= \int_G \phi(h) \int_{S^1\times S^1} \langle \pi(K_\sigma h K_\theta)v, w\rangle\, \sigma_k(K_\theta K_\sigma)\,d\theta\,d\sigma\,dh \qquad h \rightsquigarrow K_\sigma h K_\theta \qquad \int \phi(K_\sigma h K_\theta)\pi(h^{-1})w\,dh$$

$$= \int_G \langle v, \overline{\phi(h)} \int_{S^1\times S^1} \pi(K_{-\theta} h^{-1} K_{-\sigma})\sigma_{-k}(K_\theta K_\sigma)\,w\,d\theta\,d\sigma \rangle\,dh$$

$$= \langle v, \pi(\phi_0)w\rangle, \text{ where } \phi_0(h) = \int_{S^1\times S^1} \phi(K_\sigma h^{-1} K_\theta)\sigma_{-k}(K_\theta K_\sigma)\,d\theta\,d\sigma,\ \phi_0 \in C_c^\infty(K\backslash G/K, \sigma_{-k})$$

$$= 0, \text{ since } w \in H \Rightarrow \pi(\phi_0)w \in H$$

moreover, X is invariant under G

Decomposition theorem: (For $\Gamma\backslash\mathcal{H}$ compact)
The Hilbert space $L^2(\Gamma\backslash G, \chi)$ decomposes into Hilbert direct sum of subspaces irreducible invariant under the repn ρ

pf: Consider the set of the following sets

$$S = \{ H \subseteq L^2(\Gamma\backslash G) \mid H \text{ closed, irr, invariant under } \rho, \text{ and } H \perp H' \text{ if } H \neq H' \}$$

By Zorn's lemma, $\exists$ a maximal one S_{max}. then we form a subspace

$$\mathfrak{H} = \hat{\bigoplus_{H \in S_{max}}} H$$

We will show $\mathfrak{H} = L^2(\Gamma\backslash G, \chi)$. $\Longleftrightarrow$ $\overset{=V}{\mathfrak{H}^\perp}$ is zero, hence we suppose $\mathfrak{H}^\perp \neq 0$, by the definition of S_{max}, we only need to show $\underset{\overset{\shortparallel}{V}}{\mathfrak{H}^\perp}$ admits an irr invariant subspace under ρ

Take $\phi \in C_c^\infty(G)$, s.t. $\rho(\phi)$ is self-adjoint unitary & nonzero $\Rightarrow$ $\rho(\phi)$ admits a non-zero eigenvalue $\lambda \neq 0$ and since $\rho(\phi)$ is compact & self-adjoint $\Rightarrow$ $\dim_{\mathbb{C}} V_\lambda < +\infty$

We consider the smallest nonzero subspace of V_λ of the following form:

$$V_0 = V_\lambda \cap W$$

where W is a closed G-invariant subspace, and define

$$W_0 = \bigcap_{W : V_\lambda \cap W = V_0} W$$

We will prove W is irreducible

Suppose not, then $\exists W_1, \overset{\text{unitary}}{W_2} \subsetneq W$, s.t. $W = W_1 \oplus W_2$, W_i is G-invariant closed subspace
We consider $V_\lambda \cap W_i$, since $W_i \subsetneq W_0 \Rightarrow V_\lambda \cap W_i = 0$, now we choose $0 \neq f \in V_0$, then

$$f = f_1 + f_2 \quad , f_i \in W_i$$

$$\rho(\phi) f = \rho(\phi) f_1 + \rho(\phi) f_2 = \lambda f = \lambda f_1 + \lambda f_2$$

therefore we must have $f_i \in V_\lambda \Rightarrow f_i \in V_\lambda \cap W_i = 0 \Rightarrow f = 0$, contradiction

Finiteness of multiplicity: Every irr unitary repn of G, say (π, V). appears with finite multiplicity in $L^2(\Gamma\backslash G)$

pf: Consider $\mathrm{Hom}_G(V, L^2(\Gamma\backslash G))$, then $\exists k$, s.t. $V(k) \neq 0$, then by Prop 2 ③, $\exists \phi \in C_c^\infty(G)$, s.t. $\pi(\phi) v = v$
for some $0 \neq v \in V(k)$

therefore $\forall f : V \to L^2(\Gamma\backslash G)$. $\rho(\phi)(fv) = f(\rho(\phi)v) = fv \Rightarrow fv \in \underbrace{L^2(\Gamma\backslash G)_1}_{\text{finite-dimensional}}$

and obviously f is determined by fv by irreducible condition $\Rightarrow$

$$\mathrm{Hom}_G(V, L^2(\Gamma\backslash G)) \hookrightarrow L^2(\Gamma\backslash G)_1$$

But in many situations, $\Gamma\backslash\mathcal{H}$ is not compact. such as congruence subgroups, then $L^2(\Gamma\backslash G)$ fails to be separable then $\rho(\phi)$ may not be compact on $L^2(\Gamma\backslash G)$

Here comes a crucial property of cusp forms:

① $f \in L_0^2(\Gamma\backslash G/Z^+)$
② $f(\gamma g) = \chi(\gamma) f(g)$
③ $f(zg) = \omega(z) f(g)$
④ $\int_0^1 f(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g)\, dx = 0$

Theorem: Let $\phi \in C_c^\infty(G)$

1. There exists a constant C depending on ϕ. s.t. for $\forall f \in L_0^2(\Gamma\backslash G, \chi, \omega)$

$$\sup_{g\in G} |(\rho(\phi)f)(g)| \leq C \cdot \| f\|_2$$

More precisely, for $\forall g \in \mathcal{G}_{c,d}$, suppose $g = \begin{pmatrix} z & \\ & z \end{pmatrix}\begin{pmatrix} y & x \\ & 1 \end{pmatrix} k_\theta$, $\left(g \in \mathcal{G}_{c,d} := y \geq c,\ |x| \leq d\right)$, then for $\forall N>0$, $\exists C>0$, s.t.

$$\left| (\rho(\phi)f)(g) \right| \leq C \cdot |y|^{-N} \cdot \| f\|_2 \qquad (*)$$

2. $\rho(\phi)$ is compact when restricted to $L_0^2(\Gamma\backslash G, \chi, \omega)$

proof: See my notes

Rmk: The estimate $(*)$ is the origin of moderate growth condition for automorphic forms
remember the definition for Γ-automorphic forms:

- $\phi(\gamma g) = \chi(\gamma)\phi(g)$
- $\phi(gz) = \omega(z)\phi(g)$, $z \in Z(G)$
- $\phi(g k_\theta) = \phi(g) e^{-ik\theta}$ or simply right K-finite $\rightsquigarrow$ ϕ_i K-finite, hence smooth!,
  $\rightsquigarrow$ by finite multiplicity, imply $\phi = \sum_i \phi_i$, $\phi_i \in \pi_i$ (finite; irr)
- $\Delta\phi = \lambda\phi$
- moderate growth condition:

$$|\phi(g)| \leq C \cdot |g|^N, \quad g = \begin{pmatrix} z & \\ & z \end{pmatrix}\begin{pmatrix} y & x \\ & 1 \end{pmatrix} k_\theta$$

$(*)$ tells us that, for caspidal Γ-automorphic forms, ϕ is rapidly decreasing in some Sigel domain $\mathcal{G}_{c,d}$
because we can find ϕ. s.t. $\rho(\phi) f = f$

This theorem will imply:

- $L_0^2(\Gamma\backslash G, \chi)$ decomposes as Hilbert direct sum of irr closed G-subspace
- every irr repn of G appears with finite multiplicity in $L_0^2(\Gamma\backslash G, \chi)$

We also have an adelic version of this theorem:

Theorem: $\phi \in C_c^\infty(GL(n, \mathbb{A}))$

1. $\exists C > 0$, depending on ϕ, s.t. for $\forall f \in L_0^2(G_\mathbb{Q}\backslash G_\mathbb{A}, \omega)$
$$\sup_{g \in GL(n,\mathbb{A})} |(\rho(\phi) f)(g)| \leq C \cdot \| f\|_2$$
More precisely, for $g \in \mathfrak{S}^{\mathbb{A}}_{c,d}$, (which means $g_\infty \in \mathfrak{S}_{c,d}$) $g_\infty = \begin{pmatrix} z & \\ & z \end{pmatrix}\begin{pmatrix} y & x \\ & 1 \end{pmatrix} \kappa_\theta$
$$|(\rho(\phi) f)(g)| \leq C \cdot |y|^{-N} \| f\|_2 \qquad (*)$$
$GL_2(\mathbb{A}) = GL_2(\mathbb{Q}) \cdot \mathfrak{S}_{c,d}$

2. $\rho(\phi)$ is compact restricted to $L_0^2(G_\mathbb{Q}\backslash G_\mathbb{A}, \omega)$

$|(\rho(\phi) f)(g)|$, $g = \gamma g_1$,

$=$

$g_1 \in \mathfrak{S}_{c,d}$

$(g_1)_\infty = \begin{pmatrix} y & x \\ & 1 \end{pmatrix} \kappa_\theta$

$\begin{pmatrix} z & \\ & z \end{pmatrix}$

Recall: $L^2(G_\mathbb{Q}\backslash G_\mathbb{A}, \omega)$ & $L_0^2(G_\mathbb{Q}\backslash G_\mathbb{A}, \omega)$

- $\phi(\gamma g) = \phi(g)$, $\forall \gamma \in G_\mathbb{Q}$
- $\phi(z g) = \omega(z)\phi(g)$, $z \in Z(\mathbb{A})$
- $\displaystyle\int_{Z(\mathbb{A}) G_\mathbb{Q}\backslash G_\mathbb{A}} |\phi(g)|^2 \, dg < +\infty$

cuspidal condition

- $\displaystyle\int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx = 0$ for almost $\forall g \in G$

Rmk: Theorem 1 (*) will give us an interpretation of moderate growth condition:

ϕ is slowly increasing, i.e. for $\forall c > 0$, $\forall$ compact subset Ω of $G(\mathbb{A})$, $\exists C, N > 0$, s.t
$$\phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) \leq C |a|^N$$
$|\phi(g)| = |f(z) \, Im(z)^{\frac{k}{2}}| \leq M$ is bounded

for $\forall g \in \Omega$, $\forall a \in \mathbb{A}^\times$, s.t $|a| > c$

$\Longleftrightarrow$ $|\phi(g)| \leq C \cdot \|g\|^N$

Cusp forms will correspond to those elements which decrease rapidly!

$\hookrightarrow$ in a given irr subquo

This theorem will imply:

- $L_0^2(G_\mathbb{Q}\backslash G_\mathbb{A}, \omega)$ decomposes as Hilbert direct sum of irr closed G-subspace
- every irr repn of G appears with finite multiplicity in $L_0^2(G_\mathbb{Q}\backslash G_\mathbb{A}, \omega)$ multiplicity 1

· Fast decay of cusp forms

Key input: admissibility $\Rightarrow \exists \phi,\ \rho(\phi)f = f$

$$|f(g)| \le C \cdot \|g\|^{-N} \text{ when } \|g\| \text{ is large}$$

Suppose $f \in (\pi, H) \underset{\substack{\text{irr}\\\text{closed}}}{\subseteq} L_0^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$

then we know $\quad H = \widehat{\bigoplus}_{\rho \in \hat{K}} H(\rho) = \widehat{\bigoplus}_{\rho_f \in \hat{K}_f} \underbrace{\widehat{\bigoplus}_{\rho_\infty \in \hat{K}_\infty} H(\rho_\infty \otimes \rho_f)}_{\substack{\shortparallel\\ H(\rho_f)}}$

f invariant under some $K' \subset K_f$, then consider $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f) / K_0 = \coprod_i G(\mathbb{Q})_+ g_i K_0$

$$G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f)/K_f \simeq G(\mathbb{Q})_+ \backslash D^+ \times G(\mathbb{A}_f)/K_f$$

$f_i(g_\infty) := f((g_\infty, g_i))$, for $\gamma \in G(\mathbb{Q}) \cap G(\mathbb{R})^+ g_i K_0 g_i^{-1}$, we have $\Gamma_i = G(\mathbb{Q})_+ \cap g_i K_0 g_i^{-1}$

$$f_i(\gamma g_\infty) = f((\gamma g_\infty, g_i)) = f(\gamma g_\infty, \gamma\gamma^{-1} g_i)$$

$$= f(g_\infty, \gamma^{-1} g_i) = f(g_\infty, g_i k') = f_i(g_\infty)$$

i.e. $f_i(g_\infty) \in L^2(\Gamma_i \backslash G(\mathbb{R})^+, \omega)$

cuspidality: $\sigma(\infty)$

$$\int_0^1 f_i\left(\sigma\begin{pmatrix}1 & xh\\ & 1\end{pmatrix}\right)dx = \int_0^1 f\left(\sigma\begin{pmatrix}1 & xh\\ & 1\end{pmatrix}, g_i\right)dx = \int_0^1 f\left(\left(\begin{pmatrix}1 & xh\\ & 1\end{pmatrix}, \sigma^{-1} g_i\right)\right)dx = 0$$

$\Rightarrow \quad f_i \in \widehat{\bigoplus}_i (\pi_i)^{N_i} \qquad \pi_i$ is an irr repn of $G(\mathbb{R})^+ \subseteq L_0(\Gamma_i \backslash G(\mathbb{R})^+, \omega_\infty)$

then $\Delta f = \lambda f \Rightarrow$ only one π_i appears $\Rightarrow f_i \in \pi_i^{N_i}$, we take $f_i \in \pi_i(k)^{N_i}$, we

$$f_i(g\kappa_\theta) = e^{ik\theta} f_i(g) \qquad \dim \pi_i(k) = 1 \ (c + \infty)$$

$\Rightarrow \exists \phi \in C_c^\infty(K\backslash G/K, \sigma_k),\quad \rho(\phi) f_i = f_i$

$\Rightarrow \quad |f_i(g)| = |\rho(\phi) f_i(g)| \le C \cdot |y|^{-N}$, when $g \in \mathfrak{S}_{c,d}$ for some $\overset{y}{\underset{\shortparallel}{c}}, \overset{|x|}{\underset{\wedge}{d}} > 0$

$$|f(g)| = |f((g_\infty, g_f))| = |f((g_\infty, \gamma g_i k))| = |f(\gamma^{-1} g_\infty, g_i)| = |f_i(\gamma^{-1} g_\infty)|$$

$$g = \gamma \cdot (\overbrace{\gamma^{-1} g_\infty}^{g_\infty'} \cdot g_i) \cdot k \ \leadsto \qquad \le C \cdot |g_\infty'(y)|^{-N} \le C' \|g\|^{-N'}$$

Now we focus on "Adelic" representations.

First, we are curious about similar result for Prop 2

Prop 2'. (π, H) unitary representation of $G_{\mathbb{A}}$ on Hilbert space H. take $0 \neq f \in H$

① $\forall \varepsilon > 0$, $\exists \phi \in C_c^\infty(G_{\mathbb{A}})$, s.t.
$\pi(\phi)$ is self-adjoint & $|\pi(\phi)f - f| < \varepsilon$

② if $f \in H(\sigma)$, $\sigma \in \mathrm{Irr}(K)$, then we can choose ϕ in ① s.t. left & right regular action of K on $\phi \simeq \rho^* \;\&\; \rho$ respectively

③ if $\dim_{\mathbb{C}} H(\sigma) < +\infty$, then we can find ϕ, s.t. $\pi(\phi)f = f$ $(f \in H(\sigma))$

A small summary.

- Every irr unitary repn of $GL_2(\mathbb{R})^+$ / $SL_2(\mathbb{R})$ is admissible
- Suppose V is an unitary repn of $GL_2(\mathbb{R})^+ / SL_2(\mathbb{R})$. & $\rho(\phi)$ action is compact, $\forall \phi \in C_c^\infty(G)$
  then V decomposes into Hilbert direct sum of irr repns / with finite multiplicity

Admissibility $\Rightarrow$ for $\forall$ K-finite vector f, $\exists\ \phi$, s.t. $\rho(\phi)f = f$ $\Rightarrow$ (K-finite $\Rightarrow$ smooth)

$\Rightarrow$ f wsp $\Rightarrow$ f decays fast !

Admissibility for unitary irr repns of G_A

<u>Theorem</u>: Every irr unitary repn of G_A is admissible, hence factorizable

<u>Corollary</u>: Cusp forms decrease rapidly.

· Irreducible $(\mathfrak{g}, K)$-mod for $G = SL_2(\mathbb{R})$

Let's now try to classify all the irr $(\mathfrak{g}, K)$-mod for $SL_2(\mathbb{R})$

We know that $V = \bigoplus_{k \in \mathbb{Z}} V(k)$ ← finite-dimensional (admissible)

and we have a basis for $sl_2(\mathbb{R})$

$$(*) \quad R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S^2 = -I$$

$$[H, R] = 2R, \quad [H, L] = -2L, \quad [R, L] = H$$

$$\exp(tS) = \sum_{n=0}^{\infty} \frac{t^n S^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1} S}{(2n+1)!} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = K_t$$

· Suppose $V(k_0) \neq 0$ for a given $k_0 \in \mathbb{Z}$, then take $0 \neq v \in V(k_0)$

$$S v = \lim_{t \to 0} \frac{\pi(\exp(tS)) \cdot v - v}{t} = \lim_{t \to 0} \frac{K_t \cdot v - v}{t} = \lim_{t \to 0} \frac{e^{-ik_0 t} - 1}{t} v = -ik_0 v$$

We hope deal with S instead of H, thus we find eigenvector for ad_S in $\mathfrak{g}_{\mathbb{C}}$
instead of (*), we consider the following basis for $\mathfrak{g}_{\mathbb{C}}$

$$R = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad H = iS = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$[H, R] = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix} = 2R, \quad [H, L] = -2L,$$

$$[R, L] = \frac{1}{4}\begin{pmatrix} 2 & -2i \\ 2i & 2 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 2 & 2i \\ -2i & 2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & -4i \\ 4i & 0 \end{pmatrix} = H$$

$$\Delta = -\frac{1}{4}\left(H^2 + 2RL + 2LR\right) = -\frac{1}{4}\left(H^2 + 2\overset{H}{\overset{\shortparallel}{[R,L]}} + 4LR\right) \quad \text{acts}$$

$$= -\frac{1}{4}\left(H^2 - 2H + 4RL\right) \quad \Rightarrow \text{scalar, say } \lambda$$

then $Hv = k_0 v$ (H acts on $V(k)$ by k),

now consider Rv: $2Rv = [H, R]v = HRv - k_0 Rv \Rightarrow H(Rv) = (k_0 + 2)Rv$

$\Rightarrow Rv \in V(k_0 + 2)$, similarly, $Lv \in V(k_0 - 2)$

What about RL?

$$\lambda \cdot v = -\frac{1}{4}\left(k^2 - 2k + 4RL\right)v$$

$$RL \cdot v = \left(-\frac{k}{2}\left(\frac{k}{2} - 1\right) - \lambda\right) v \in \mathbb{C}v \qquad \frac{k(2-k)}{4}$$

$$LR\, v = (RL - \underset{H}{\underset{\shortparallel}{[R, L]}})v = \left(-\frac{k}{2}\left(\frac{k}{2} + 1\right) - \lambda\right) v \in \mathbb{C}v$$

$$\frac{-k(k+2)}{4}$$

Case 1, $\lambda \neq \frac{k(2-k)}{4}$ for any $k \in \mathbb{Z}$.

Start from

$$\mathbb{C}v \oplus \underbrace{\mathbb{C}Rv}_{\downarrow} \oplus \underbrace{\mathbb{C}R^2v} \oplus \cdots \oplus \mathbb{C}R^n v \oplus \cdots$$

$Rv \neq 0$ because $LRv \neq 0$ $\Rightarrow$ same argument shows $R^n v \neq 0$, $\forall n \geq 0$

now consider Lv, $Lv \neq 0$ because $RLv \neq 0 \Rightarrow$ same argument shows $L^n v \neq 0$, $\forall n \geq 0$

$$\Rightarrow \bigoplus_{n\geq 1} L^n v \oplus \mathbb{C}v \bigoplus_{n \geq 1} \mathbb{C}R^n v \text{ invariant under } (\mathfrak{g}, K)\text{-action}$$

Case 2: $\lambda = \frac{k(2-k)}{4}$ for some $k \geq 1$

generally, take $v \in V(k_0)$, $v \neq 0$.

consider $\bigoplus_{n\geq 1} \mathbb{C}L^n v \oplus \mathbb{C}v \bigoplus_{n\geq 1} \mathbb{C}R^n v$

$\frac{-k_0(2+k_0)}{4}$

suppose $R^{n-1}v \neq 0$, but $R^n v = 0 \Rightarrow$ just assume $n = 1$

i.e. $Rv = 0 \Rightarrow 0 = LRv = \left(-\frac{k_0}{2}\left(\frac{k_0}{2}+1\right) - \lambda\right)v \Rightarrow \lambda = -\frac{k_0}{2}\left(\frac{k_0}{2}+1\right)$

$-k_0 = k \Rightarrow k_0 = -k$

or $-k_0 = 2-k \Rightarrow k_0 = k-2$

Note: $\underset{V(k_0)}{\underset{\cap}{Rv}} = 0 \Longrightarrow LRx = 0 \Rightarrow \lambda = -\frac{k_0}{2}\left(\frac{k_0}{2}+1\right) \Longleftrightarrow -k_0 = k$ or $2-k$

$k_0 = -k$ or $k-2$

$\underbrace{V(-k). \quad V(k-2)}_{\text{may exist } R=0} \quad V(k)$

$Lv = 0 \Longrightarrow RLx = 0 \Rightarrow \lambda = -\frac{k_0}{2}\left(\frac{k_0}{2}-1\right). \Rightarrow k_0 = k$ or $2-k$

$L = 0$: $V(2-k)$ $\quad V(k)$

$R = 0$: $V(-k)$ $\quad V(k-2)$ $\quad RLv = 0$

$v \in V(k)$, Rv, R^2v, ...

Consider 3 cases. ① $V(k) \neq 0$ $\quad k, k+1, \ldots \rightsquigarrow$

② $V(k) = 0$, $V(k-2) \neq 0$ $\quad 2-k, 4-k, \ldots, k-4, k-2$

③ $V(k) = 0$, $V(k-2) = 0$. $\quad -(k+4), -(k+2), -k$.

<u>Thm</u>: Irreducible $(\mathfrak{g}, K)$-mod V for $GL_2(\mathbb{R})^+ / SL_2(\mathbb{R})$

Suppose Δ acts by scalar multiplication λ, I acts by scalar μ, then

PS   <u>Case 1</u>: $\lambda \neq \frac{k}{2}(1-\frac{k}{2})$, $k \in \mathbb{Z}$, there are two possibilities

① V is even type, then

$$V = \bigoplus_{l \in \mathbb{Z}} \mathbb{C} v_l \qquad H v_l = 2l \cdot v_l, \qquad \text{all even integer}$$

② V is odd type, then

$$V = \bigoplus_{l \in \mathbb{Z}} \mathbb{C} v_l. \qquad H v_l = (2l-1) v_l \qquad \text{all odd integer}$$

<u>Case 2</u>: $\lambda = \frac{k}{2}(1-\frac{k}{2})$ for some $k \geq 1$, then

① k is even

odd type:

PS $\quad V = \bigoplus_{l \in \mathbb{Z}} \mathbb{C} v_l. \qquad H v_l = (2l-1) v_l$

even type:

DS $\quad \Sigma^-(k), \quad \Sigma^0(k), \quad \Sigma^+(k)$

② k is odd

even type:

PS $\quad V = \bigoplus_{l \in \mathbb{Z}} \mathbb{C} v_l \qquad H v_l = 2l \cdot v_l,$

odd type:

DS $\quad \Sigma^-(k), \quad \Sigma^0(k), \quad \Sigma^+(k)$

now if $G = GL_2(\mathbb{R})$, we also has the notion of irreducible $(\mathfrak{g}, O(2))$-module, the only difference is: when $\lambda = \frac{k}{2}(1-\frac{k}{2})$, $k \geq 1$, and V is the same type as k, then:

$$\Sigma^+(k) \oplus \Sigma^-(k) \quad \text{or} \quad \Sigma^0(k)$$

because $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ permutes $\Sigma^+(k)$ & $\Sigma^-(k)$

There is a natural question

- Which irreducible $(\mathfrak{g}, K)$-mod arises from an irreducible representation?

We will see that every irr $(\mathfrak{g}, K)$-mod in the previous theorem arises from a principal series representation, this means that over Archimedean places, there are no cuspidality!

Now for $s_1, s_2 \in \mathbb{C}$, $\varepsilon = 0$ or 1, we define a space $H(s_1, s_2, \varepsilon)$ as follows

$$f \in H(s_1, s_2, \varepsilon) \iff \begin{cases} f \text{ is a measurable function on } G = GL_2(\mathbb{R})^+ \text{ or } SL_2(\mathbb{R}) \\ f\left(\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} g\right) = \operatorname{sgn}(y_1)^{\varepsilon} \left|\frac{y_1}{y_2}\right|^{\frac{1}{2}} \cdot |y_1|^{s_1} \cdot |y_2|^{s_2} \cdot f(g) \end{cases} \quad (y_1, y_2 \text{ must have the same sign})$$

Since we have Iwasawa decomposition $G = BK$, where $K = SO(2) \simeq S^1$, we see that $f \in H(s_1, s_2, \varepsilon)$ is totally determined by its restriction on $K = SO(2)$, then by the classical theory for L^2 space of S^1, we know $H(s_1, s_2, \varepsilon)$ is admissible, and we consider the following function:

$$f_k(\kappa_\theta) = e^{ik\theta}$$

this function can be extended to a function in $H(s_1, s_2, \varepsilon)$ iff $f_k(\kappa_\pi) = e^{ik\pi} = (-1)^\varepsilon$, i.e. $k \equiv \varepsilon \mod 2$.

and obviously, $H(s_1, s_2, \varepsilon)(k)$ consists of extension of this function f_k, i.e.

$$f_k\left(\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} \kappa_\theta\right) = \left|\frac{y_1}{y_2}\right|^{\frac{1}{2}} |y_1|^{s_1} |y_2|^{s_2} \operatorname{sgn}(y_1)^\varepsilon e^{ik\theta}$$

therefore the $(\mathfrak{g}, K)$-mod for $H(s_1, s_2, \varepsilon)$ is all the integers congruent to ε mod 2

To know its irreducibility, we should know the action of Δ and I on f_k. By the standard Iwasawa decomposition, we have:

$$f_k\left(\begin{pmatrix} u & \\ & u \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & y^{-\frac{1}{2}}x \\ & y^{-\frac{1}{2}} \end{pmatrix} \kappa_\theta\right) = u^{s_1+s_2} \cdot y^{\frac{1}{2}(s_1-s_2+1)} \cdot e^{ik\theta}, \text{ where } u, y > 0, \ x \in \mathbb{R}, \ \theta \in [0, 2\pi)$$

note: λ & μ will determine s_1 & s_2

<u>Prop</u>: Denote $s = \frac{1}{2}(s_1 - s_2 + 1)$, $\lambda = s(1-s)$, $\mu = s_1 + s_2$. We have:

$$H f_l = l f_l, \qquad R f_l = \left(s + \frac{l}{2}\right) f_{l+2}, \qquad L f_l = \left(s - \frac{l}{2}\right) f_{l-2}$$

$$\Delta f_l = \lambda f_l, \qquad Z f_l = \mu f_l$$

Therefore, when $s \notin \frac{\mathbb{Z}}{2}$, i.e. $2s \notin \mathbb{Z}$, $H(s_1, s_2, \varepsilon)$ gives rise to an irr $(\mathfrak{g}, K)$-mod, we need more careful analysis when $s = \frac{k}{2}$ for some $k \in \mathbb{Z}$.

- $k \not\equiv \varepsilon \mod 2$, then $H(s_1, s_2, \varepsilon)$ is also irreducible
- $k \equiv \varepsilon \mod 2$, then $H(s_1, s_2, \varepsilon)$ must be reducible,

1. when $k \geq 1$, it has <u>two</u> invariant subspaces:

$$\bigoplus_{l \geq k} \mathbb{C} f_l, \quad \bigoplus_{l \leq -k} \mathbb{C} f_l \qquad \simeq \qquad \Sigma^+(k), \ \Sigma^-(k)$$

the quotient is also irreducible:

$$H(s_1, s_2, \varepsilon)_{fin} \Big/ \bigoplus_{l \leq -k} \mathbb{C} f_l \oplus \bigoplus_{l \geq k} \mathbb{C} f_l \qquad \simeq \qquad \Sigma^0(k)$$

2. when $k \leq 1$, it has <u>one</u> invariant irreducible space

$s = 1 - \frac{m}{2}$, $m \geq 1$

$k = 2 - m$

$$\bigoplus_{2-m \leq l \leq m-2} \mathbb{C} f_l \qquad \simeq \qquad \Sigma^0(m)$$

the quotient space decomposes into direct sum of two irreducible spaces

$$\bigoplus_{l \geq m} \mathbb{C} \bar{f}_l, \quad \bigoplus_{l \leq -m} \mathbb{C} \bar{f}_l \qquad \simeq \qquad \Sigma^+(k), \ \Sigma^-(k)$$

Actually, we only write the K-finite functions. the full space is the following

Let $H^\infty(s_1, s_2, \epsilon)$ be the space of smooth functions $f \in C^\infty(G)$ that satisfy

$$f\left(\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} g\right) = y_1^{s_1+1/2} y_2^{s_2-1/2} f(g), \qquad y_1, y_2 > 0, \qquad (5.8)$$

$$f\left(\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} g\right) = (-1)^\epsilon f(g). \qquad (5.9)$$

We let G act by right translation:

$$(\pi(g)f)(x) = f(xg), \qquad (5.10)$$

for g as in Eq. (5.11). We give $H^\infty(s_1, s_2, \epsilon)$ a Hermitian inner product by defining

$$(f_1, f_2) = \frac{1}{2\pi} \int_0^{2\pi} f_1(\kappa_\theta) \overline{f_2(\kappa_\theta)} \, d\theta. \qquad (5.14)$$

Let $H(s_1, s_2, \epsilon)$ be the Hilbert space completion of this space. It can be identified with $L^2[-\pi/2, \pi/2]$ via the map $\theta \mapsto \kappa_\theta$.

· Question: what's the corresponding functions on $\mathcal{H}$

$f(g\kappa_\theta) = f(g) e^{ik\theta} \Rightarrow$ consider $\tilde{f}(z) = f(g_z) \cdot j(g_z, i)^k$

the reverse direction is:

$f(g) = \varphi(gi) \cdot j(g, i)^{-k}$

$f(g\kappa_\theta) = \varphi(gi) j(g\kappa_\theta, i)^{-k} = \varphi(gi) j(g,i)^{-k} e^{ik\theta} = f(g) e^{ik\theta}$

$j(g\kappa_\theta, i) = j(g, i) \cdot \underset{e^{-i\theta}}{\underset{\shortparallel}{j(\kappa_\theta, i)}}$ $\qquad \underset{e^{-i\theta}}{-i\sin\theta + \cos\theta}$

for a different choice of g_z, g_z'

$g_z' = g_z \cdot \kappa_\theta \Rightarrow$

$f(g_z') \cdot j(g_z', i)^k = f(g_z \cdot \kappa_\theta) \cdot j(g_z, i)^k \cdot j(\kappa_\theta, i)^k$

$= f(g_z) e^{ik\theta} \cdot j(g_z, i)^k e^{-ik\theta} = f(g_z) \cdot e^{ik\theta}$

thus $\tilde{f}$ may not be Γ-invariant, since f may not be $G(\mathbb{Q})_1$-invariant

Unitaricity & Intertwining integrals

Let's first derive some necessary condition for unitary representation. Suppose $\mathfrak{H}$ is an irr repn of G, then there exists a G-invariant Hermitian inner product on $\mathfrak{H}$, i.e.

$$\langle g\cdot v, g\cdot w\rangle = \langle v, w\rangle, \quad \forall v, w \in \mathfrak{H}$$

How does $\mathfrak{g}$ act on it? $\forall X \in \mathfrak{g}$,

$$\langle Xv, w\rangle = \lim_{t\to 0}\langle \exp(tX)v, w\rangle = \lim_{t\to 0}\langle v, \exp(-tX)w\rangle = \langle v, -Xw\rangle$$

therefore we see that X action on $\mathfrak{H}$ is skew-symmetric, now if $Xv = \mu v$, then:

$$\mu\langle v, v\rangle = \langle Xv, v\rangle = \langle v, -Xv\rangle = \langle v, -\mu v\rangle = -\bar{\mu}\langle v, v\rangle$$

i.e. $\mu = -\bar{\mu} \Rightarrow \mu$ is purely imaginary.

Prop: Suppose $\mathfrak{H}$ is an irr unitary repn of $G = GL_2(\mathbb{R})^+$ or $SL_2(\mathbb{R})$, Δ acts by λ, Z acts by μ, then we have:

- Δ is symmetric $\Rightarrow \lambda$ is real
- Z is skew-symmetric $\Rightarrow \mu$ is purely imaginary

Remember in the previous section, we construct the space $H(s_1, s_2, \varepsilon)$, in that space, $\lambda = s(1-s)$, $\mu = s_1 + s_2$

λ is real, μ is purely imaginary $\Longrightarrow$ $s_1 = \frac{\mu \pm \sqrt{1-4\lambda}}{2}$, $s_2 = \frac{\mu \mp \sqrt{1-4\lambda}}{2}$

When $\lambda \geq \frac{1}{4}$, then s_1 & s_2 are both purely imaginary, in this case the induced representation $H(s_1, s_2, \varepsilon)$ is automatically unitary since it is induced from a unitary representation, How to prove this? We need the following lemma

Lemma: Suppose G is a unimodular topological group, P & K are closed subgroups, s.t.

- K is compact when G is reductive, take K to be the maximal compact, P parabolic
- $G = PK$

δ_P is the modular character of P, i.e. $d_R(p) = \delta(p)\, d_L(p)$, and let $C(P\backslash G, \delta)$ denotes the space of continuous functions on G, s.t.

$$f(pg) = \delta(p)\, f(g), \quad p \in P$$

Then the following functional is invariant under the right regular repn of G

$$I(f) = \int_K f(k)\, dk$$

Then we can construct a Hermitian pairing on the space $H(s_1, s_2, \varepsilon)$ when s_1 & s_2 are purely imaginary, actually the following construction is valid for any unitary repn of parabolic subgroup, suppose (σ, V) is an unitary representation of the group P, consider the following induced representation:

$$\mathrm{Ind}_P^G V = \{ f: G \to V \mid f(pg) = \delta(p)^{\frac{1}{2}}\sigma(p) f(g),\ \forall p \in P,\ g \in G.\ f \text{ continuous} \}$$

We also construct a pairing on the induced space:

$$\langle f, g \rangle := \int_K \langle f(k), g(k) \rangle \, dk$$

This pairing is G-invariant, because the function $h \longmapsto \langle f(h), g(h) \rangle$ belongs to $C(P\backslash G, \delta)$ then the Hilbert completion of $\mathrm{Ind}_P^G V$ w.r.t this pairing is a unitary representation of the group G i.e. we have shown the general principle: induced repn of unitary repn is still unitary!

Therefore we have shown:

<u>Case A</u>: when λ is real, $\lambda \geqslant \frac{1}{4}$, μ is purely imaginary, then $H(s_1, s_2, \varepsilon)$ is unitary

Now the remaining case is $\lambda < \frac{1}{4}$ & μ is purely imaginary, we first show that $\lambda > 0$,

Lemma: If $\mathfrak{H}$ is a unitary repn of $GL_2(\mathbb{R})^+/SL_2(\mathbb{R})$, and $\lambda \neq \frac{k}{2}(1-\frac{k}{2})$ for some $k \equiv \varepsilon \mod 2$, which is equivalent to the $(\mathfrak{g}, K)$-mod associated to $\mathfrak{H}$ is all the odd/even numbers, then we have

$$\lambda > 0$$

Laplacian on unitary space is positive!

pf: By the assumption, $f_\varepsilon \in \mathfrak{H}$, then

$$0 < \langle L f_\varepsilon, L f_\varepsilon \rangle = -\langle RL f_\varepsilon, f_\varepsilon \rangle = \langle (\Delta + \frac{H^2 - 2H}{4}) f_\varepsilon, f_\varepsilon \rangle = \left(\lambda + \frac{\varepsilon^2 - 2\varepsilon}{4}\right) \langle f_\varepsilon, f_\varepsilon \rangle$$

therefore $\lambda > \frac{2\varepsilon - \varepsilon^2}{4}$,

$\varepsilon = 0 \Rightarrow \lambda > 0$

$\varepsilon = 1 \Rightarrow \lambda > \frac{1}{4}$ already know it is unitarizable

Therefore essentially there are two cases we still don't know

Case B1. $\varepsilon = 0$, $0 < \lambda < \frac{1}{4}$, μ purely imaginary

Case B2. $\lambda = \frac{k}{2}(1 - \frac{k}{2})$, μ purely imaginary for some $k \geqslant 2$

Next, let's show in Case B1, the repn $H(s_1, s_2, 0)$ is unitary.

In this case $s_1 = \frac{1}{2}(\mu \pm \sqrt{1-4\lambda})$, $s_2 = \frac{1}{2}(\mu \mp \sqrt{1-4\lambda})$, it is not unitary, i.e. $H(s_1, s_2, 0)$ is not induced from a unitary repn

Then the pairing we construct before, which depends on the fact that

$$h \mapsto \langle f(h), g(h) \rangle \in C(P\backslash G, \delta)$$

but in our case $h \overset{\varphi}{\mapsto} \langle f(h), g(h) \rangle$ satisfies: $\varphi(ph) = \langle f(ph), g(ph) \rangle = \delta(p) |\chi(p)|^2 \varphi(h)$, here χ is not unitary

but we have, if the pairing is given by: $H(s_1, s_2, \varepsilon) \times H(-\bar{s}_1, -\bar{s}_2, \varepsilon)$, then

$$\varphi(ph) = \langle f(ph), g(ph) \rangle = \delta(p) \cdot \langle \chi(p) f(h), \bar{\chi}^{-1}(p) g(h) \rangle = \delta(p) \varphi(h) \Rightarrow \varphi \in C(P\backslash G, \delta)$$

Therefore we obtained the following G-invariant Hermitian pairing:

$$\langle f, g \rangle = \int_K \langle f(k), g(k) \rangle \, dk$$

We also observed that, in Case B1, $-\bar{s}_1 = s_2$, $-\bar{s}_2 = s_1$, i.e. we get a G-invariant pairing:

$$H(s_1, s_2, \varepsilon) \times H(s_2, s_1, \varepsilon) \longrightarrow \mathbb{C}$$

If we can show that $H(s_2, s_1, \varepsilon) \simeq H(s_1, s_2, \varepsilon)$, then the unitaricity will follow immediately.
Let's first look at the $(\mathfrak{g}, K)$-mod associated to $H(s_1, s_2, \varepsilon)$ & $H(s_2, s_1, \varepsilon)$ respectively, we see $s \to 1-s$, λ & μ are invariant under the interchange of s_1 & s_2.

Def: (Intertwining operator) For $f \in H(s_1, s_2, \varepsilon)$, we define

$$(M(s)f)(g) = \int_{\mathbb{R}} f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g)\, dx$$

$$s = \frac{1}{2}(s_1 - s_2 + 1) = \frac{1}{2}\left(1 + \sqrt{1-4\lambda}\right) > \frac{1}{2}$$

Prop: Convergence & Analytic continuation of $M(s)$ (obviously as usual, w.r.t a flat section of $H(s_1, s_2, \varepsilon)$)

1. if f is smooth, then $M(s)f$ converges absolutely for $\forall g \in G$ when $\mathrm{re}(s) > \frac{1}{2}$ and the resulting function $M(s)f$ is also smooth, belongs to $H(s_2, s_1, \varepsilon)$, and we have
$$M(s) \circ \rho(g) = \rho(g) \circ M(s) \quad \text{on } H^\infty(s_1, s_2, \varepsilon) \text{ when } g \in G,\ \mathrm{re}(s) > \tfrac{1}{2}$$
moreover, $M(s)$ preserves the K-finite vectors
2. When $\mathrm{re}(s) > \frac{1}{2}$, we have
$$M(s) f_{k,s} = (-i)^k \sqrt{\pi}\, \frac{\Gamma(s)\Gamma(s-\frac{1}{2})}{\Gamma(s+\frac{k}{2})\Gamma(s-\frac{k}{2})}\, f_{k,1-s} \qquad (*)$$

This proposition suffices to show that Case B1 is unitary, because we could define a pairing:

$$H^\infty(s_1, s_2, \varepsilon) \times H^\infty(s_1, s_2, \varepsilon) \longrightarrow \mathbb{C}$$

$$\langle f_1, f_2 \rangle = \int_K f_1(k)\, \overline{i^\varepsilon M(s) f_2(k)}\, dk$$

this pairing is G-invariant, moreover,

$$\langle f_{k,s}, f_{k,s} \rangle = (-1)^{\frac{k}{2}} \sqrt{\pi}\, \frac{\Gamma(s)\Gamma(s-\frac{1}{2})}{\Gamma(s-\frac{k}{2})\Gamma(s+\frac{k}{2})}$$

in Case B1, $\varepsilon = 0$, k must be even, then $\mathrm{sgn}\,\Gamma(s-\frac{k}{2}) = (-1)^{\frac{k}{2}}$ when $\frac{1}{2} < s < 1$ $\left(s = \frac{1}{2}(s_1 - s_2 + 1) = \frac{1}{2}(1 + \sqrt{1-4\lambda})\right)$
hence $\langle f_{k,s}, f_{k,s} \rangle > 0$, we can normalize $f_{k,s}$ to get $\langle f'_{k,s}, f'_{k,s} \rangle = 1$, then we get a Hermitian pairing:

$$H^\infty(s_1, s_2, 0) \times H^\infty(s_1, s_2, 0) \longrightarrow \mathbb{C}$$

Now this pairing can be extended to $H(s_1, s_2, 0)$ by identifying it with $L^2([-\pi, \pi])$ $(C^\infty([-\pi, \pi]))$
i.e. the Case B1 is unitary

Rmk: 1. it's tempting to think that $M(s)$ gives an isomorphism between $H(s_1, s_2, \varepsilon)$ and $H(s_2, s_1, \varepsilon)$, but this is not ture, because the constants in $(*)$ decay slowly when $k \to +\infty$, but this slowly decay constant is fast enough to show that, $H^\infty(s_1, s_2, \varepsilon)$ will go to a non-dense subspace of $H(s_2, s_1, \varepsilon)$, hence can't extend to the whole space
2. the same argument applies to Case B2

但 $H(s_2, s_1, \varepsilon)$ 不可约，如果取
$H^\infty(s_1, s_2, \varepsilon)$ 之闭包，可以吗？
我们只要说明：取极限时，会跑出 $H(s_2, s_1, \varepsilon)$?

Let's state the Unitaricity Theorem

Thm: Unitary repn for $GL_2(\mathbb{R})^+$. The following is a complete list of the $(\mathfrak{g}, K)$-mod of irr admissible unitary repn of $GL_2(\mathbb{R})^+$, there is a unique representative of irr unitary repn in each infinitesimal equivalence class:

1. One-dimensional. $g \mapsto |\det(g)|^{\mu}$
2. Principal series : $P_{\mu}(\lambda, \varepsilon)$, here μ is purely imaginary, $\lambda \geq \frac{1}{4}$. $\varepsilon = 0$ or 1 — A
3. Complentary series : $P_{\mu}(\lambda, 0)$, here $0 < \lambda < \frac{1}{4}$. μ purely imaginary — B1
4. Discrete series : $D^{\pm}_{\mu}(k)$, for $k \geq 1$, μ purely imaginary — B2

Rmk: Why we focus on unitary representation, One of the most important reasons it that automorphic repns are all unitary, hence both the infinite place & finite place irr repn are unitary. Another reason is unclear, it is that for unitary repn, $(\mathfrak{g}, K)$-mod can reflect the whole repn space:

Thm: Suppose G is a reductive Lie group, $(\pi, \mathfrak{H})$, $(\pi', \mathfrak{H}')$ are two irr unitary (hence admissible) repns, then

$$(\pi, \mathfrak{H}) \simeq (\pi', \mathfrak{H}') \iff \text{they are infinitesimally equivalent}$$

$\Rightarrow$ so λ & μ essentially determines the repn

Comparison

1. One-dimensional . $g \mapsto |\det(g)|^{\mu}$
2. Principal series : $P_{\mu}(\lambda, \varepsilon)$, here μ is purely imaginary, $\lambda \geq \frac{1}{4}$. $\varepsilon = 0$ or 1 A
3. Complentory series : $P_{\mu}(\lambda, 0)$, here $0 < \lambda < \frac{1}{4}$. μ purely imaginary B1
4. Discrete series : $D_{\mu}^{\pm}(k)$, for $k \geq 1$, μ purely imaginary B2

1 $\sim$

2. $s_1 + s_2 = \mu$, $\frac{1}{2}(s_1 - s_2 + 1) = s$, $\lambda = s(1-s)$

$$s_1 = \frac{\mu + \sqrt{1-4\lambda}}{2}, \quad s_2 = \frac{\mu - \sqrt{1-4\lambda}}{2} \Rightarrow \text{purely imaginary}$$

$P_{\mu}(\lambda, 0) \sim \pi^{+}_{\sqrt{1-4\lambda}}$

$P_{\mu}(\lambda, 0) \sim \pi^{-}_{\sqrt{1-4\lambda}}$

$P_{\mu}(\lambda, \varepsilon)$: induced from $\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} \to |y_1|^{s_1} \cdot |y_2|^{s_2} \cdot \operatorname{sgn}(y_1)^{\epsilon}$

$\begin{pmatrix} a & x \\ & a^{-1} \end{pmatrix} \to |a|^{s_1 - s_2} \operatorname{sgn}(a)^{\epsilon} = |a|^{\sqrt{1-4\lambda}} \cdot \operatorname{sgn}(a)^{\epsilon}$ $\rightsquigarrow$ purely imaginary $\quad i\mathbb{R}$

3. $P_{\mu}(\lambda, 0)$: induced from $\begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} \to |y_1|^{s_1} \cdot |y_2|^{s_2}$ s_1 & s_2 are not purely imaginary

$\begin{pmatrix} a & x \\ & a^{-1} \end{pmatrix} \to |a|^{s_1 + s_2} = |a|^{\sqrt{1-4\lambda}} \rightsquigarrow \sqrt{1-4\lambda} \in (0,1)$ $\quad (0, 1)$

$\pi^{C}_{\sqrt{1-4\lambda}}$ $\quad \cup (-1, 0)$

interchg s_1 & s_2

4. $\lambda = \frac{k}{2}\left(1 - \frac{k}{2}\right)$ for some $k \geq 2$

$\begin{pmatrix} a & x \\ & a^{-1} \end{pmatrix} \to |a|^{\pm(k-1)} \cdot \operatorname{sgn}(a)^{\epsilon}$ $\quad (-\infty, -1] \cup [1, +\infty)$

$4\lambda = k(2-k)$

$1 - 4\lambda = 1 - 2k + k^2$

Remaining Problems:

1. Why $M(s)$ can't be extended to $H(s_1, s_2, s)$?
2. Realization of discrete series $D_\mu^{\pm}(k)$.

Gelbart chapter 3

In this section we realize cusp forms as functions on the adele group $GL_2(\mathbb{A})$

1. Basic Notions $\quad G = GL_2/_{\mathbb{Q}}$

- $GL_2(\mathbb{A})$ is the restricted product of $GL_2(\mathbb{R})$ & $GL_2(\mathbb{Q}_p)$ w.r.t $\underset{\text{compact at }\infty}{O(2,\mathbb{R})} \times \prod_p \underset{\text{compact at } p}{GL_2(\mathbb{Z}_p)}$

  or it can be recognized as $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a,b,c,d \in \mathbb{A},\ ad-bc \in \mathbb{A}^* \right\}$

- $Z(\mathbb{A}) = C(GL_2(\mathbb{A})) = \left\{ \begin{pmatrix} t & \\ & t \end{pmatrix} \middle| \; t \in \mathbb{A}^* \right\}$

- $G_{\mathbb{Q}} = GL_2(\mathbb{Q}) \xhookrightarrow{\text{diagonal}} GL_2(\mathbb{A})$

local Haar measure glue together to get a global Haar measure on $GL_2(\mathbb{A})$
under this measure

$$Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A}) \text{ is finite}$$

2. Strong approximation

G algebraic group $/\mathbb{Q}$. G is semi-simple, simply connected, non-compact type, then

$$G(\mathbb{Q}) \text{ is dense in } G(\mathbb{A}_f)$$

Apply to $G = SL_2 \Rightarrow SL_2(\mathbb{Q})$ is dense in $SL_2(\mathbb{Q})$, take any open compact $K \subseteq SL_2(\mathbb{A}_f)$, we get

$$SL_2(\mathbb{Q}) \cdot K = SL_2(\mathbb{A}_f)$$

then $\quad SL_2(\mathbb{A}) = SL_2(\mathbb{Q}) \cdot SL_2(\mathbb{R}) \cdot K \longrightarrow$ not $\mathbb{A}_f^\times$

this implies that, if $K \overset{\text{open compact}}{\subseteq} GL_2(\mathbb{A}_f)$, s.t. $\det(K) = \prod \mathbb{Z}_p^\times$, then

$$GL_2(\mathbb{A}) = GL_2(\mathbb{Q}) \cdot GL_2(\mathbb{R})^+ \, K$$

usually, we will take $\quad K = K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; c \equiv 0 \bmod N,\ a,b,c,d \in \hat{\mathbb{Z}},\ ad-bc \in \hat{\mathbb{Z}}^\times \right\}$

then $\quad Z(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A}) \simeq GL_2(\mathbb{Q})_+ \backslash PGL_2(\mathbb{R})^+ \cdot K/_{Z(\mathbb{A}_f)} \longrightarrow$ already finite volume

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \underbrace{SL_2(\mathbb{R})}_{\text{finite measure, roughly } \mathcal{H} \times SO(2,\mathbb{R})} \to PGL_2(\mathbb{R})^+ \to 1$$

- Grossencharacter

Def: Grossencharacter is a continuous homomorphism:

$$\psi: \mathbb{A}_F^\times \to F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$$

i.e. character of $\mathbb{A}_F^\times$ trivial on $F^\times$

any Grossencharacter admits a factorization

$$\psi(x) = \chi(x) \cdot |x|^s$$

here $|\cdot|$ is idele norm, $s \in \mathbb{C}$, χ is unitary, i.e. a Hecke character
Hecke characters are well-understood by $\underbrace{\text{Tate's thesis, they are actually characters on ray class group.}}_{\text{By existence theorem of CFT}}$

In our case, we will use the character induced from $\psi: (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^*$

$\forall p$, $\psi_p: \mathbb{Z}_p^\times \to (\mathbb{Z}_p/N\mathbb{Z}_p)^\times \to \mathbb{C}^*$, obviously it is trivial when $p \nmid N$

these piece together to get:

$$\psi: \prod_{p<\infty} \mathbb{Z}_p^\times \xrightarrow{\prod \psi_p} \mathbb{C}^*$$

this gives a Grossencharacter on $\mathbb{A}_\mathbb{Q}^\times$, because $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \simeq (0,+\infty) \times \prod_{p<\infty} \mathbb{Z}_p^\times$

We can therefore define a character on $K_0(N)$ by:

$$\psi: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \psi(a)$$

- Norm on $G(\mathbb{A})$

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, local norm $\begin{cases} \text{at } \infty, \ \max\{|a|, |b|, |c|, |d|, |ad-bc|^{-1}\} & \|g\|_\infty \\ \text{at } p, \ \max\{|a|, |b|, |c|, |d|, |ad-bc|^{-1}\} & \|g\|_p \end{cases}$

$\|g\| := \prod_v \|g\|_v$ (almost every one is 1)

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} \right\|_\infty = \left\| \begin{pmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{pmatrix} \right\|_\infty$$

$\le \max\{|ax|+|bz|, |ay|+|bw|, |cx|+|dw|, |cy|+|dw|, \det A_1^{-1} \cdot \det A_2^{-1}$

$\le 2\max\{|ax|, |by|, |ay|, |bw|, |cx|, |dw|, |c|$

$\le 2 \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right\|$

- $\|g \cdot h\| \le 2 \cdot \|g\| \cdot \|h\|$

- From cusp forms to functions on $G(\mathbb{A})$

$\forall f \in S_k(N, \psi)$, we define ϕ_f on $G(\mathbb{A})$ by:

$\forall g \in G(\mathbb{A})$, $g = \gamma g_\infty k$, where $\gamma \in G(\mathbb{Q})$, $g_\infty \in G(\mathbb{R})^+$, $k \in K_0(N)$

$$\phi_f(g) = f(g_\infty(i))\, j(g_\infty, i)^{-k} \psi(k)$$

the condition $f \in S_k(N, \psi)$ guarantees ϕ_f is well-defined, Let's now check properties of ϕ_f

(a) $\phi(\gamma g) = \phi(g), \ \forall \gamma \in G(\mathbb{Q})$

(b) $\phi(gk) = \psi(k)\phi(g), \ \forall k \in K_0(N)$

(c) $\phi(g k_\theta) = \phi(g) e^{-ik\theta}, \ \forall k_\theta \in SO(2,\mathbb{R}) \subset G(\mathbb{R})$

(d) view ϕ as functions on $GL_2(\mathbb{R})^+$, it satisfies:

$$\Delta \phi = -\frac{k}{2}\left(\frac{k}{2}-1\right)\phi$$

(e) $\phi(zg) = \psi(z)\phi(g), \ \forall z \in Z(\mathbb{A}) \simeq \mathbb{A}^\times$

(f) ϕ is slowly increasing, i.e. for $\forall c>0$, $\forall$ compact subset Ω of $G(\mathbb{A})$, $\exists C, N>0$, s.t

$$\phi\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} g\right) \leq C|a|^N \qquad |\phi(g)| = |f(z) \, Im(z)^{\frac{k}{2}}| \leq M \text{ is bounded}$$

for $\forall g \in \Omega$, $\forall a \in \mathbb{A}^\times$, s.t. $|a|>c$

(g) ϕ is cuspidal, i.e.

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ & 1\end{pmatrix} g\right) dx = 0 \quad \text{for almost all } g \in G(\mathbb{A})$$

pf: Let's check (f), We prove a more succinct form: $\exists C, N>0$, s.t.

$$|\phi(g)| \leq C \cdot \|g\|^N, \ \forall g \in G(\mathbb{A})$$

suppose $g = \gamma g_\infty k$, $g_\infty = \begin{pmatrix} u & \\ & u\end{pmatrix}\begin{pmatrix} y & x \\ & 1\end{pmatrix} k_\theta$, $u>0$

then $\phi(g) = f(x+iy)\, j(g_\infty, i)^{-k} \psi(k)$ $\qquad j\left(\begin{pmatrix} u & \\ & u\end{pmatrix}\begin{pmatrix} y & x \\ & 1\end{pmatrix}, i\right) = u \cdot (u^2 y)^{-\frac{1}{2}} = y^{-\frac{1}{2}}$

$= f(z) y^{\frac{k}{2}} \psi(k)$

$\Rightarrow |\phi(g)| = |f(z)| \cdot |y|^{\frac{k}{2}}$, we should relate $\|g\|$ with these terms

$$g_\infty = \begin{pmatrix} u & \\ & u\end{pmatrix}\begin{pmatrix} y & x \\ & 1\end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix} = \begin{pmatrix} u & \\ & u\end{pmatrix}\begin{pmatrix} y\cos\theta + x\sin\theta & -y\sin\theta + x\cos\theta \\ \sin\theta & \cos\theta\end{pmatrix}$$

$\Rightarrow \|g\|_\infty = \|\gamma g_\infty\|_\infty \geq \frac{\sqrt{2}}{2} \left\|\gamma \cdot \begin{pmatrix} u & \\ & u\end{pmatrix}\begin{pmatrix} y & x \\ & 1\end{pmatrix}\right\|_\infty$ $\qquad \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} y & x \\ & 1\end{pmatrix}$

$\geq \frac{\sqrt{2}}{2} \max\{|\gamma_{11}|_\infty \cdot |uy|, \ |\gamma_{21}|_\infty \cdot |u|, \ |\det\gamma|_\infty^{-1} y^{-1} \cdot u^{-2}\}$ $\qquad = \begin{pmatrix} ay & ax+b \\ cy & cx+d\end{pmatrix}$

$\|g\|_p = \|\gamma k_p\|_p = \|\gamma\|_p \geq \max\{|\gamma_{11}|_p, \ |\det(\gamma)|_p^{-1}\}$

$\Rightarrow \|g\| = \prod_v \|g\|_v \geq \frac{\sqrt{2}}{2} \max\{|uy|, \ y^{-1}u^{-2}\} \geq C' \cdot y^{\frac{1}{3}} \quad (*)$ $\qquad |\phi(g)| = |\phi(\gamma g)|$

we know that $f(z) = e^{2\pi i z} \cdot \hat{f}(z) \Rightarrow |\phi(g)| \leq M \cdot e^{-2\pi y} y^{\frac{k}{2}}$ when $y \gg 0$

$g = \gamma \cdot \epsilon g_\infty \cdot k$ $\qquad \leq C \cdot \|g\|^N$ for some C, N when $y \gg 0$

$|\phi(g)| = |f(z) y^{\frac{k}{2}}| = |f(\epsilon z) (\mathrm{Im}\, \epsilon z)^{\frac{k}{2}}| \leq C \|\gamma \cdot \epsilon g_\infty \cdot k\| \leq C \cdot \prod C' |g|_\infty \cdot \prod |g|_p = C'' \|g\|$, $\underbrace{\text{only finitely many } \epsilon}_{\text{(why)}}$

$\Rightarrow |\phi(g)| \leq C \cdot \|g\|^N$

- $L^2(G_\mathbb{Q}\backslash G_A, \psi)$, this is the Hilbert space of measurable functions ϕ, s.t.

  (a) $\phi(\gamma g) = \phi(g) \quad \forall \gamma \in G_\mathbb{Q}$

  (b) $\phi(zg) = \psi(z)\phi(g), \forall z \in Z(\mathbb{A})$

  (c) square-integrable:

$$\int_{Z_A G_\mathbb{Q}\backslash G_A} |\phi(g)|^2 \, dg < +\infty$$

  (ϕ_f satisfy this condition, because ϕ_f is bounded, and $Z_A G_\mathbb{Q}\backslash G_A$ finite vol)

<u>Def</u>: Automorphic Forms on $G(\mathbb{A})$

(a) $\phi(\gamma g) = \phi(g)$, $\forall \gamma \in G(\mathbb{Q})$

(b) $\exists$ Grossencharacter ψ, s.t.

$$\phi(zg) = \psi(z)\phi(g), \quad \forall z \in Z_A$$

(c) ϕ is right K-finite, here $K = K_\infty \times \prod_p K_p$

(d) As a function on infinite place, i.e. function on $G_\infty = GL_2(\mathbb{R})$,

$$\phi \text{ is smooth \& } Z(U(\mathfrak{g}))\text{-finite} \quad \left(Z(U(\mathfrak{g})) = \mathbb{C}[D_I, \Delta]\right)$$

(e) ϕ is slowly increasing: $\exists C, N > 0$, s.t.

$$|\phi(g)| \le C \cdot \|g\|^N$$

*(f) cuspidal condition:

$$\int_{\mathbb{Q}\backslash A} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx = 0 \quad \text{for almost every } g \in G(A)$$

$\Rightarrow$ will imply square-integrable (note. not every automorphic forms is square-integrable, such as Eisenstein series)

Rmk: 1. $A_0(\psi) \subseteq \underbrace{L^2(G_\mathbb{Q}\backslash G_A, \psi)}_{\substack{\text{square-integrable} \\ \text{on } Z_A G_\mathbb{Q}\backslash G_A}}$, and $\overline{A_0(\psi)} = L_0^2(G_\mathbb{Q}\backslash G_A, \psi)$

cuspidal square-integrable

2. Fourier expansion

For almost every $g \in G(\mathbb{A})$,

$$x \longmapsto \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \text{ is square-integrable on } A/\mathbb{Q}$$

then by Pontryagin duality, $(A/\mathbb{Q})^* \simeq \mathbb{Q}$ $\left((\mathbb{R}/\mathbb{Z})^* \simeq \mathbb{Z}\right)$

$$\phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \sum_{r \in \mathbb{Q}} \phi_r(g)\, e(rx), \qquad e: A/\mathbb{Q} \longrightarrow \mathbb{C}^*$$

$$e((x_v)) := e^{2\pi i x_\infty} \cdot \prod_p e^{-2\pi i \{x_p\}}$$

fraction part

almost all $= 1$

here,

$$\phi_r(g) = \int_{A/\mathbb{Q}} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) e(-rx)\, dx$$

then cusp form $\Leftrightarrow$ $\phi_0(g) = 0$ for almost every $g \in G(\mathbb{A})$

We have two notions of Moderate growth / Slowly increasing

[Gelbart]: For $\forall c>0$, and compact subset $\omega \subseteq G(\mathbb{A})$, $\exists$ constant $C, N>0$, s.t.

$$\left|\phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)\right| \leq C \cdot |a|^N$$

for $\forall g \in \omega$, $\forall a \in \mathbb{A}^\times$, s.t $|a| > c$

[Goldfeld]: $\exists C, N>0$, s.t.

$$|\phi(g)| \leq C \cdot \|g\|^N$$

Are these two definitions agree? here

$$\|g\| = \prod_v \|g\|_v, \quad \|g\|_v = \max\{|a|_v, |b|_v, |c|_v, |d|_v, |ad-bc|_v^{-1}\}$$

We will see that, they are all equivalent to:

[Gelbart]': For $\forall \omega \subset G(A)$ compact, $\exists C, N>0$, s.t

$$\left|\phi\left(i_\infty\begin{pmatrix} y & \\ & 1 \end{pmatrix} \cdot g\right)\right| \leq C \cdot |y|^N \text{ when } |y| \to +\infty, \ \forall g \in \omega$$

pf: [Gelbart] $\Rightarrow$ [Gelbart]' is trivial

[Goldfeld] $\Rightarrow$ [Gelbart]', this is because

$$\left|\phi\left(i_\infty\begin{pmatrix} y & \\ & 1 \end{pmatrix} \cdot g\right)\right| \leq C_N \cdot \left\| i_\infty\begin{pmatrix} y & \\ & 1 \end{pmatrix} \cdot g \right\|^N$$

$$\leq C_N' \cdot \|g\|^N \cdot \left\| i_\infty\begin{pmatrix} y & \\ & 1 \end{pmatrix} \right\|^N \leq C_N'' \cdot |y|^N$$

[Gelbart]' $\Rightarrow$ [Goldfeld]: since ϕ is K-finite, suppose in K-repn, ϕ only appear in $\rho_1, \dots, \rho_r$, $\phi = \sum_{i=1}^r c_i \phi_i$

then suppose ρ_i is trivial on $K_i \subseteq K$, then, ϕ is invariant by $\bigcap_{i=1} K_i = K_0$

$\forall g \in G(A)$, $g = \gamma \cdot g_\infty \cdot k$, $k \in K$. $\Rightarrow \|g\| \geq |y|^{\frac{1}{3}}$

then $\phi(g) = \sum c_i \phi_i(\gamma \cdot g_\infty)$, since K/K_0 finite, c_i can only rge over a finite set

therefore, if we can show $\phi_i(g_\infty)$ is rapidly decreasing, results on g will follows

$$\phi_i(g_\infty) = \phi_i\left(i_\infty\left(\begin{pmatrix} z & \\ & z \end{pmatrix}\begin{pmatrix} y & x \\ & 1 \end{pmatrix} k_\theta\right), 1, \dots, 1, \dots\right) = \omega(z) e^{ik\theta} \phi_i\left(\begin{pmatrix} y & x \\ & 1 \end{pmatrix}\right)$$

Note: ϕ_i are linear combinations of $\underbrace{K\text{-translates of } \phi}_{\text{compact}}$, hence [Gelbart]' holds for ϕ_i

we get $|\phi_i(g_\infty)| \leq C \cdot |y|^N \leq C \cdot \|g\|^{3N} \Rightarrow |\phi(g)| \leq C \cdot \|g\|^{3N}$

[Gelbart]' $\Rightarrow$ [Gelbart] : We know by finiteness of class group,

$$\mathbb{A}^\times = F^\times \cdot \coprod_i \left(\mathbb{A}_\infty \times a_i \prod_v \mathcal{O}_{F,v}^\times \right) = \coprod_i F^\times \mathbb{A}_i^\times$$

suppose $a = x \cdot a_\infty \cdot a_i$, then $|a| = |a_\infty| \cdot |a_i| \sim |a_\infty|$

$$\left| \phi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g \right) \right| = \left| \phi\left(i_\infty \begin{pmatrix} a_\infty & \\ & 1 \end{pmatrix} \cdot \underbrace{i_f \begin{pmatrix} a_i & \\ & 1 \end{pmatrix} g}_{\text{only ranges over a compact subset}} \right) \right|$$

$$\leq C \cdot |a_\infty|^N \leq C' |a|^N$$

Rmk : $\|g\|$ & y. We know that

$$G_{\mathbb{Q}} \backslash G_{\mathbb{A}} / K$$

has representatives: $\begin{pmatrix} r_\infty & \\ & r_\infty \end{pmatrix} \begin{pmatrix} y_\infty & x_\infty \\ & 1 \end{pmatrix}$ with $r_\infty > 0$, $y_\infty > 0$, $-\frac{1}{2} \leq x_\infty \leq \frac{1}{2}$, $x_\infty^2 + y_\infty^2 \geq 1$

then $\|g\| \geq C' \cdot y_\infty^{\frac{1}{2}}$

Rmk : the argument [Gelbart]' $\Rightarrow$ [Gelbart] can be applied to the rapid decrease case! (i.e. $N < 0$)
in this case, we have:

$$\left| \phi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g \right) \right| = \left| \phi\left(i_\infty \begin{pmatrix} a_\infty & \\ & 1 \end{pmatrix} \cdot \underbrace{i_f \begin{pmatrix} a_i & \\ & 1 \end{pmatrix}}_{\text{over a compact subset}} g \right) \right| \leq C \cdot |a_\infty|^N \leq C' \cdot |a|^N \quad (N < 0)$$

In the case of rapid decay, do we still have these three equivalences?

Rmk : By [GGSP] Thm, we actually proved rapid decay version for cusp automorphic forms, this will imply the rapid decay case for [Gelbart], which will be used in Whittaker model when computing the Fourier coefficient

Moderate growth

Adelic : $\exists\, C, N > 0$, s.t.

$$|\phi(g)| \leq C \cdot \|g\|^N$$

here $\qquad \|g\| = \prod_v \|g\|_v, \quad \|g\|_v = \max\{|a|_v, |b|_v, |c|_v, |d|_v, |ad-bc|_v^{-1}\}$

Classical: $\quad \mathfrak{a}$ is a cusp, $\sigma_\mathfrak{a}(\infty) = \mathfrak{a}$,

$$|\phi(\sigma_\mathfrak{a}(x+iy))| \leq C \cdot \|y\|^M \quad \text{for some } C, M > 0$$

relation between $\|g\|$ & $\|y\|$

- known : $\boxed{\|g\| \geq \frac{\sqrt{2}}{2} |y_\infty|^{\frac{1}{3}}}$ $\quad \left(g = \gamma \cdot \left(\begin{pmatrix} y_\infty & x_\infty \\ & 1 \end{pmatrix} \begin{pmatrix} r_\infty & \\ & r_\infty \end{pmatrix}, 1, 1, \dots \right) \cdot k \right)$

Do we have a reverse one: $\boxed{\|g\| \leq C \cdot |y_\infty|^K}$ $\quad (y_\infty \gg 0)$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$g_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_\infty & x_\infty \\ & 1 \end{pmatrix} \begin{pmatrix} r_\infty & \\ & r_\infty \end{pmatrix}$$

$$= \begin{pmatrix} a y_\infty & a x_\infty + b \\ c y_\infty & c x_\infty + d \end{pmatrix} \cdot \begin{pmatrix} r_\infty & \\ & r_\infty \end{pmatrix} \qquad (ad-bc) \cdot y_\infty r_\infty^2$$

$$\|g_\infty\| = \max\left\{ r_\infty \cdot \max\{|a y_\infty|, |c y_\infty|, |a x_\infty + b|, |c x_\infty + d|\}, \; r_\infty^{-2} y_\infty^{-1} |ad-bc|^{-1} \right\}$$

when $y_\infty \gg 0$, then this

$$= \max\left\{ r_\infty \cdot \max\{|a|, |c|\} \cdot y_\infty, \; r_\infty^{2} y_\infty^{-1} |ad-bc|^{-1} \right\}$$

$$\leq \max\{ r_\infty \cdot y_\infty, \; r_\infty^2 \cdot y_\infty^{-1} \} \cdot \|\gamma\|_\infty$$

$$\|g_p\| = \|\gamma \cdot k_p\| = \|\gamma\|_p$$

$$\Rightarrow \|g\| \leq \max\{ r_\infty y_\infty, \; r_\infty^2 \cdot y_\infty^{-1} \}$$

· Global Hecke operators

$\Gamma = \Gamma_0(N)$, take $\forall p \nmid N$, consider the following double coset:

$$K_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_p = H_p$$

here $K_p = K_p^N = GL_2(\mathbb{Z}_p)$

take ϕ function on $G(\mathbb{A})$, right invariant under K_p, we define:

$$(\tilde{T}(p)\phi)(g) = \int_{H_p} \phi(gh)\,dh = \int_{G_p} \phi(gh) 1_{H_p}(h)\,dh$$

· $\tilde{T}(p)$ maps $L_0^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}}, \psi)$ into $L_0^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}}, \psi)$, mainly because $\tilde{T}(p)$ is a "local operator"

We prove more: $\tilde{T}(p)$ takes (cuspidal) Automorphic forms to (cuspidal) Automorphic forms

(a) $$(\tilde{T}(p)\phi)(\gamma g) = \int_{H_p} \phi(\gamma g h)\,dh = \int_{H_p} \phi(gh)\,dh = (\tilde{T}(p)\phi)(g)$$

(b) $$(\tilde{T}(p)\phi)(gk) = \int_{H_p} \phi(gkh)\,dh = \int_{H_p} \phi(gh\,h^{-1}kh)\,dh = \int_{H_p} \phi(gh)\,\psi(h^{-1}kh)\,dh$$

$$gh\,\underbrace{h^{-1}kh}_{\in K_0(N)}$$

$$= \psi(k)(\tilde{T}(p)\phi)(g)$$

since $K/K_0(N)$ is finite, $\tilde{T}(p)\phi$ must be K-finite

(c) $(\tilde{T}(p)\phi)(zg) = \psi(z)(\tilde{T}(p)\phi)(g)$, $\forall z \in Z_{\mathbb{A}}$

(d) Since G_∞ & K_p commutes, this is automatic

(e) obviously

(f) $$\int_{\mathbb{Q}\backslash\mathbb{A}} (\tilde{T}(p)\phi)\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx = \int_{\mathbb{Q}\backslash\mathbb{A}} \int_{H_p} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} gh\right) dh\,dx \equiv 0 \quad \text{a.e. } g$$

Next, we show local operator $\tilde{T}(p)$ agrees with usual Hecke operator

Thm: take $f \in S_k(\Gamma)$, then

$$p^{\frac{k}{2}-1}\,\tilde{T}(p)\phi_f = \phi_{T_p f}$$

pf: See note in the book

Question: What's others $T(p)$? $p \mid N$

We consider $\widetilde{T}(p)$ for $p \mid N$, naively, it should have the same definition as previous case

$$H_p = K_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} K_p, \qquad K_p = K_0(N)_p$$

$$\widetilde{T}(p) : \phi \mapsto \left(g \mapsto \int_{H_p} \phi(gh)\, dh \right)$$

here, since $p \mid N \Rightarrow K_p \backslash H_p = \coprod_{b=0}^{p-1} K_p \begin{pmatrix} p & b \\ & 1 \end{pmatrix}$

Now, what happened to ϕ_f, where $f \in S_k(N, \psi)$?

$$\left(\widetilde{T}(p)\, \phi_f \right)(g) = \int_{H_p} \phi_f(gh)\, dh$$

$$= \sum_{b=0}^{p-1} \int_{K_p \begin{pmatrix} p & b \\ & 1 \end{pmatrix}} \phi_f(gh)\, dh$$

suppose $g = \gamma g_\infty \cdot k$, obviously, we could ignore γ, i.e. $g = g_\infty \cdot k$

then $g \cdot h = g_\infty \cdot kh = g_\infty \cdot kk' \begin{pmatrix} p & b \\ & 1 \end{pmatrix}$

$$\begin{pmatrix} p^{-1} & -jp^{-1} \\ & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & i \\ & 1 \end{pmatrix} = \begin{pmatrix} ap^{-1} & (b-jd)p^{-1} \\ c & d \end{pmatrix} \begin{pmatrix} p & i \\ & 1 \end{pmatrix} = \begin{pmatrix} a & (b-jd+ia)p^{-1} \\ cp & d+ci \end{pmatrix}$$

$$= g_\infty \cdot \begin{pmatrix} p & b' \\ & 1 \end{pmatrix} kk'$$

$$= \begin{pmatrix} p & b' \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} p^{-1} & -b'p^{-1} \\ & 1 \end{pmatrix} g_\infty \cdot kk'$$

$$= \sum_{b=0}^{p-1} \int_{K_p} f\left(\frac{z-b'}{p} \right) j\left(\begin{pmatrix} p^{-1} & -b'p^{-1} \\ & 1 \end{pmatrix} g_\infty, i \right)^{-k} \psi(k)\psi(k')\, dk'$$

Connection with classical q-expansion

if $f \in S_k(N, \psi)$, we have at ∞

$$f = \sum_{n=1}^{\infty} a_n q^n$$

we also have the adelic lifting $\phi_f \in L_0^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}}, \psi)$

for $\forall r \in \mathbb{Q}$, we compute the following:

$$\phi_r(g) = \int_{\mathbb{Q}\backslash \mathbb{A}_{\mathbb{Q}}} \phi_f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) e(-rx)\, dx = ?$$

general g is hard to compute, let's try $g = \begin{pmatrix} y & \\ & 1 \end{pmatrix} \in GL_2(\mathbb{R})$, with $y > 0$

then $\phi_r\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{cases} a_n e^{-2\pi n y} \cdot y^{\frac{k}{2}}, & \text{if } r \in \mathbb{Z} \\ 0 & \text{, otherwise} \end{cases} \Rightarrow \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \sum_n \phi_n(y)\, e(nx)$

pf: if $r \notin \mathbb{Z}$, $\exists p, m \geq 1$, s.t. $r = a p^{-m}$, where $(a, p) = 1$,

then since ϕ_f is K-invariant.

$$\phi_r\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \int_{\mathbb{Q}\backslash \mathbb{A}_{\mathbb{Q}}} \phi_f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \overset{\nearrow \text{ finite part}}{p^m} \\ & 1 \end{pmatrix}\right) e(-rx)\, dx$$

$$= \int_{\mathbb{Q}\backslash \mathbb{A}_{\mathbb{Q}}} \phi_f\left(\begin{pmatrix} 1 & x + p^m \\ & 1 \end{pmatrix}\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) e(-rx)\, dx$$

$$= \int_{\mathbb{Q}\backslash \mathbb{A}_{\mathbb{Q}}} \phi_f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) e(-rx + rp^{m}) \, dx$$

$$= e(rp^m)\, \phi_r\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)$$

here $rp^m = (\underset{\infty}{r}, r, \ldots, \underset{p}{rp^m}, \ldots)$, if $e(rp^m) = 1$

then $(0, 0, \ldots, r(p^m - 1), 0, \ldots 0)$ also in the kernel of e

but $r(p^m - 1) = \underset{\underset{\mathbb{Z}_p^\times}{\Uparrow}}{a} - \underset{\hookrightarrow p^{-m}\mathbb{Z}_p}{r}$ $\Rightarrow$ non-trivial $\Rightarrow$ $\phi_r\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$ for $y \notin \mathbb{Z}$

now if $r = n \in \mathbb{Z}_{>0}$

$$\phi_r\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = \int_{\mathbb{Q}\backslash \mathbb{A}_{\mathbb{Q}}} \phi_f\Bigg(\underbrace{\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}}_{L}\Bigg) e(-nx)\, dx$$

$$\infty: \begin{pmatrix} y & x_\infty \\ 0 & 1 \end{pmatrix}, \quad \text{finite: } \begin{pmatrix} 1 & x_f \\ & 1 \end{pmatrix}$$

so this value $= f(x_\infty + iy)\, y^{\frac{k}{2}}$

$$= \int_0^1 f(x_\infty + iy)\, y^{\frac{k}{2}} e^{-2\pi i n x_\infty}\, dx = a_n \cdot y^{\frac{k}{2}} e^{-2\pi n y}$$

$\{\mathcal{H}_G\text{-mod}\} - \{\text{irr admissible repn of } G_A\}$

Tensor Product Theorem

In this section, we will state and explain an important theorem of the structure of automorphic representation

Restricted tensor product:

Def: Suppose we have a family of vector spaces $\{V_i\}_{i\in I}$, and a distinguished vector ξ_i° for almost all $i\in I$, then the restricted tensor product of $\{V_i\}$ w.r.t $\{\xi_i^\circ\}$ is a vector space spanned by vectors of the following type:

$$\bigotimes_i \xi_i, \quad \text{for almost all } i,\ \xi_i = \xi_i^\circ$$

Rmk: another definition, set S to be the finite set where we didn't associate a distinguished vector, then define an inclusion relation for those subsets of I containing S, denote this set by Ω. now if $S_1 \subset S_2$, then define

$$\lambda_{S_1,S_2}: \bigotimes_{i\in S_1} V_i \longrightarrow \bigotimes_{i\in S_2} V_i$$
$$\otimes \xi_i \longmapsto \bigotimes_{i\in S_1}\xi_i \otimes \bigotimes_{i\in S_2-S_1}\xi_i^\circ$$

then we form the direct limit $\varinjlim_{S'\in\Omega} \bigotimes_{i\in S'} V_i$

Now suppose on each V_i, there is an Hermitian form, then we further assume each ξ_i° has length 1, then we can transfer this Hermitian form to each $\bigotimes_{i\in S'} V_i$, and finially to the space $\bigotimes_i V_i$

Restricted tensor product for representations

Let's come back to the situation we are working on, suppose F is a # fied, Σ = places on F, Suppose for almost all $v\in\Sigma$, we are given a irreducible admissible representation (π_v, H_v), s.t.

For almost all v, π_v is spherical (unramified)

Then by all the local theory we have developed, $\dim_{\mathbb{C}} H_v^{K_v} = 1$, then pick $\xi_v^\circ \in H_v^{K_v}$, for almost all v. We can form the restricted tensor product:

$$\bigotimes_v H_v$$

this space has a natural G_A-action, since $\forall g\in G_A$, almost all components are in K_v

Moreover, if each H_v carries a Hermitian inner product, we can normalize ξ_v°, s.t. $\|\xi_v^\circ\| = 1$ and if in this case, each π_v is unitary, then the resulting representation G_A is unitary

Representation of G_A

We define a representation (π, H) of G_A to be admissible by the same principle:

- View H as a representation of compact group $\widehat{\bigotimes_v} K_v = K$, then

$$\hat{H} = \widehat{\bigoplus} \hat{H}(\sigma)$$

Hilbert completion w.r.t an inner product on the grp K

then each $\hat{H}(\sigma)$ is finite-dimensional

Theorem 1. Cuspidal automorphic representations are admissible

Theorem 2. Every irreducible admissible unitary representation π of G_A is factorizable. i.e, it is isomorphic to:

$$\pi \simeq \bigotimes_v \pi_v$$

here every π_v is an irr unitary admissible repn of G_v, almost all of them are spherical

Rmk: actually, we make a Hilbert completion on the RHS, because $\bigotimes_v \pi_v$ is not complete w.r.t the inner product

General theory: modules over idempotented algebra

In this subsection, we state the main theorem of modules over idempotented algebra without proof

Thm: Let k be an algebraically closed field, $(H_v, \mathcal{E}_v)$, $v \in \Sigma$ be a family of idempotented k-algebras.
For almost all v, let e_v° be a spherical element of H_v, define H to be the restricted tensor product of H_v w.r.t e_v°, it is itself an idempotented algebra.
Now for $\forall v \in \Sigma$, we specify a simple admissible H_v-module M_v, and for almost all $v \in \Sigma$, we specify a non-zero element $m_v^\circ \in M_v[e_v^\circ]$, then we form the restricted tensor product $\bigotimes_v M_v$ w.r.t. m_v°, then

- $\bigotimes_v M_v$ is a simple admissible H-module
- every simple admissible H-module is of this form

Explain: 1. Idempotented algebra $(H, \mathcal{E})$

- H is a k-algebra, not necessarily have unit
- $\mathcal{E}$ is a set of idempotents, s.t. if $e_1, e_2 \in \mathcal{E}$, $\exists\, e_0 \in \mathcal{E}$, s.t $e_0 e_1 = e_1 e_0 = e_1$, $e_0 e_2 = e_2 e_0 = e_2$
- $\forall \phi \in H$, $\exists\, e \in \mathcal{E}$, s.t $e\phi = \phi e = \phi$

this can be defined as an "order" on $\mathcal{E}$
$e_0 \geq e_1$, if $e_1 e_0 = e_0 e_1 = e_1$
因为 $e_0 = e_1 + \underbrace{(e_0 - e_1)}_{\text{also idempotent}}$

although H may not have unit, $\underbrace{eHe}_{= H[e]}$ has a unit e

2 Modules for idempotented algebra

Suppose $(H, \mathcal{E})$ is an idempotented algebra, and M is an H-module, an important "submodule" is $M[e] := eM$

- $M[e] = eM$ is $H[e]$-module

We call M is smooth if $M = \bigcup_{e \in \mathcal{E}} M[e]$.

We call M is admissible if: M is smooth, $\dim_k M[e] < +\infty$

3. Spherical idempotent e°

e° is a Spherical idempotent for the idempotented algebra $(H, \mathcal{E})$, if

- $e^\circ \in \mathcal{E}$
- $\exists$ anti-involution ι on H, s.t. $\iota|_{H[e^\circ]} = id \Rightarrow H[e^\circ]$ is commutative

$(xy)^\iota = y^\iota x^\iota$,

$(x^\iota)^\iota = x$

Examples: a. G locally compact, totally disconnected group, s.t. $GL_n(F)$. F NA local field

- $\mathcal{H}(G) := C_c^\infty(G)$
- $\mathcal{E}$ = for each compact open subgp K, $\frac{1}{vol(K)} 1_K$

$g \to g^T$

We know in this case, we have category equivalence:

$$\{\text{smooth (admissible) } G\text{-repns}\} \longleftrightarrow \{\text{smooth (admissible) } \mathcal{H}(G)\text{-modules}\}$$

In the case of $GL_n(F)$, we can take the spherical idempotent to be $1_{GL(n, \mathcal{O}_v)}$

b. K is a compact Lie group

- $\mathcal{H}(K)$ = Smooth, bi-K-finite functions on K, then by Peter-Weyl theorem

$$L^2(K) = \hat{\bigoplus_{\sigma \in \hat{K}}} \underbrace{V_\sigma \otimes V_\sigma^*}_{\text{matrix coefficients}} \Rightarrow \mathcal{H}(K) = \bigoplus_{\sigma \in \hat{K}} V_\sigma \otimes V_\sigma^*,$$ i.e. finite linear combinations of matrix coefficients

and we know $V_\sigma \otimes V_\sigma^* \simeq End_{\mathbb{C}}(V_\sigma)$, then there is an idempotent $e_\sigma \in \mathcal{H}(K)$

We also have a category equivalence:

$$\{K\text{-finite, admissible } K\text{-repns}\} \longleftrightarrow \{\text{Smooth, admissible } \mathcal{H}_K\text{-mod}\}$$

C. G is a reductive Lie group, we fix a maximal compact subgroup K

- $\mathcal{H}(G) = \{f * D \mid f \in \mathcal{H}_K, D \in U(\mathfrak{g}_{\mathbb{C}})\} \simeq \mathcal{H}_K \otimes_{U(\mathfrak{k})} U(\mathfrak{g}_{\mathbb{C}})$
- $\mathcal{E} = \mathcal{E}$ of $\mathcal{H}(K)$

We have a category equivalence:

$$\{\text{admissible } (\mathfrak{g}, K)\text{-modules}\} \longleftrightarrow \{\text{smooth, admissible } \mathcal{H}_G\text{-mod}\}$$

Now we state some interesting and useful results on $(\mathcal{H}, \mathcal{E})$-mod

We don't need to work with full $\mathcal{E}$, we work with $\mathcal{E}^\circ$, which is a cofinal system of $\mathcal{E}$, i.e. $\forall e_0 \in \mathcal{E}^\circ, \exists e_1 \in \mathcal{E}^\circ$ $e_1 \geq e_0$

<u>Prop 1</u>: M is a simple $(H, \mathcal{E})$-mod $\Leftrightarrow M[e] = 0$ or a simple $\mathcal{H}[e]$-mod for $\forall e \in \mathcal{E}^\circ$

Now if M is simple admissible, then $M[e^\circ]$ is simple $\mathcal{H}[e^\circ]$-mod, for a spherical element $e^\circ \in \mathcal{H}$ then it must be <u>one-dimensional</u> or 0! hence there is a character $\mathcal{H}[e^\circ] \to \mathbb{C}$

<u>Prop 2</u>: M & N are simple admissible module of $(H, \mathcal{E})$, then
$$M \simeq N \Leftrightarrow M[e] \simeq N[e] \text{ as } H[e]\text{-mod for } \forall e \in \mathcal{E}^\circ$$

Now if e° is a spherical element, and $M[e^\circ], N[e^\circ] \neq 0$, then
$$M \simeq N \Leftrightarrow M[e^\circ] \simeq N[e^\circ]$$

<u>Prop 3</u>: If M_1, M_2 are simple admissible $(H_1, \mathcal{E}_1), (H_2, \mathcal{E}_2)$-mod, then:
- $M_1 \otimes M_2$ is simple admissible $(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{E}_1 \otimes \mathcal{E}_2)$-mod
- every simple admissible $(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{E}_1 \otimes \mathcal{E}_2)$-mod takes this form

How does the <u>general tensor product theorem</u> imply tensor product theorem for cuspidal automorphic repn?

For a cuspidal automorphic repn (π, H), we know H is an irr closed subspace of $L_0^2(G_{\mathbb{Q}} \backslash G_{\mathbb{A}}, \omega)$ then we take the K-finite vectors. they form a dense subspace of H, i.e.
$$H = \widehat{\bigoplus_{\rho \in \hat{K}}} H(\rho), \quad H^\circ = \bigoplus_{\rho \in \hat{K}} H(\rho)$$

then H° is $(\mathfrak{g}, K_\infty) \times GL_2(\mathbb{A}_f)$-module, which must be admissible, irreducible
H° is also a $\mathcal{H}_{GL(\mathbb{R})} \otimes \bigotimes_v \mathcal{H}_{G_v}$-module, it is also admissible, simple $\Big\} \Rightarrow$ works for $GL(n)$

$\hookrightarrow$ restricted tensor product w.r.t $e_v^\circ = 1_{GL(2,\mathcal{O}_v)}$

$\hookrightarrow$ consider $e = \bigotimes_v e_v$, $e_v = e_v^\circ$ for almost all v and $H[e]$, $\forall v \in H[e]$, v is fixed by $\bigotimes_v e_v$, hence for almost all v, K_v fixes v

for other place v, $\exists K_v' \leq K_v$ fixes v (K_v' depends on e_v)

hence v is invariant under $\prod_{v \neq \infty} K_v'$
and the e_∞ tells us v lies in ρ_∞ for some $\rho_\infty \in \hat{K}_\infty$ $\Big\} \Rightarrow$ then are only finitely many ρ

because repn of K are of the form $\bigotimes_v \rho_v$
see lemma 3.3.1

We implicitly use the following.
$$\mathcal{H}_G\text{-mod} \leftrightsquigarrow GL(2, \mathbb{A}) \text{ representation}$$

is this one-to-one?

Admissibility of cuspidal automorphic representation and more

In this section, we prove

<u>Theorem</u>: Cuspidal automorphic representations are admissible

<u>proof and more</u>: Suppose $(\pi, H) \subseteq L_0^2(G_{\mathbb{Q}}\backslash G_{\mathbb{A}}, \omega)$ is a cuspidal automorphic representation, then

$$H = \widehat{\bigoplus_{\rho \in \hat{K}}} H(\rho)$$

this theorem is saying that each piece $H(\rho)$ is finite-dimensional

We can also single out the K-finite piece, i.e.

$$H_{fin} = \bigoplus_{\rho \in \hat{K}} H(\rho)$$

<u>Claim</u>: H_{fin} is the space of cuspidal automorphic forms on $GL_2(A)$

pf: We should check each $\phi \in H_{fin}$ satisfies the following conditions

- ϕ is smooth
- ϕ is $\mathcal{Z}$-finite
- ϕ is K-finite
- ϕ is of moderate growth

Actually, we only need to show $\forall \phi \in H(\rho)$ satisfies these conditions

Let's focus on another subspace instead of $H(\rho)$:

$$H = \widehat{\bigoplus_{\rho_f \in \hat{K}_f}} H(\rho_f), \quad H^f_{fin} = \bigoplus_{\rho_f \in \hat{K}_f} H(\rho_f)$$

the relation between $H(\rho_f)$ & $H(\rho)$ is, $H(\rho_f) = \widehat{\bigoplus_{\rho_\infty \in \hat{K}_\infty}} H(\rho_\infty \otimes \rho_f)$

Next, we will realize $H(\rho_f)$ in some $L_0^2(\Gamma\backslash G, \chi)$ which we are familiar with

We know that, since K_f is totally disconnected, ρ_f is trivial on some open compact $K_0 \subseteq K_f$ $\ker \rho_f$

then $\forall \phi \in H(\rho_f)$ is actually a function on

$$G_{\mathbb{Q}}\backslash G_{\mathbb{A}}/K_0$$

We consider $G_{\mathbb{Q}} GL_2(\mathbb{R})^+ \backslash G_{\mathbb{A}}/K_0$ is finite, take $g_i \in G_f$ as a set of representative, $1 \le i \le h$

then define $\Phi_i(g_\infty) = \phi((g_\infty, g_i))$ for $g_\infty \in GL_2(\mathbb{R})^+$, define Γ to be the projection onto $GL(2,\mathbb{R})^+$ of $G_{\mathbb{Q}} \cap GL_2(\mathbb{R})^+ K_f$, it is a congruence subgroup, and $\forall \gamma \in \Gamma$

$$\Phi_i(\gamma g_\infty) = \Phi_i(g_\infty), \quad \forall g_\infty \in GL_2(\mathbb{R})^+$$

and $\phi \in L_0^2(G_{\mathbb{Q}}\backslash G_{\mathbb{A}}, \omega) \Rightarrow \Phi_i \in L_0^2(\Gamma\backslash G, 1, \omega_\infty)$, hence we get

$$H(\rho_f) \longrightarrow L_0^2(\Gamma\backslash G, 1, \omega_\infty)^{\oplus h}$$

then $\rho(f)$, $f \in C_c^\infty(GL_2(\mathbb{R})^+)$ is a compact operator on $H(\rho_f)$, then

$$H(\rho_f) = \widehat{\bigoplus_i} \pi_i$$

here π_i is irr unitary (hence admissible) repn of $GL_2(\mathbb{R})^+$, and with finite multiplicity

then by the theory concerning the structure of π_i. we know that
for any fixed $\rho_\infty \in \hat{K}_\infty$, in the space $H(\rho_\infty \otimes \rho_f)$.

smooth, K_∞-finite, Z-finite functions are dense, because we have

$$H(\rho_\infty \otimes \rho_f) = \hat{\bigoplus_i} \pi_i(\underbrace{\rho_\infty^+}_{SO(2)\text{-part}})$$

therefore we take $\phi \in H(\rho)$ to be smooth, K-finite, Z-finite

Now, we should be care of another thing: Δ acts as scalar multiplication on (π, H)
or equivalently, all the π_i are isomorphic to each other

pf: we have already note that $\exists$ dense subspace $H_f^\infty \subseteq H$, consisting of smooth, K-finite functions
now Δ acts as an endomorphism of $(\mathfrak{g}, K_\infty) \times GL_2(\mathbb{A}_f)$-mod H_f^∞, then (π, H) is irreducible implies that H_f^∞ is an irr $(\mathfrak{g}, K_\infty) \times GL_2(\mathbb{A}_f)$-mod, hence Δ must acts as a scalar! because just take λ-eigenspace of Δ in H_f^∞. i.e. $H_f^\infty(\lambda)$, we should have $H_f^\infty(\lambda)$ is invariant under the action of $(\mathfrak{g}, K_\infty) \times GL_2(\mathbb{A}_f)$, it must be the whole space!

hence $\Delta\phi = \lambda\phi$, and $\Delta\Phi_i = \lambda\Phi_i \Rightarrow$ by the property of finite multiplicity, $H(\rho) = \pi_i(\rho_\infty^+)^N$, $N < +\infty$
i.e. it is finite-dimensional, then by H-S section, it must decay very fast $\Rightarrow$ moderate growth

Rmk: This proof tells us much more that admissibility itself, we get:

- Δ acts as a scalar on each cuspidal automorphic representation

- cuspidal automorphic forms are just the K-finite part of a cuspidal automorphic repn
  i.e. we have the same situation as the real case $\Rightarrow$ K-finite $\Rightarrow$ smooth

- the condition "Z-finite" is saying that $f \in \mathcal{A}_0(G_{\mathbb{Q}} \backslash G_{\mathbb{A}}, \omega)$ only lies in finitely many irr subspaces

More on the local Hecke algebra

In the course of studying the spherical part V^K of a smooth $GL_2(F_v)$-repn, we use the spherical Hecke algebra

$$\mathcal{H}_K = e_K * \mathcal{H}_G * e_K = \{ f \in C_c^\infty(G) \mid f \text{ is } K\text{-bi-invariant} \}$$

We have seen that this is commutative ring with unit e_K. Moreover, if V is an irr admissible G-repn, then either $V^K = 0$, or V^K is a simple $\mathcal{H}_K$-mod. hence $\mathcal{H}_K$ acts by a character on V^K, and this character uniquely determines the original module V. Therefore, the structure of $\mathcal{H}_K$ is important for us to study spherical representation

Iwasawa decomposition tells us that

$$G = \coprod_{n \geq m} K \begin{pmatrix} \varpi^n & \\ & \varpi^m \end{pmatrix} K$$

then $\forall f \in \mathcal{H}_K$ should be linear combination of $1_{G_{n,m}}$, where $G_{n,m} = K \begin{pmatrix} \varpi^n & \\ & \varpi^m \end{pmatrix} K$

We define several important functions:

$$T(p^k) = \sum_{\substack{a+b=k \\ a \geq b}} 1_{a,b}$$

$$R(p) = 1_{1,1}$$

$$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} \varpi^n & \\ & \varpi^m \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} & -\varpi^m \\ \varpi^n & \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = \begin{pmatrix} \varpi^m & \\ & \varpi^n \end{pmatrix}$$

Prop: If $k \geq 1$, we have

$$T(p)T(p^k) = T(p^{k+1}) + q R(p) T(p^{k-1})$$

pf: $$(T(p) \cdot T(p^k))(g) = \int_G T(p)(gh)\, T(p^k)(h^{-1})\, dh, \quad \left.\begin{matrix} \operatorname{supp} T(p) = G_{1,0} \amalg G_{0,1} \\ \operatorname{Supp} T(p^k) = \coprod_{a+b=k} G_{a,b} \end{matrix}\right\} \Rightarrow \text{it is supported on } \coprod_{a+b=k+1} G_{a,b}$$

$T(p^{k+1})$, $R(p)T(p^{k-1})$ are also supported on this subset, hence we evaluate at

$$\begin{pmatrix} \varpi^{k+1-r} & \\ & \varpi^r \end{pmatrix}, \quad 0 \leq r \leq k+1$$

$$\begin{aligned}
(T(p) \cdot T(p^k)) \begin{pmatrix} \varpi^{k+1-r} & \\ & \varpi^r \end{pmatrix} &= \int_G T(p)\left(\begin{pmatrix} \varpi^{k+1-r} & \\ & \varpi^r \end{pmatrix} h \right) T(p^k)(h^{-1})\, dh \\
&= \int_G T(p)(h)\, T(p^k)\left(h^{-1} \begin{pmatrix} \varpi^{k+1-r} & \\ & \varpi^r \end{pmatrix} \right) dh \\
&= \int_{K \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} K} T(p^k)\left(h^{-1} \begin{pmatrix} \varpi^{k+1-r} & \\ & \varpi^r \end{pmatrix} \right) dk
\end{aligned}$$

$$= T(p^k)\left(\begin{pmatrix} \varpi^{k+1-r} & \\ & \varpi^{r-1} \end{pmatrix} \right) + \sum_{b \bmod p} T(p^k)\left(\begin{pmatrix} \varpi & b \\ & 1 \end{pmatrix}^{-1} \begin{pmatrix} \varpi^{k+1-r} & \\ & \varpi^r \end{pmatrix} \right)$$

$$K \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} K = \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix} K \amalg \coprod_{b \bmod p} \begin{pmatrix} \varpi & b \\ & 1 \end{pmatrix} K$$

$$K \begin{pmatrix} \varpi^{-1} & \\ & 1 \end{pmatrix} K = K \begin{pmatrix} 1 & \\ & \varpi^{-1} \end{pmatrix} \amalg \coprod_{b \bmod p} K \begin{pmatrix} \varpi & b \\ & 1 \end{pmatrix}^{-1}$$

$$T(p^k)\left(\begin{pmatrix} w^{k+1-r} & \\ & w^{r-1}\end{pmatrix}\right) + \sum_{b \bmod p} T(p^k)\left(\begin{pmatrix} w & b \\ & 1\end{pmatrix}^{-1}\begin{pmatrix} w^{k+1-r} & \\ & w^{r}\end{pmatrix}\right)$$

$$= T(p^k)\left(\begin{pmatrix} w^{k+1-r} & \\ & w^{r-1}\end{pmatrix}\right) + \sum_{b \bmod p} T(p^k)\Big($$

$$\begin{pmatrix} w^{-1} & -bw^{-1} \\ & 1\end{pmatrix}\begin{pmatrix} w^{k+1-r} & \\ & w^{r}\end{pmatrix} = \begin{pmatrix} w^{k-r} & -bw^{r-1} \\ & w^{r}\end{pmatrix} = \begin{pmatrix} w^{k-r} & \\ & w^{r}\end{pmatrix}\begin{pmatrix} 1 & -bw^{2r-1-k} \\ & 1\end{pmatrix}$$

① $k-r \le r-1 \rightsquigarrow k-r,\ r$

② $k-r > r-1$

$$\downarrow$$

$r-1,\ k-r+1$

$$\begin{pmatrix} w^{k-r} & -bw^{r-1} \\ & w^{r}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} = \begin{pmatrix} bw^{r-1} & w^{k-r} \\ -w^{r} & 0\end{pmatrix}$$

$2r \ge k+1$

$r \ge \frac{k+1}{2}$

$$\begin{pmatrix} 1 & \\ b^{-1}w & 1\end{pmatrix}\begin{pmatrix} bw^{r-1} & w^{k-r} \\ -w^{r} & 0\end{pmatrix} = \begin{pmatrix} bw^{r-1} & w^{k-r} \\ 0 & b^{-1}w^{k-r+1}\end{pmatrix}$$

$$\begin{pmatrix} bw^{r-1} & w^{k-r} \\ & b^{-1}w^{k-r+1}\end{pmatrix}\begin{pmatrix} 1 & -b^{-1}w^{k-2r+1} \\ & 1\end{pmatrix} = \begin{pmatrix} bw^{r-1} & \\ & b^{-1}w^{k-r+1}\end{pmatrix}$$

$$= \begin{cases} 1, & \text{when } 1 \le r \le k+1 \\ 0, & \text{when } r = 0 \end{cases}$$

$$+ \begin{cases} 2, & \text{when } 0 \le r \le k \\ 0, & \text{when } r = k+1 \end{cases}$$

$$+ \sum_{\substack{b \bmod p \\ \ne 0}} \begin{cases} r \ge \frac{k+1}{2} & \begin{cases} 1, & \frac{k+1}{2} \le r \le k \\ 0, & r = k+1 \end{cases} \\ r < \frac{k+1}{2} & \begin{cases} 1, & 1 \le r < \frac{k+1}{2} \\ 0, & r = 0 \end{cases} \end{cases}$$

$\Rightarrow$ when $r = 0 \Rightarrow$ the value is 1

when $r = k+1 \Rightarrow$ the value is 1

when $1 \le r \le k \Rightarrow 2 + q - 1 = q + 1$

Prop: $\mathcal{H}_K$ is generated by $T(p)$, $R(p)$, $R(p)^{-1}$

pf: Only need to show $\mathbb{1}_{a,b}$ is generated by these

$$\mathbb{1}_{a,b} = \mathbb{1}_{K\begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix}K}$$

<u>Claim</u>: $\mathbb{1}_{K\begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix}K} = R(p)^b \cdot \mathbb{1}_{K\begin{pmatrix} \varpi^{a-b} & \\ & 1 \end{pmatrix}K}$

pf:
$$\left(R(p)^b \cdot \mathbb{1}_{K\begin{pmatrix} \varpi^{a-b} & \\ & 1 \end{pmatrix}K}\right)(g) = \int_G R(p)^b(gh)\, \mathbb{1}_{K\begin{pmatrix} \varpi^{a-b} & \\ & 1 \end{pmatrix}K}(h^{-1})\, dh$$
$$= \int_G R(p)^b(h)\, \mathbb{1}_{K\begin{pmatrix} \varpi^{a-b} & \\ & 1 \end{pmatrix}K}(h^{-1}g)\, dh$$
$$= \int_{K\begin{pmatrix} \varpi^b & \\ & \varpi^b \end{pmatrix}K} \mathbb{1}_{K\begin{pmatrix} \varpi^{a-b} & \\ & 1 \end{pmatrix}K}(h^{-1}g)\, dh$$
$$= \int_K \mathbb{1}_{K\begin{pmatrix} \varpi^{a-b} & \\ & 1 \end{pmatrix}K}\left(\begin{pmatrix} \varpi^{-b} & \\ & \varpi^{-b} \end{pmatrix} g\right) dk$$

this is nonzero $\Leftrightarrow$ $g \in K\begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix}K$, and equals to 1

$$\Rightarrow \mathbb{1}_{K\begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix}K} = R(p)^b \cdot \mathbb{1}_{K\begin{pmatrix} \varpi^{a-b} & \\ & 1 \end{pmatrix}K}$$

Now we focus on the function $\mathbb{1}_{K\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix}K}$, $n \geq 2$ $\begin{cases} n=0, & 1 \\ n=1, & T(p) \end{cases}$

<u>Claim</u>: $\mathbb{1}_{K\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix}K} = T(p^n) - R(p)\underbrace{T(p^{n-2})}$

$\begin{pmatrix} \varpi^a & \\ & \varpi^b \end{pmatrix}$, $a \geq b \geq 0$ $a+b=n-2$

乘上 $R(p)$ 后为: $\begin{pmatrix} \varpi^{a'} & \\ & \varpi^{b'} \end{pmatrix}$, $a' \geq b' \geq 1$, $a'+b'=n$

Now we consider the character $\chi : \mathcal{H}_K \to \mathbb{C}$

Suppose $\chi(T(p^k)) = a_k$, obviously $a_0 = 1$, then

$$T_{p^r} = T_p T_{p^{r-1}} - p^{k-1} \langle p \rangle T_{p^{r-2}}, \quad \text{for } r \geq 2.$$

$$a_1 \cdot a_k = a_{k+1} + q\,\omega(p)\,a_{k-1}$$

in classical language, a_k is the eigenvalue of $T_{p^k} f$,

and usually on $S_n(\Gamma, \tilde{\chi})$, $\omega(p) = \tilde{\chi}(p) \cdot p^{n-2}$

because $\underset{\text{Hecke}}{\underset{\uparrow}{T(p)}} = p^{\frac{k}{2}-1} \underset{\text{classical}}{\underset{\uparrow}{\tilde{T}(p)}} \Rightarrow a_1 = q^{\frac{k}{2}-1} b_1$

$$T(p)^2 = T(p^2) + q R(p)$$

$$p^{k-2} \tilde{T}(p)^2 = T(p^2) + q R(p) q^{\frac{k}{2}-1} \tilde{T}(p) = T(p^2) + q^{\frac{k}{2}} R(p) \tilde{T}(p)$$

$$\tilde{T}(p^2) = \tilde{T}(p) \tilde{T}(p) - q^{k-1} \tilde{R}(p)$$

Question. for a newform $f \in S_k(\Gamma_0(N), \chi)^{new}$, we obtain $\phi_f \in A_0(G_\mathbb{Q}\backslash G_\mathbb{A}, \chi)$, $\rightsquigarrow \pi$

What's π ?

· First, we recall the construction of ϕ_f

$$f \rightsquigarrow \phi_\infty : GL_2(\mathbb{R})^+ \to \mathbb{C}, \quad \phi_\infty(g_\infty) = f(g_\infty i)\cdot j(g_\infty, i)^{-k}\cdot \det(g_\infty)^{\frac{k}{2}}$$

$$\phi_\infty(\gamma g_\infty) = f(\gamma g_\infty i)\cdot j(\gamma g_\infty, i)^{-k}\cdot \det$$
$$= f(\gamma g_\infty i)\cdot j(\gamma, z)^{-k} j(g_\infty, i)^{-k}$$
$$= f(\gamma z)\cdot (cz+d)^{-k}\cdot j(g_\infty, i)^{-k}$$
$$= \chi(\gamma)\cdot f(z)\cdot j(g_\infty, i)^{-k}$$
$$= \chi(\gamma)\,\phi_\infty(g_\infty)$$

$$\phi_\infty\left(\begin{pmatrix} u & \\ & u\end{pmatrix} g_\infty\right) = f(g_\infty i)\cdot j\left(\begin{pmatrix} u & \\ & u\end{pmatrix} g_\infty, i\right)^{-k}$$

now $\forall g \in GL_2(\mathbb{A})$,

$$g = \gamma g_\infty k, \quad \gamma \in GL_2(\mathbb{Q})_+,\ g_\infty \in GL_2(\mathbb{R})^+,\ k \in K_0(N)$$

$$\phi_f(g) = \chi(k)^{-1}\phi_\infty(g_\infty)$$

well-defined: $\gamma g_\infty k = \gamma' g_\infty' k' \Rightarrow \gamma g_\infty = \gamma' g_\infty',\ \gamma k = \gamma' k'$

$$\chi(k)^{-1}\phi_\infty(g_\infty) = \chi(\gamma^{-1}\gamma' k')^{-1}\phi_\infty(\gamma^{-1}\gamma' g_\infty') = \chi(k')^{-1}\chi(\gamma^{-1}\gamma')^{-1}\chi(\gamma^{-1}\gamma')\phi_\infty(g_\infty') = \chi(k')^{-1}\phi_\infty(g_\infty')$$

What's the central character?

Suppose $z \in \mathbb{A}_\mathbb{Q}^\times$, then $z = r\cdot u_\infty \cdot s$, $s \in \prod_p \mathbb{Z}_p^\times$, $r \in \mathbb{Q}^\times$, $u_\infty > 0$

$$\Rightarrow \phi_f\left(\begin{pmatrix} z & \\ & z\end{pmatrix} g\right) = \phi_f\left(\begin{pmatrix} r & \\ & r\end{pmatrix}\gamma \begin{pmatrix} u_\infty & \\ & u_\infty\end{pmatrix} g_\infty \cdot \begin{pmatrix} s & \\ & s\end{pmatrix} k\right) = \phi_\infty\left(\begin{pmatrix} u_\infty & \\ & u_\infty\end{pmatrix} g_\infty\right)\cdot \chi\left(\begin{pmatrix} s & \\ & s\end{pmatrix} k\right)^{-1}$$
$$= \phi_\infty(g_\infty)\cdot \chi(s)^{-1}\chi(k)^{-1}$$
$$= \chi(s)^{-1}\phi_f(g)$$

$\Rightarrow$ central character $= \chi^{-1}$

Should be: $\phi_f = \phi_\infty \otimes \bigotimes_p \phi_p$

Claim: For $p \nmid N$, ϕ_p is K_p-invariant

pf: $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in K_p \Rightarrow \left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}_p \phi_f\right)(g) = \phi_f\left(g\begin{pmatrix} a & b \\ c & d\end{pmatrix}_p\right) \Rightarrow \phi_p\left(g\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) = \phi_p(g)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad \phi_f(g)$ by definition

Hence For $p \nmid N$, $\pi \simeq B(\chi_1, \chi_2)$, with χ_1, χ_2 unramified. $\chi_1(p) = \alpha_1$, $\chi_2(p) = \alpha_2$

suppose $T_p f = a_p f$, then $T(p) = p^{\frac{k}{2}-1} T_p \Rightarrow T(p) f_p = p^{\frac{k}{2}-1} a_p = p^{\frac{1}{2}}(\alpha_1 + \alpha_2)$

and $(1, \ldots, p, \ldots 1) = p\cdot(p^{-1}, \ldots, 1, \ldots p^{-1})$

$$\underset{p}{\uparrow}$$

$$\left(\begin{pmatrix} p & \\ & p\end{pmatrix}_p \phi_f\right)(g) = \chi\left((p^{-1}, p^{-1}, \ldots, 1, \ldots p^{-1})\right)^{-1}\phi_f(g) = \chi(p)\,\phi_f(g) \Rightarrow \alpha_1\alpha_2 = \chi(p)$$

$$\alpha_1 + \alpha_2 = p^{\frac{k-3}{2}} a_p$$

For $p \mid N$ $\quad \phi_f$ is invariant under $K(p^{v_p(N)})_p$

$v_p(N) = 1$, $\Rightarrow$ π_p is special $\quad \pi_p \simeq \pi(\omega, \omega|\cdot|)$ $\quad \omega^2 \cdot |\cdot| = \chi_p$

$v_p(N) \geq 2$. $\Rightarrow$ π_p is supercuspidal

$p = \infty$, $\quad D_0^+(k)$

$$s = \frac{1}{2}(s_1 - s_2 + 1) = \frac{1}{2}(2s_1 + 1) = s_1 + \frac{1}{2}$$

$$\lambda = s(1-s) = \left(\frac{1}{2} + s_1\right)\left(\frac{1}{2} - s_1\right)$$

$$\mu = s_1 + s_2 = 0 \Rightarrow s_2 = -s_1$$

$$s_1 = \frac{k-1}{2} \qquad s_2 = \frac{1-k}{2}$$

$$f\left(\begin{pmatrix} x_1 & b \\ & x_2 \end{pmatrix} g\right) = |x_1|^{\frac{k}{2}} |x_2|^{-\frac{k}{2}} f(g)$$

Why $\quad \phi_f = \bigotimes_v \phi_v$?

we will prove this by the follow:

① $\mathcal{H}(G_{\mathbb{A}})$-mod $\mathcal{H}(G_{\mathbb{A}})\phi_f$ is irr

pf : if it's not.

$$\mathcal{H}(G_{\mathbb{A}})\phi_f = \pi_1 \overset{\{0\}}{\oplus} \pi_2$$

$\pi_{1\infty} = \pi_{2\infty}$ & $\underbrace{\pi_{1v} \text{ \& } \pi_{2v} \text{ almost coincide!}}$

$\hookrightarrow$ take $f_1 \in \pi_1$, $\quad f_2 \in \pi_2$

$$f_1 = \bigotimes_v f_{1,v} \qquad f_2 = \bigotimes_v f_{2,v}$$

$$f_1 = \sum_k \underbrace{\rho(k)}_{\rho_\infty(X_1 X_2 \cdots)} \phi_f \quad , \quad f_2 = \sum \underbrace{\rho(k')}_{\rho_\infty(X_1 X_2 \cdots)} \phi_f$$

$\Rightarrow$ $T(p)$ on f_1 = $T(p) f_2$ almost coincide

$\Rightarrow$ $\pi_1 = \pi_2$, then by Multiplicity One, no. such decomposition exists

(Whittaker function is Fourier coefficient!)

$$\Rightarrow \mathcal{H}(G_{\mathbb{A}})\phi_f \simeq \pi_\infty \otimes \bigotimes_p \pi_p = \pi$$

$$\dim \pi^{K_0(N), \chi, k} = 1. \quad \left(\pi_p^{K_0(N)_p, \chi_p} = 1 \right)$$

$$\mathbb{C}\phi_f = \phi_\infty \otimes \bigotimes_p \phi_p$$

global multiplicity one

$\Downarrow$

local multiplicity one !

i.e. $\pi^{K_0(N), \chi, k} = \pi_\infty(k) \otimes \bigotimes_p \pi_p^{K_0(N)_p, \chi_p}$

Satake isomorphism

Whittaker model & L- functions

Def: For a simple $\mathcal{H}_G$-mod (π, V), and a fixed character $\psi : F \to \mathbb{C}^*$, F is a local field
Whittaker model $\mathcal{W}$ is a space of functions on $GL_2(F)$, s.t. $\forall W \in \mathcal{W}$

1. $W\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \psi(x) W(g)$

2. W is a smooth function on $GL_2(F)$ & K- finite

3. $\exists$ isomorphism $V \xrightarrow{\sim} \mathcal{W}$, $v \mapsto W_v$, s.t.

   $W_{\pi(\phi)v} = \rho(\phi) W_v$, here $\rho(\phi)$ is the convolution operator

   i.e. the following diagram commutes:

$$\begin{array}{ccc} V & \longrightarrow & \mathcal{W} \\ {\scriptstyle \pi(\phi)}\downarrow & & \downarrow{\scriptstyle \rho(\phi)} \\ V & \longrightarrow & \mathcal{W} \end{array}$$

in [Gelbart], it is also assumed that $W\left(\begin{pmatrix} z & \\ & z \end{pmatrix} g\right) = \omega(z) W(g)$
this simply follows from 3. and the fact that (π, V) is simple
W is K-finite is also implicitly deduced by 3.

4. (only for Archimedean places) growth condition: for fixed $g \in G$, $\exists N, C > 0$, s.t.

   $W\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} g\right) \leq C \cdot |y|^N$, when $|y| \to +\infty$

The first important thing about Whittaker model is: Existence & Uniqueness for automorphic reprs
this is a consequence of local existence & uniqueness

Local Multiplicity One theorem :

F is a local field (either A or NA), ψ non-trivial character of F, $G = GL_2(F)$
Suppose (π, V) is a simple admissible $\mathcal{H}_G$-mod, then there exists a unique Whittaker model, w.r.t ψ
at most one

pf: F NA: $\dim V_\psi = 1$, here $V_\psi = V/V(\psi)$. $V(\psi)$ = space spanned by $\{\pi(n)v - \psi(n)v\}_{n \in N}$ for $GL(n, F)$
then

$\mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G \psi) \simeq \mathrm{Hom}_N(\pi|_N, \psi) \simeq \mathrm{Hom}_N(V, \psi) \simeq \mathrm{Hom}_{\mathbb{C}}(V_\psi, \mathbb{C})$ is one-dimensional
hence there is only one embedding $(\pi, V) \hookrightarrow \mathrm{Ind}_N^G \psi$

F A: Here $\mathcal{H}_G$ is the linear space of $\{f * D \mid f \in \mathcal{H}_K, D \in U(\mathfrak{g}_{\mathbb{C}})\}$, Simple admissible $\mathcal{H}_G$-mod $\Leftrightarrow$ irr admissible $(\mathfrak{g}, K)$-mod
We know irr $(\mathfrak{g}, K)$-mod structure is $\mathbb{Z}$, $\mathbb{Z}_{|\cdot| \geq k}$, $\mathbb{Z}_{|\cdot| \leq k}$, $(GL_2(\mathbb{R}) / GL_2(\mathbb{C}))$
each piece $V(m)$ corresponds to (by condition 4) a function space, $\mathcal{W}(\lambda, \mu, m)$ necessarily $\dim = 1$
Moreover, function in $W(\lambda, \mu, m)$ necessarily rapid decrease! $W(g) = O(e^{-r} y^{\frac{m}{2}})$

Explicit description of NA local Whittaker model

Prop: Fix a pair of unitary character $\xi_1, \xi_2 : F_v^\times \to \mathbb{C}^\times$, and a pair $s_1, s_2 \in \mathbb{C}^2$, define $\chi_i(x) = \xi_i(x)\cdot|x|^{s_i}$,
Consider a functional $B(\chi_1,\chi_2) \longrightarrow \mathbb{C}$ by the following formula:

$$\Lambda f = \int_{F_v} f\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) \psi(-x)\, dx$$

1. When $\mathrm{Re}(s_1 - s_2) > 0$, this integral absolutely converges
2. This morphism is nonzero, hence gives a Whittaker functional $\begin{cases} \text{for } \pi(\chi_1,\chi_2) & \text{when } \chi_1^{-1}\chi_2(x) \neq |x|^{-1}, \\ \text{for } \sigma(\chi_1,\chi_2) = \chi\cdot St_G & \text{when } \underbrace{\chi_1^{-1}\chi_2(x) = |x|^{-1}}_{\text{[illegible] modality-F}}. \end{cases}$ $\begin{matrix} \xi_1 = \xi_2 \\ \mathrm{Re}(s_1 - s_2) = 1 \end{matrix}$
3. For a given flat section f_0, then $\Lambda f_{s_1,s_2}$ can be analytic continued to all $s_1, s_2 \in \mathbb{C}^2$

pf: $f\left(w_0\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right) = f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right) = f\left(\begin{pmatrix} x^{-1} & -1 \\ & x\end{pmatrix}\begin{pmatrix} 1 & \\ x^{-1} & 1\end{pmatrix}\right) = \chi_1(x^{-1})\chi_2(x)|x|^{-1} f\left(\begin{pmatrix} 1 & \\ x^{-1} & 1\end{pmatrix}\right)$, $\forall x \neq 0$

when $|x|$ is sufficiently large, this equals to $|x|^{s_2 - s_1 - 1}\cdot(\xi_1^{-1}\xi_2)(x) f(1)$, then

$$\Lambda f = \int_{v(x) > N} + \int_{v(x) \leq N} = \underbrace{\text{analytic}}_{\substack{f \rightsquigarrow f(w_0) \\ \text{compact domain}}} + \sum_{k=N}^{-\infty} \int_{v(x)=k} q^{k(1+s_1-s_2)} (\xi_1^{-1}\xi_2)(x)\, f(1)\, \psi(-x)\, dx \quad (*)$$

$$= \text{analytic} + \sum_{k=N}^{-\infty} f(1)\cdot\int_{v(x)=k} q^{k(s_1-s_2)}(\xi_1^{-1}\xi_2)(x)\,\psi(-x)\, d^\times x$$

if $\xi_1^{-1}\xi_2$ is ramified, then the last term $= 0$, Λf is analytic for $\forall s_1, s_2$
if $\xi_1^{-1}\xi_2$ is unramified, then suppose $\xi_1^{-1}\xi_2(p_v) = q^\alpha$, here α is pure imaginary, the last term is dominated by

$$f(1)\cdot \sum_{k=N}^{-\infty} q^{k(s_1-s_2)}\, q^{k\alpha} \int_{v(x)=k} d^\times x = f(1)\cdot\frac{q^{-N(s_2-s_1-\alpha)}}{1 - q^{s_2-s_1-\alpha}}, \text{ when } \mathrm{Re}(s_1 - s_2) > 0$$

the only obstruction arises when $q^{s_2 - s_1} = q^\alpha$, i.e. $\chi_1 = \chi_2$ case, but this can be overcomed by choosing $f(1) = 0$
hence we choose a "flat" section: (Bruhat decomposition: $G = B \amalg B w_0 N$) 所以本质仍是红外发散

$x_1 = x_2 \Rightarrow (*)$ 不收敛

$\int q^k \psi(-x)\, dx$ 仍会收敛

$$f\left(\begin{pmatrix} y_1 & z \\ & y_2\end{pmatrix} w_0 \begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right) = \chi_1(y_1)\chi_2(y_2)\left|\frac{y_1}{y_2}\right|^{\frac{1}{2}}, \text{ when } y_1, y_2 \in \mathcal{O}_v^\times,\ x \in \mathfrak{p}_v^{c(\psi)},\ z \in \mathcal{O}_v$$

then $\Lambda f = \int_{\mathfrak{p}_v^c} \psi(-x)\, dx = \mathrm{vol}(\mathfrak{p}_v^c) \neq 0 \Rightarrow \Lambda \neq 0$

now let's suppose $\chi_1 = \chi_2$, then $g(x) = f\left(w_0\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right) = f\left(\begin{pmatrix} 1 & \\ x^{-1} & 1\end{pmatrix}\right)$, then $g(x) = \begin{cases} f(w_0), & \text{when } |x| \to 0 \\ f(1), & \text{when } |x| \to \infty \end{cases}$

therefore, $\Lambda f = \lim_{n\to\infty} \int_{\mathfrak{p}^{-n}} g(x)\psi(-x)\, dx$, when $N, M \gg 0$, $\int_{-M \leq v(x) \leq -N} g(x)\psi(-x)\, dx = f(1)\underbrace{\int_{-M \leq v(x) \leq -N}}\psi(-x)\, dx = f(1)\int_{-M\leq v(x) \leq -N} \psi(-x+y)\, dx$

$= \psi(y)\, f(1) \int_{-M \leq v(x) \leq -N} \psi(-x)\, dx$

choose $v(y) > -N$, and $\psi(y) \neq 1$, then this implies the integral $= 0$

hence Λf also converges for $\chi_1 = \chi_2$

therefore for a given flat section f_0, then $\Lambda f_{s_1,s_2}$ can be analytically continued to $\forall s_1, s_2$

Rmk: here, flat section for $B(\chi_1,\chi_2)$ is a function $f_0 : GL_2(\mathcal{O}_v) \longrightarrow \mathbb{C}$, s.t.

$$f_0\left(\begin{pmatrix} a_1 & b \\ & a_2\end{pmatrix} k\right) = \xi_1(a_1)\xi_2(a_2) f_0(k), \text{ for } \forall a_i \in \mathcal{O}_v^\times,\ b \in \mathcal{O}_v,\ k \in GL_2(\mathcal{O}_v)$$

and we extend f_0 to be in $B(\chi_1,\chi_2)$ for various s_1, s_2. 3 means $\Lambda f_{s_1,s_2}$ has analytic continuation to all of $s_1, s_2 \in \mathbb{C}^2$

Now we get an explicit description Whittaker model for NA, principal series repns $\pi(\chi_1, \chi_2)$, with the condition that $|\chi_1(\varpi)/\chi_2(\varpi)| < 1$ ($\Leftrightarrow \mathrm{Re}(s_1 - s_2) > 0$), now we would like to extend this description to $|\chi_1(\varpi)/\chi_2(\varpi)| \le 1$ case.

Let's use the same notation, $\chi_i = \xi_i \cdot |\ |^{s_i}$, where ξ_i is unitary, $s_i \in \mathbb{C}$. choose a flat section f_{s_1,s_2}, then we define:

$$\Lambda f_{s_1,s_2} = \lim_{k \to +\infty} \int_{\mathfrak{p}^{-k}} f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}) \psi(-x)\, dx.$$ ψ must be non-trivial!

Claim: This limit exists, and Λ defines a Whittaker functional

pf: suppose the conductor of ψ is $\mathfrak{p}^{m_0}$, $m_0 \in \mathbb{Z}$, then we will show, when k is sufficiently large, then

$$\int_{v(x)=-k} f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}) \psi(-x)\, dx = 0$$

We can first assume that $f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix})\psi(-x) = |x|^{s_2 - s_1 - 1} \cdot (\xi_1^{-1}\xi_2)(x) \cdot f(1)$, this is right when k is sufficiently large, consider the subset $v(x) = -k$ divided by the condition, $y' \equiv y \mod \mathfrak{p}^{m_0 - 1}$, i.e. $y' - y \in \mathfrak{p}^{m_0-1} \Leftrightarrow 1 - y^{-1}y' \in \mathfrak{p}^{k+m_0-1}$ when $k >> 0$, we have $|y'|^{s_2-s_1-1}(\xi_1^{-1}\xi_2)(y') = |y|^{s_2-s_1-1}(\xi_1^{-1}\xi_2)(y)$. hence we divid $v(x) = -k$ by mod $\mathfrak{p}^{m_0-1}$ condition, over each piece, we must have essentially the integral

$$\int_{\substack{v(y)=-k \\ y \equiv x + \mathfrak{p}^{m_0-1}}} \psi(-y)\, dy = \int_{\substack{v(y)=-k \\ y \equiv x + \mathfrak{p}^{m_0-1}}} \psi(-y+y_0)\, dy = \psi(y_0) \int_{\substack{v(y)=-k \\ y \equiv x + \mathfrak{p}^{m_0-1}}} \psi(-y)\, dy$$

for $\forall y_0 \in \mathfrak{p}^{m_0-1}$, but ψ is non-trivial on $\mathfrak{p}^{m_0-1}$, hence the integral is 0

the fact that Λ defines a Whittaker functional is a trivial computation

Therefore, we get Whittaker functional/model for any irr principal series representation, i.e.

$$(\Lambda f)(g) = \lim_{k \to +\infty} \int_{\mathfrak{p}^{-k}} f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}) \psi(-x)\, dx$$

By all the analysis, we see that this limit is absolutely convergent, essentially, it only involves finitely many terms!

Actually, we have already obtained the Whittaker functional for Principal series rpn & Steinberg repn!

And the corresponding Whittaker model is:

$$\begin{aligned} W_f(g) &= \Lambda(\rho(g) f) \\ &= \lim_{k \to +\infty} \int_{\mathfrak{p}^{-k}} f(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g)\, \psi(-x)\, dx \end{aligned}$$

- Now we have constructed explicit Whittaker functional for St_G, $\pi(\chi_1, \chi_2)$ where $|\chi_1^{-1}\chi_2(x)| = |x|^{\beta}$, with $Re(\beta) < 0$, and $\chi_1 = \chi_2$

The remaining cases are $\pi(\chi_1, \chi_2)$, where $|\chi_1^{-1}\chi_2(x)| = |x|^{\beta}$ s.t. $Re(\beta) \geq 0$, in this case, the functional can't be computed directly by the integral formula

Recall that from Whittaker functional, we can construct Whittaker model w.r.t ψ by:

$$W_f(g) = \Lambda(\pi(g) f) = \int_{F_v} f\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \right) \psi(-x)\, dx$$

Now we are going to give an intertwining operator $B(\chi_1, \chi_2) \to B(\chi_2, \chi_1)$, where $Re(s_1 - s_2) > 0$

<u>Prop</u>: When $Re(s_1 - s_2) > 0$, we can define an operator $M: B(\chi_1, \chi_2) \to B(\chi_2, \chi_1)$ by:

$$(Mf)(g) = \int_{F_v} f\left(w_0 \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \right) dx$$

Moreover, for a given flat section f_0,

$(M f_{s_1, s_2})(g)$ has analytic continuation to all s_1, s_2 s.t. $\chi_1 \neq \chi_2$

<u>pf</u>: exactly the same as the previous proposition, note here "ψ" is trivial, so we can't extend to the whole plane $\mathbb{C}^2$

<u>Rmk</u>: $M(s)$ gives an isomorphism of G-repns: $B(\chi_1, \chi_2) \xrightarrow[\sim]{M} B(\chi_2, \chi_1)$, when $\chi_1 \neq \chi_2$

pf: We have shown this for $Re(s_1 - s_2) > 0$, and by the theory of analytic continuation,

$$M\, \pi(g) f - \pi(g)\, M f \quad \text{is zero for } Re(s_1 - s_2) > 0,$$

thus it must vanish for $\forall s_1, s_2$, s.t. $\chi_1 \neq \chi_2$, therefore, it is a G-repn isomorphism when $\chi_1 \neq \chi_2$

$\mathcal{H}(G_A)$-mod
$\simeq$
need exp, not actually a $GL(2,A)$-repn, rather $(\mathfrak{g}, K)\times GL(2, A_f)$-mod

<u>Def</u>: (Global Whittaker model)

(π, V) is an irr admissible repn of $GL(2,A)$ (or equivalently, V is an irr admissible $\mathcal{H}_{G_A}$-mod)
Whittaker model of π w.r.t a non-trivial character ψ of A/F is a space $\mathcal{W}$ of functions on $GL(2,A)$

1. $W\left(\begin{pmatrix} 1 & x \\ & 1\end{pmatrix} g\right) = \psi(x)\, W(g), \ \forall x \in A$

2. W is smooth & K-finite

3. $\exists$ isomorphism $V \xrightarrow{\sim} \mathcal{W}$, $v \mapsto W_v$, s.t

$$W_{\pi(\phi)v} = \rho(\phi)\, W_v$$, here $\rho(\phi)$ is the convolution operator

i.e. the following diagram commutes:

$$\begin{tikzcd} V \arrow[r] \arrow[d, "\pi(\phi)"'] & \mathcal{W} \arrow[d, "\rho(\phi)"] \\ V \arrow[r] & \mathcal{W} \end{tikzcd}$$

in [Gelbart], it is also assumed that $W\left(\begin{pmatrix} z & \\ & z\end{pmatrix} g\right) = \omega(z)\, W(g)$
this simply follows from 3. and the fact that (π, V) is simple
W is K-finite is also implicitly deduced by 3.

4. growth condition: for fixed $g \in G$, $\exists N, C > 0$, s.t.

$$W\left(\begin{pmatrix} y & \\ & 1\end{pmatrix} g\right) \leq C \cdot |y|^N, \text{ when } |y| \to +\infty, \ y \in A^\times$$ $\leadsto$ essentially a condition only in terms of y_∞

Rmk on 2: • K-finite is an obvious definition
• meaning of smooth: $\forall g \in G$, $\exists$ a nbd $N \subseteq GL(2,A)$, s.t
$$W(h) = W_g(h_\infty). \text{ for } \forall h \in N$$
here W_g is a smooth function on $GL(2, A_\infty)$

$\Rightarrow$ the def includes smooth at infinite place & locally constant at finite places

<u>Global Multiplicity One Theorem</u>

(π, V) is an irr admissible repn of $GL(2,A)$ (or equivalently, V° is an irr admissible $\mathcal{H}_{G_A}$-mod)
Then (π, V) admits a Whittaker model w.r.t a non-trivial character ψ of A_F if and only if:

(π_v, V_v) admits a Whittaker model w.r.t. ψ_v, $\forall v$

here, we should know that ψ_v is non-trivial)
actually, this follows from ψ non-trivial (can be constructed by e_{F_v})
and $A_F^* \simeq F$, $\psi(x) = e_F(ax)$
$a \in F$

if this is the case, then $\mathcal{W}$ is unique and spanned by functions of the following type

$$W(g) = \prod_v W_v(g_v).$$

here $W_v \in \mathcal{W}_v$, and $W_v = W_v^\circ$ for almost all v, where W_v° is a spherical element in spherical repn π_v, and

$$W_v^\circ(g_v) = 1 \quad \text{for } g_v \in GL_2(\mathcal{O}_v)$$

and if $\phi = \otimes \phi_v$, then $W_\phi = \prod W_v$, W_v corresponds to ϕ_v

Whittaker model for Automorphic representation

By local multiplicity one theorem, there exists a unique Whittaker model for a cuspidal automorphic repn (π, V)
We can also write down this model explicitly!

<u>Thm</u>: Suppose (π, V) is a cuspidal automorphic repn $\subseteq L_0^2(G_F\backslash G_{\mathbb{A}}, \omega)$, then (π, V^0) admits a unique Whittaker model
Take $\phi \in V^0 \subset \mathcal{A}_0(G_F\backslash G_{\mathbb{A}}, \omega)$, $g \in GL(2, \mathbb{A})$, let

$$W_\phi(g) = \int_{\mathbb{A}/F} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi(-x)\, dx$$

Note: $\mathbb{A}/F$ is compact

Then the Whittaker model for π, say $\mathcal{W}$, consists of functions of this type
We also have Fourier expansion:

$$\phi(g) = \sum_{\alpha \in F^\times} W_\phi\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\right)$$

pf: W_ϕ obviously satisfies 1, 2 (because 2 essentially is "right action", but we integral by "left action"; they commute)
to show 3, we should prove $W_\phi = 0 \Rightarrow \phi = 0$, which is a consequence of the <u>Fourier expansion</u>
Now we consider the matter of moderate growth, this essentially follows from the moderate growth condition of automorphic forms.
because for fixed $g \in G_{\mathbb{A}}$, consider

$$\begin{aligned} W_\phi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} g\right) &= \int_{\mathbb{A}/F} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y & \\ & 1 \end{pmatrix} g\right) \psi(-x)\, dx \\ &= \int_{\mathbb{A}/F} \phi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & y^{-1}x \\ & 1 \end{pmatrix} g\right) \psi(-x)\, dx \\ &= |y| \int_{y^{-1}\cdot\mathbb{A}/F} \phi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi(-xy)\, dx \end{aligned}$$

since $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g$ when $x \in \mathbb{A}/F$ ranges over a compact subset, hence for $\forall N > 0$, $\exists C_N$, s.t.

$$\left|\phi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)\right| \leq C_N |y|^N, \text{ when } |y| \to +\infty$$

in fact, we could take $\leq C_N |y|^{-N}$ since it is a <u>cusp form</u>!

then we get:

$$\left|W_\phi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix} g\right)\right| \leq C_N' |y|^{1+N} \text{ when } |y| \to +\infty$$

$\leq C_N' |y|^{1-N}$, when $|y| \to +\infty$

Now we try to establish the Fourier expansion:
Observation:

$$f: x \longmapsto \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right)$$

is a period function on $\mathbb{A}$, period $= F$, i.e. $f(x+a) = f(x)$, $\forall a \in F$.
and since ϕ is smooth $\Rightarrow f$ is a smooth function on $\mathbb{A}/F$, then [Goldfeld 1.8.10] tells us:

$$f(x) = \sum_{\alpha \in F} \hat{f}(\alpha) \psi(\alpha x)$$

here

$$\hat{f}(\alpha) = \int_{\mathbb{A}/F} f(x)\psi(-\alpha x)\, dx = \int_{\mathbb{A}/F} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi(-\alpha x)\, dx$$

Note: $\alpha = 0$, we got by definition, $\hat{f}(0) = 0$

$\alpha \neq 0$, $\displaystyle\int_{\mathbb{A}/F} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \psi(-\alpha x)\, dx = \int_{\mathbb{A}/F} \phi\left(\begin{pmatrix} 1 & \alpha^{-1}x \\ & 1 \end{pmatrix} g\right) \psi(-x)\, dx$

$$\phi\left(\begin{pmatrix} \alpha^{-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} \alpha & x \\ & 1 \end{pmatrix} g\right) = \phi\left(\begin{pmatrix} \alpha & x \\ & 1 \end{pmatrix} g\right) = \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\right)$$

$$= \int_{\mathbb{A}/F} \phi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\right) \psi(-x)\, dx$$

$$= W_\phi\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\right)$$

i.e. we have the equality:

$$\phi(g) = \sum_{\alpha \in F^\times} W_\phi\left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g\right)$$

this summation essentially only involves those α, s.t. $\alpha \cdot N \in \mathcal{O}_F$, i.e. $\alpha \in N^{-1}\mathcal{O}_F$

Rmk: In this situation, ϕ & W_ϕ both rapidly decrease!

Uniqueness of Whittaker model will give us the famous multiplicity one theorem for cuspidal automorphic repns

Multiplicity One Theorem:

Suppose (π, V) & (π', V') are two irreducible subrepresentations of $L_0^2(G_F \backslash G_{\mathbb{A}}, \omega)$

Then if $\pi_v \simeq \pi_v'$ for:

- all archimedean places
- almost all finite places

then we necessarily have: $(\pi, V) = (\pi', V')$

Rmk: latter we will see that the information about all the finite places determine the infinite places!

pf: We only have to show, there exists a common nonzero function $\phi \in V \cap V'$ $\Rightarrow$ the Fourier expansion tells us $\pi_v \cap \pi_v' \neq 0$

Consider finite subset S, s.t.

$\forall v \notin S$, π_v is spherical, $\pi_v \simeq \pi_v'$

then for $\forall v \notin S$, $\mathcal{W}_{\pi_v} = \mathcal{W}_{\pi_v'}$, hence we take W_v, s.t. W_v is stable under K_v-action, and $W_v(k_v) = 1$, $\forall k_v \in K_v$

now we consider those $v \in S$, recall:

$$\mathrm{Hom}_M(\pi_v|_M, \mathrm{Ind}_N^M \psi_v) \simeq \mathrm{Hom}_N(\pi_v|_N, \psi_v) \text{ is one-dimensional}$$

we choose this unique $\varphi: \pi_v \to \mathrm{Ind}_N^M \psi_v$, and denote its image by $\mathcal{K}_v$, $\forall f \in \mathcal{K}_v$ is determined by its restriction on $\begin{pmatrix} y & \\ & 1 \end{pmatrix} \in M$

Claim: $c\text{-}\mathrm{Ind}_N^M \psi_v \subseteq \mathcal{K}_v \subseteq \mathrm{Ind}_N^M \psi_v$, the first equality holds iff π_v is cuspidal (36.1. Corollary 2) $\sim$ Weil repn

Note: As a vector space, $\mathcal{K}_v = \mathcal{W}_v$, but we could view $\mathcal{K}_v$ as a function space on $F_v^\times$, and $c\text{-}\mathrm{Ind}_N^M \psi_v \subseteq \mathcal{K}_v$ simply means $C_c^\infty(F_v^\times) \subseteq \mathcal{K}_v = \mathcal{W}_v$,

now we take $W_v \in \mathcal{W}_v$, s.t. $W_v\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) \in C_c^\infty(F_v^\times)$, then $\exists W_v' \in \mathcal{W}_v'$, s.t. $W_v'\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = W_v\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)$, now we consider

$$W(g) = \prod W_v(g) \in \mathcal{W}_\pi, \quad W'(g) = \prod W_v'(g) \in \mathcal{W}_{\pi'},$$ and we consider ϕ & ϕ' similarly

ϕ & ϕ' are right-invariant under some $K_0 \underset{\text{compact}}{\overset{\text{open}}{\subseteq}} GL(2, \mathbb{A}_f)$, and they agree on $\begin{pmatrix} y & x \\ & 1 \end{pmatrix} g_\infty$, here $y \in \mathbb{A}^\times$, $x \in \mathbb{A}$, $g_\infty \in GL(2, \mathbb{A}_\infty)$

ϕ & ϕ' are also G_F-invariant (automorphic! if we argue by W & W' this may not be true by first sight) $\Rightarrow$ by Strong approximation (det must be surj) we get $\phi = \phi'$

Local repns: Jacquet module & Classification of irr repns

We first work over a non-Archimedean local field F, and a smooth admissible repn (π, V) of $GL_2(F)$

<u>Def</u>: For a character $\psi: F \to \mathbb{C}^\times$, which may be trivial. we consider the following

$$V(\psi) = \operatorname{Span}_{\mathbb{C}}\left\{ \pi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right)v - \psi(x)v \right\}$$

$$\text{and } \quad J_\psi(V) := V/V(\psi)$$

<u>Rmk</u>: • Whittaker functional corresponds to the smooth dual of $J_\psi(V)$

because if $\Lambda: V \to \mathbb{C}$ satisfy $\Lambda(\pi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}v) = \psi(x)\Lambda(v) \rightsquigarrow \Lambda$ vanishes on $V(\psi)$

hence Λ factors through $V/V(\psi) = J_\psi(V)$

• For ψ = trivial character, $J(V)$ admits a repn of B, essentially T

because $V(N)$ is B-stable:

$$\pi\begin{pmatrix} a & v \\ & b \end{pmatrix} \cdot \left(\pi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}v - v \right) = \pi\begin{pmatrix} 1 & b^{-1}ax \\ & 1 \end{pmatrix} \left(\pi\begin{pmatrix} a & v \\ & b \end{pmatrix}v \right) - \pi\begin{pmatrix} a & v \\ & b \end{pmatrix}v$$

$$\begin{pmatrix} a & v \\ & b \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \begin{pmatrix} a & ax+v \\ & b \end{pmatrix} = \begin{pmatrix} 1 & b^{-1}ax \\ & 1 \end{pmatrix}\begin{pmatrix} a & v \\ & b \end{pmatrix}$$

This is not true for other non-trivial characters of F!

• Now the principal series comes into sight:

fix a repn, or a character $\sigma: T \simeq (F^\times)^2 \to \mathbb{C}^\times$, we consider

Frobenius reciprocity

$$\operatorname{Hom}_T(J(V), \sigma) \simeq \operatorname{Hom}_B(J(V), \sigma) \simeq \operatorname{Hom}_B(V, \sigma) \simeq \operatorname{Hom}_G(V, \operatorname{Ind}_B^G \sigma)$$

then suppose V is an irr smooth admissible repn & $\exists$ non-zero T-homo $J(V) = V/V(N) \to \sigma$

we get $V \hookrightarrow \operatorname{Ind}_B^G \sigma$, so we obtain

$$\begin{array}{c} J(V) \neq 0 \\ \Updownarrow \\ V(N) \subsetneq V \end{array} \Longrightarrow \pi \text{ is a subrepn of some induced repn } \operatorname{Ind}_B^G \sigma$$

Actually, in almost all cases, $\operatorname{Ind}_B^G \sigma$ is already irreducible, so in a lot of cases, π is just $\operatorname{Ind}_B^G \sigma$

ind repn in the usual sense
not the modified version

• Funtorial point of view

$$J: \quad V \longrightarrow V_N = V/V(N)$$

gives a functor from:

$$\begin{matrix} G \\ \text{or } B \\ \text{or } M \end{matrix} \text{-repns} \longrightarrow T\text{-repns}$$

easy to understand

it has the following property:

J is exact

Actually, J_ψ is also exact, but we should view J_ψ as: $\begin{matrix} G \\ \text{or } B \\ \text{or } M \end{matrix}$ -repns $\longrightarrow$ vector spaces.

J_ψ is exact

the exactness is not difficult to show, the only tricky point is:

$$0 \to V \to W \text{ exact } \Rightarrow 0 \to J_\psi(V) \to J_\psi(W) \text{ exact}$$

this can be done by using the " Whittaker functional crietrion"

Lemma. *Let μ_N be a Haar measure on N and ϑ a character of N.*

(1) *Let (π, V) be a smooth representation of N and $v \in V$. The vector v lies in $V(\vartheta)$ if and only if there is a compact open subgroup N_0 of N such that*

$$\int_{N_0} \vartheta(n)^{-1}\pi(n)v \, d\mu_N(n) = 0. \tag{8.1.1}$$

independent which vector space it lies in

(2) *The process $(\pi, V) \mapsto V_\vartheta$ is an exact functor from* Rep(N) *to the category of complex vector spaces.*

Our next task is to understand the following irr criterion of induced repns

Irreducibility Criterion. *Let $\chi = \chi_1 \otimes \chi_2$ be a character of T, and set $(\Sigma, X) = \mathrm{Ind}_B^G \chi$.*

(1) *The representation (Σ, X) is reducible if and only if $\chi_1\chi_2^{-1}$ is either the trivial character or the character $x \mapsto \|x\|^2$ of $F^\times$.*

(2) *Suppose that (Σ, X) is reducible. Then:*

(a) *the G-composition length of X is 2;*

(b) *one composition factor of X has dimension 1, the other is of infinite dimension;*

(c) *X has a 1-dimensional G-subspace if and only if $\chi_1\chi_2^{-1} = 1$;*

(d) *X has a 1-dimensional G-quotient if and only if $\chi_1\chi_2^{-1}(x) = \|x\|^2$, $x \in F^\times$.*

The argument we are going to use is Assuming X reducible, then see what can we say about χ_1 & χ_2

We first consider the statement about G-composition

Instead of directly considering G, we consider some smaller subgroups

$$0 \longrightarrow V \longrightarrow X \longrightarrow \chi \longrightarrow 0 \qquad \mathrm{Ind}_B^G \chi \longrightarrow \chi \text{ the canonical map}$$

$$f \longmapsto f(1)$$

this is an exact sequence of B-repns, χ is already irr, now we focus on V

We have the following decomposition

$$G = B \sqcup BwN, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

then $f(1) = 0 \Leftrightarrow \mathrm{Supp} f \subseteq BwN$, actually we can say more: $\exists$ compact open $N_0 \subset N$, s.t.

$$\mathrm{Supp} f \subseteq BwN_0$$

Actually, we have the following identification

$$V \xrightarrow{\sim} C_c^\infty(F)$$

$$f \longmapsto f_N : x \mapsto f\left(w\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right)$$

pf: when $|x|$ small enough, we have

$$0 = f\left(\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} -x^{-1} & -1 \\ & -x \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & x^{-1} \\ & 1 \end{pmatrix}\right)$$

$$= \chi_1(-x^{-1})\chi_2(-x) f_N(x^{-1})$$

Now we still focus on V, V is a B-repn, under the isomorphism $V \simeq C_c^\infty(F)$
We describe the action of B on $C_c^\infty(F)$

$$\left(\pi\begin{pmatrix} a & v \\ & b\end{pmatrix} f_N\right)(x) = f\left(w\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\begin{pmatrix} a & v \\ & b\end{pmatrix}\right) = f\left(w\begin{pmatrix} a & bx+v \\ & b\end{pmatrix}\right)$$

$$= f\left(\begin{pmatrix} b & \\ & a\end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & a^{-1}(bx+v) \\ & 1\end{pmatrix}\right) = \chi_1(b)\chi_2(a) f_N\left(a^{-1}(bx+v)\right)$$

We would like to understand $J(V)$.

By previous proposition, $f \in V(N) \iff \int_{N_0} \pi(n) f \, d\mu(n) = 0$ for some open compact $N_0 \subseteq N$

so we consider for $x \in T$, use $f(B) = 0$

$$\tilde{f}_{N_0}(x) = \int_{N_0} (\pi(n) f)(xw) \, d\mu(n) = \int_{N_0} f(xwn) \, d\mu(n) = \chi(x) \tilde{f}_{N_0}(1)$$

since $f(xwn)$ is actually compactly supported in terms of N, we can just consider

$$\begin{aligned} C_c^\infty(F) \simeq V &\longrightarrow \mathbb{C} \\ f &\longmapsto \int_N f(wn) \, dn \end{aligned}$$

this gives $V/V(N) \simeq \mathbb{C}$, the T-repn on $\mathbb{C}$ is $\chi^w \cdot \delta_B^{-1}$, hence we have $V_N \simeq \chi^w \cdot \delta_B^{-1}$

i.e. we have $\quad 0 \to V(N) \to V \to \chi^w \cdot \delta_B^{-1} \to 0$ as T-repns, also B-repn

The next crucial thing is:

<u>Lemma</u>: $V(N)$ is irreducible over B, even over M.

Here as repn of B or M, X has length ≤ 3, recall:

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] & X \arrow[r] & \chi \arrow[r] & 0 \\ 0 \arrow[r] & \underbrace{V(N)}_{\substack{\text{irr as } B(M) \\ \text{-repn}}} \arrow[r] & V \arrow[u] \arrow[r] & V_N \simeq \underbrace{\chi^w \cdot \delta_B^{-1}}_{\substack{\text{irr as } B(M) \\ \text{repn}}} \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & V(N) \arrow[r] & V \arrow[r] & X \\ & \substack{\text{quotient} \\ \chi^w \cdot \delta_B^{-1}} \arrow[u] & \substack{\text{quotient} \\ \chi} \arrow[u] & \end{tikzcd}$$

Therefore as a G-repn, X has composition length ≤ 3, and either X has a finite-dim G-irr subrepn
or X has a finite-dim G-irr quotient

Now we first focus on the case: X has a finite-dim G-irr subspace

<u>Claim</u>: Suppose (π, V) is an irr smooth admissible G-repn, then $\pi \simeq \phi \circ \det$

pf: this needs to inputs

- every finite-dim irr repn of G contains a N-fixed vector
- $SL_2(F)$ is generated by N & any element in $SL_2(F) - B(F)$

Now we have a one-dim subrepn of X, but this will rule out a lot of choices by the following:

<u>Prop</u>: X contains a N-stable subspace $\Leftrightarrow \chi_1 = \chi_2$
moreover, this subspace is also unique!

pf: Suppose $f \in X$ is N-stable, since $\operatorname{supp} f$ is left B-invariant, $\Rightarrow$ by $G = B \sqcup BwN$
we have $\operatorname{Supp} f = G$ or B or BwN but $f(1)$ & $f(w)$ are related by

$$B \Leftrightarrow f(1) \neq 0,\ f(w) = 0 \qquad BwN \Leftrightarrow f(1) = 0,\ f(w) \neq 0$$

$$f(1) = \chi_1^{-1}\chi_2(-x)\psi(x^{-1})f(w)$$

$$f(1) \underset{\substack{|x|\\ \text{small}}}{=} f\left(\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} -x^{-1} & -1 \\ & -x \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & x^{-1} \\ & 1 \end{pmatrix}\right)$$

$$= \chi_1(-x^{-1})\chi_2(-x)\psi(x^{-1})f(w)$$

therefore $\operatorname{Supp} f = G \Rightarrow f(1) \neq 0$
hence $f \notin V$, we consider $\mathbb{C}f$

$$0 \to V \to X \to \mathbb{C}f \to 0 \quad \text{as } B\text{-repn}$$

then since $\mathbb{C}f \simeq X \Rightarrow N$ acts trivially on $\mathbb{C}f$, i.e. f invariant under $N(F)$

since

$$f(1) = \chi_1^{-1}\chi_2(-x)f(w) \quad \text{for } x \text{ very small} \Rightarrow \chi_1(x) = \chi_2(x) \text{ for } x \text{ very small}$$

$$\chi_1 = |\cdot|^{s_1}\cdot\xi_1,\quad \chi_2 = |\cdot|^{s_2}\cdot\xi_2 \Rightarrow s_1 = s_2 \bmod \frac{2\pi i}{\log q}$$
$$\& \ \xi_1 = \xi_2$$

hence we get $\underset{\substack{\| \\ \phi}}{\chi_1} = \chi_2$ & $f(g) = \phi(\det g)\cdot f(1) \Rightarrow$ uniqueness

Therefore it tells us that

① X has a finite-dim G-irr subspace $\Rightarrow \chi_1 = \chi_2$

now we consider the opposite: X has a finite-dim G-irr subquotient, obviously it's still one-dim!

We use the following isomorphism:

$$\left(\mathrm{Ind}_B^G \chi\right)^\vee \simeq \mathrm{Ind}_B^G \chi^{-1} \cdot \delta_B^{-1}$$

$$\delta_B : \begin{pmatrix} y_1 & x \\ & y_2 \end{pmatrix} \to \left|\frac{y_2}{y_1}\right|$$

then $\mathrm{Ind}_B^G \chi^{-1} \cdot \delta_B^{-1}$ has a finite-dim irr G-subrepn, hence

$$\tilde{\chi}_1 = \chi_1^{-1} \cdot |\cdot|$$

$$\tilde{\chi}_2 = \chi_2^{-1} \cdot |\cdot|^{-1}$$

② hence $\tilde{\chi}_1 = \tilde{\chi}_2 \iff \chi_1 \cdot \chi_2^{-1} = |\cdot|^2$

Now we consider the next question: G-composition length of X when it is reducible

in the case $\chi_1 = \chi_2 = \phi$, we have

$$0 \to \mathbb{C}f \to X \to X/\mathbb{C}f \to 0$$

since $V \cap \mathbb{C}f = 0 \Rightarrow$ As B-repn, $V \simeq X/\mathbb{C}f$.

if $X/\mathbb{C}f$ is not irr as G-repn $\Rightarrow V$ has G- composition length ≥ 2

but we have already seen, V has B-composition length $= 2$

then suppose $X/\mathbb{C}f$ is not G-irr, we get

$$0 \to X_1 \to X/\mathbb{C}f \to X_2 \to 0 \quad \text{as } G\text{-repn}, \ X_1 \ \& \ X_2 \text{ irr}$$

this is also a B-repn, and if we replace $X/\mathbb{C}f$ by V,

$$0 \to V_1 \to V \to V_2 \to 0 \quad \text{as } B\text{-repn}$$

then if $\dim V_1 = 1 \Rightarrow \dim X_1 = 1 \Rightarrow$ we again get a one-dim repn of G, contradiction to previous

hence $\dim V_2 = \dim X_2 = 1$, and $V_2 \simeq \chi^w \cdot \delta_B^{-1} \Rightarrow X_2 = \phi' \cdot \det$,

but $\begin{pmatrix} a & \\ & 1 \end{pmatrix} \xrightarrow{\chi^w \cdot \delta_B^{-1}} = \phi(a) \cdot |a|. = \phi'(a)$

$\begin{pmatrix} 1 & \\ & a \end{pmatrix} \xrightarrow{\chi^w \cdot \delta_B^{-1}} = \phi(a) \cdot |a|^{-1} = \phi'(a)$

$\Rightarrow$ contradiction $\Rightarrow X/\mathbb{C}f$ is G-irr

Therefore, in the case $\chi_1 = \chi_2$, we have

$$\begin{tikzcd} 0 \arrow[r] & \phi\cdot\det \arrow[r] & X \arrow[r] & V \arrow[r] & 0 \\ & & & \phi\cdot St_G \arrow[u] & \end{tikzcd}$$

if $\chi_1\cdot\chi_2^{-1} = |\cdot|^2$, we have

$$0 \to V' \to X \to \phi'\cdot\det \to 0$$

Now we show V' is essentially $\tilde{\phi}\cdot St_G$ for some $\tilde{\phi}$

Prop: **Proposition.** *Let χ, ξ be characters of T. The space $\mathrm{Hom}_G(\mathrm{Ind}_B^G\chi, \mathrm{Ind}_B^G\xi)$ has dimension 1 if $\xi = \chi$ or $\chi^w\delta_B^{-1}$, zero otherwise.*

pf: $\mathrm{Hom}_G(\mathrm{Ind}_B^G\chi, \mathrm{Ind}_B^G\xi) \simeq \mathrm{Hom}_B(\mathrm{Ind}_B^G\chi, \xi)$

$\simeq \mathrm{Hom}_T((\mathrm{Ind}_B^G\chi)_N, \xi)$ $(*)$

we know that

$$0 \to \chi^w\cdot\delta_B^{-1} \to (\mathrm{Ind}_B^G\chi)_N \to \chi \to 0$$

hence $(*) \neq 0 \Leftrightarrow \xi = \chi$ or $\chi^w\cdot\delta_B^{-1}$, and the intertwing space is 1-dim'l

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Then Starting from $0 \to 1_G \to \mathrm{Ind}_B^G 1_B \to St_G \to 0$

we get $0 \to St_G^\vee \to \mathrm{Ind}_B^G\delta_B^{-1} \to 1_G \to 0$

then since we have isomorphism as G-mps:

$$\mathrm{Ind}_B^G\delta_B^{-1} \to \mathrm{Ind}_B^G 1_B$$

we have $St_G^\vee \xrightarrow{inj} \mathrm{Ind}_B^G\delta_B^{-1} \xrightarrow{\sim} \mathrm{Ind}_B^G 1_B \twoheadrightarrow St_G$

this is not zero, because otherwise $St_G^\vee \hookrightarrow 1_G \Rightarrow$ impossible, hence

$$St_G^\vee \simeq St_G$$

Now we arrive at the Classification theorem

<u>Thm</u>: The followings is a complete list of irr smooth admissible repn of $GL_2(F)$

- $\phi \cdot \det$
- $\mathrm{Ind}_B^G \chi$, with $\chi_1\chi_2^{-1} \neq 1$ & $|\cdot|^2$
- $\phi \cdot \mathrm{St}_G$
- cuspidal

<u>Rmk</u>: Sometimes we don't use the usual induced repn, we use a modified version of induced

$$\iota_B^G \chi := \mathrm{Ind}_B^G \delta_B^{-\frac{1}{2}} \cdot \chi$$

this has the advantage that

$$(\iota_B^G \chi)^\vee \simeq \mathrm{Ind}_B^G \chi^{-1} \cdot \delta_B^{-\frac{1}{2}} = \iota_B^G \chi^{-1}$$

examples: $\chi_1 = \xi_1 \cdot |\cdot|^{s_1}$, $\chi_2 = \xi_2 \cdot |\cdot|^{s_2}$

then $\iota_B^G \chi$: $f: G \to \mathbb{C}$ $f\left(\begin{pmatrix} b_1 & x \\ & b_2 \end{pmatrix} g\right) = |b_1|^{s_1+\frac{1}{2}} |b_2|^{s_2-\frac{1}{2}} \xi_1(b_1)\xi_2(b_2) f(g)$

- Whittaker functional

Let's now turn our eye to the Whittaker functional of principal series repn

We have $0 \to V \to X \to \chi \to 0$

then take the Jacquet functor J_ψ, $\psi: F \to \mathbb{C}^\times$ non-trivial,

$$0 \to V_\psi \to X_\psi \to \chi_\psi \to 0$$

$$\chi_\psi = \chi / \chi(\psi), \quad \chi(\psi) = \mathrm{span}_{\mathbb{C}} \underbrace{\chi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) v - \psi(x) v}_{(1-\psi(x))v} \Rightarrow \chi(\psi) = \chi \Rightarrow \chi_\psi = 0$$

hence $V_\psi \simeq X_\psi$

now we have $0 \to V(N) \to V \to V_N \simeq \chi^{w} \cdot \delta_B^{-1} \to 0$

we get $V(N)_\psi \simeq V_\psi$ by the same argument, i.e.

$$V(N)_\psi \simeq V_\psi \simeq X_\psi$$

Now $V(N)_N = 0$. and if $V(N)_\psi = 0$ for some non-trivial ψ, this will imply $V(N) = 0 \Rightarrow X_\psi \neq 0$

and actually, $V(N)_\psi \cong (c\text{-}\mathrm{Ind}_N^M \psi)_\psi \simeq \psi$ is one-dim $\Rightarrow$ uniqueness for principal series

· A Technical lemma

We have repeatedly used the following lemma

<u>Lemma</u>: Let (π, V) be a smooth repn of M, then

$$V_\psi = 0 \text{ for all } \psi \Rightarrow V = 0$$

or we can state it as the following:

$$V_N = 0 \ \& \ V_\psi = 0 \text{ for "some" } \psi \Rightarrow V = 0$$