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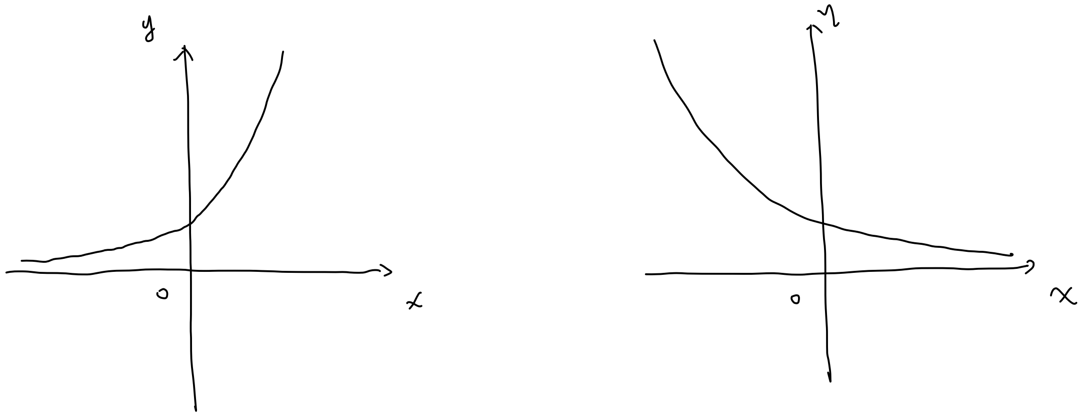
Exponential function (continued)

Recall: $f(x) = a^x$, $(a > 0)$

Graph



· Natural Exponential function

$$f(x) = e^x$$

What is e?

The number e is defined as the value that $(1 + 1/n)^n$ approaches as n becomes large. (In calculus this idea is made more precise through the concept of a limit.) The table shows the values of the expression $(1 + 1/n)^n$ for increasingly large values of n.

n	$\left(1 + \frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

It appears that, rounded to five decimal places, $e \approx 2.71828$; in fact, the approximate value to 20 decimal places is

$$e \approx 2.71828182845904523536$$

It can be shown that e is an irrational number, so we cannot write its exact value in decimal form.

Logarithmic function

Recall: inverse function

For any one-to-one function, f, we have defined f^{-1}

$x \neq y \Rightarrow f(x) \neq f(y)$

$f^{-1}(y) = x$, x is the unique number s.t. $f(x) = y$

Exponential function is one-to-one!

Def:

DEFINITION OF THE LOGARITHMIC FUNCTION

Let a be a positive number with $a \neq 1$. The **logarithmic function with base a**, denoted by $\log_a$, is defined by

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

So $\log_a x$ is the *exponent* to which the base a must be raised to give x.

Rmk: inverse function of the exponential function

$a = e$ $\quad \log_e x = \ln x$ (natural logarithmic function)

EXAMPLE 1 ■ Logarithmic and Exponential Forms

The logarithmic and exponential forms are equivalent equations: If one is true, then so is the other. So we can switch from one form to the other as in the following illustrations.

Logarithmic form	Exponential form
$\log_{10} 100{,}000 = 5$	$10^5 = 100{,}000$
$\log_2 8 = 3$	$2^3 = 8$
$\log_2\left(\frac{1}{8}\right) = -3$	$2^{-3} = \frac{1}{8}$
$\log_5 s = r$	$5^r = s$

x	$\log_{10} x$
10^4	4
10^3	3
10^2	2
10	1
1	0
10^{-1}	-1
10^{-2}	-2
10^{-3}	-3
10^{-4}	-4

It is important to understand that $\log_a x$ is an *exponent*. For example, the numbers in the right-hand column of the table in the margin are the logarithms (base 10) of the numbers in the left-hand column. This is the case for all bases, as the following example illustrates.

EXAMPLE 2 ■ Evaluating Logarithms

(a) $\log_{10} 1000 = 3$ because $10^3 = 1000$

(b) $\log_2 32 = 5$ because $2^5 = 32$

(c) $\log_{10} 0.1 = -1$ because $10^{-1} = 0.1$

(d) $\log_{16} 4 = \frac{1}{2}$ because $16^{1/2} = 4$

When we apply the Inverse Function Property described on page 222 to $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, we get

$$\log_a(a^x) = x \qquad x \in \mathbb{R}$$

$$a^{\log_a x} = x \qquad x > 0$$

We list these and other properties of logarithms discussed in this section.

PROPERTIES OF LOGARITHMS

Property	Reason
1. $\log_a 1 = 0$	We must raise a to the power 0 to get 1.
2. $\log_a a = 1$	We must raise a to the power 1 to get a.
3. $\log_a a^x = x$	We must raise a to the power x to get a^x.
4. $a^{\log_a x} = x$	$\log_a x$ is the power to which a must be raised to get x.

$\log_a(xy) = \log_a x + \log_a y$: multiplication $\Rightarrow$ addition

$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$: division $\Rightarrow$ substraction

EXAMPLE 3 ■ Applying Properties of Logarithms

We illustrate the properties of logarithms when the base is 5.

$\log_5 1 = 0$	Property 1	$\log_5 5 = 1$	Property 2
$\log_5 5^8 = 8$	Property 3	$5^{\log_5 12} = 12$	Property 4

Graphs

Recall that if a one-to-one function f has domain A and range B, then its inverse function f^{-1} has domain B and range A. Since the exponential function $f(x) = a^x$ with $a \neq 1$ has domain $\mathbb{R}$ and range $(0, \infty)$, we conclude that its inverse function, $f^{-1}(x) = \log_a x$, has domain $(0, \infty)$ and range $\mathbb{R}$.

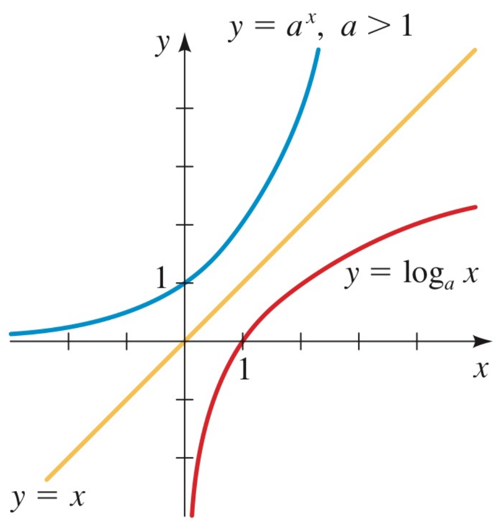
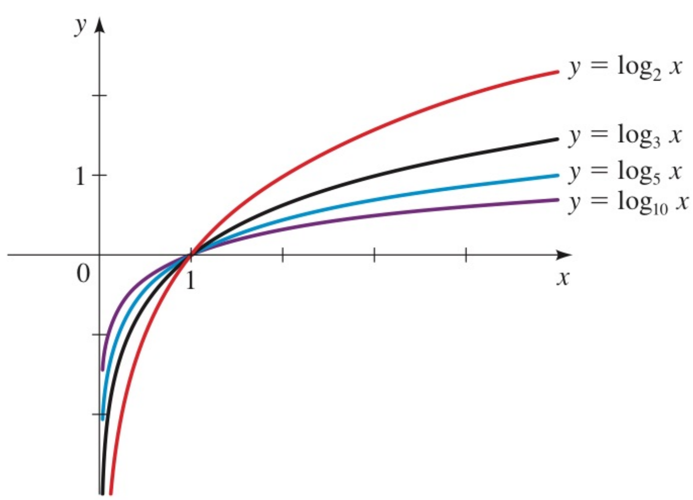


FIGURE 2 Graph of the logarithmic function $f(x) = \log_a x$

Figure 4 shows the graphs of the family of logarithmic functions with bases 2, 3, 5, and 10. These graphs are drawn by reflecting the graphs of $y = 2^x$, $y = 3^x$, $y = 5^x$, and $y = 10^x$ (see Figure 2 in Section 4.1) in the line $y = x$. We can also plot points as an aid to sketching these graphs, as illustrated in Example 4.



Some special one:

COMMON LOGARITHM

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

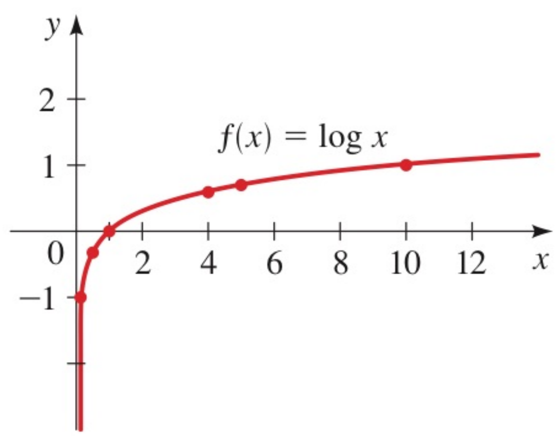
$$\log x = \log_{10} x$$

EXAMPLE 7 ■ Evaluating Common Logarithms

Use a calculator to find appropriate values of $f(x) = \log x$, and use the values to sketch the graph.

SOLUTION We make a table of values, using a calculator to evaluate the function at those values of x that are not powers of 10. We plot those points and connect them by a smooth curve as in Figure 8.

x	$\log x$
0.01	−2
0.1	−1
0.5	−0.301
1	0
4	0.602
5	0.699
10	1



NATURAL LOGARITHM

The logarithm with base e is called the **natural logarithm** and is denoted by **ln**:

$$\ln x = \log_e x$$

The natural logarithmic function $y = \ln x$ is the inverse function of the natural exponential function $y = e^x$. Both functions are graphed in Figure 9. By the definition of inverse functions we have

$$\boxed{\ln x = y \quad \Leftrightarrow \quad e^y = x}$$

If we substitute $a = e$ and write "ln" for "$\log_e$" in the properties of logarithms mentioned earlier, we obtain the following properties of natural logarithms.

Laws of logarithms

LAWS OF LOGARITHMS

Let a be a positive number, with $a \neq 1$. Let A, B, and C be any real numbers with $A > 0$ and $B > 0$.

Law	Description
1. $\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
2. $\log_a\left(\dfrac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.
3. $\log_a(A^C) = C \log_a A$	The logarithm of a power of a number is the exponent times the logarithm of the number.

EXAMPLE 1 ■ Using the Laws of Logarithms to Evaluate Expressions

Evaluate each expression.

(a) $\log_4 2 + \log_4 32$

(b) $\log_2 80 - \log_2 5$

(c) $-\frac{1}{3} \log 8$

EXAMPLE 2 ■ Expanding Logarithmic Expressions

Use the Laws of Logarithms to expand each expression.

(a) $\log_2(6x)$ **(b)** $\log_5(x^3y^6)$ **(c)** $\ln\left(\dfrac{ab}{\sqrt[3]{c}}\right)$

EXAMPLE 3 ■ Combining Logarithmic Expressions

Use the Laws of Logarithms to combine each expression into a single logarithm.

(a) $3 \log x + \frac{1}{2} \log(x + 1)$

(b) $3 \ln s + \frac{1}{2} \ln t - 4 \ln(t^2 + 1)$

WARNING:

Warning Although the Laws of Logarithms tell us how to compute the logarithm of a product or a quotient, *there is no corresponding rule for the logarithm of a sum or a difference*. For instance,

$$\log_a(x + y) \neq \log_a x + \log_a y$$

In fact, we know that the right side is equal to $\log_a(xy)$. Also, don't improperly simplify quotients or powers of logarithms. For instance,

$$\frac{\log 6}{\log 2} \neq \log\left(\frac{6}{2}\right) \qquad \text{and} \qquad (\log_2 x)^3 \neq 3 \log_2 x$$

Change of base

Change of Base Formula

For some purposes we find it useful to change from logarithms in one base to logarithms in another base. Suppose we are given $\log_a x$ and want to find $\log_b x$. Let

$$y = \log_b x$$

We write this in exponential form and take the logarithm, with base a, of each side.

$$\begin{aligned} b^y &= x && \text{Exponential form} \\ \log_a(b^y) &= \log_a x && \text{Take } \log_a \text{ of each side} \\ y \log_a b &= \log_a x && \text{Law 3} \\ y &= \frac{\log_a x}{\log_a b} && \text{Divide by } \log_a b \end{aligned}$$

This proves the following formula.

CHANGE OF BASE FORMULA

$$\log_b x = \frac{\log_a x}{\log_a b}$$

69. Change of Base Formula Use the Change of Base Formula to show that

$$\log e = \frac{1}{\ln 10}$$

70. Change of Base Formula Simplify: $(\log_2 5)(\log_5 7)$

71. A Logarithmic Identity Show that

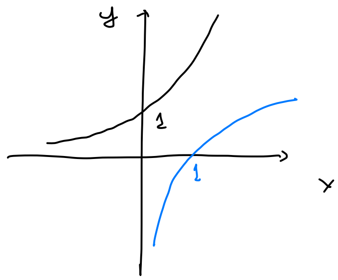
$$-\ln(x - \sqrt{x^2 - 1}) = \ln(x + \sqrt{x^2 - 1})$$

$$\log_2 5 \cdot \log_5 8 = \log_2 5 \cdot \frac{\log_2 8}{\log_2 5} = \log_2 8 = 3$$

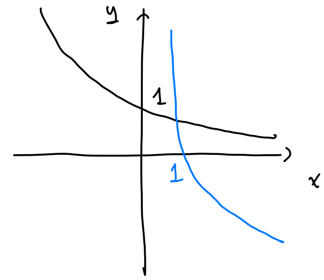
Short summary on exponential functions & logarithmic functions

Given $a > 0$

- $a > 1$



$0 < a < 1$



common exponential / logarithmic : 10^x $(a = 10)$, $\log x = \log_{10} x$

natural exponential / logarithmic : e^x, $\ln x = \log_e x$

- $a^{x+y} = a^x \cdot a^y$ $\qquad$ $\log_a (xy) = \log_a x + \log_a y$

$$\text{addition} \underset{\log}{\overset{\exp}{\rightleftarrows}} \text{multiplication}$$

- $a^0 = 1$ $\qquad$ $\log_a 1 = 0$

$$\underset{\text{in addition}}{\underset{\uparrow}{0}} \rightsquigarrow \underset{\text{in multiplication}}{\underset{\uparrow}{1}}$$

- $a^x = a^y \Leftrightarrow x = y$ $\qquad$ $\log_a x = \log_a y \Leftrightarrow x = y \; (> 0)$

$$\frac{\log_a x}{\log_b x} = \log_a b$$

Exponential & logarithmic equations

· Exponential equations

$$a^x = a^y \quad \Rightarrow \quad x = y$$

GUIDELINES FOR SOLVING EXPONENTIAL EQUATIONS

1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of each side, then use the Laws of Logarithms to "bring down the exponent."
3. Solve for the variable.

EXAMPLE 1 ■ Exponential Equations

Solve the exponential equation.

(a) $5^x = 125$ **(b)** $5^{2x} = 5^{x+1}$

SOLUTION

(a) We first express 125 as a power of 5 and then use the fact that the exponential function $f(x) = 5^x$ is one-to-one.

$5^x = 125$	Given equation
$5^x = 5^3$	Because $125 = 5^3$
$x = 3$	One-to-one property

The solution is $x = 3$.

(b) We first use the fact that the function $f(x) = 5^x$ is one-to-one.

$5^{2x} = 5^{x+1}$	Given equation
$2x = x + 1$	One-to-one property
$x = 1$	Solve for x

The solution is $x = 1$.

EXAMPLE 2 ■ Solving an Exponential Equation

Consider the exponential equation $3^{x+2} = 7$.

(a) Find the exact solution of the equation expressed in terms of logarithms.

(b) Use a calculator to find an approximation to the solution rounded to six decimal places.

EXAMPLE 3 ■ Solving an Exponential Equation

Solve the equation $8e^{2x} = 20$.

SOLUTION We first divide by 8 to isolate the exponential term on one side of the equation.

$$
\begin{aligned}
8e^{2x} &= 20 && \text{Given equation} \\
e^{2x} &= \tfrac{20}{8} && \text{Divide by 8} \\
\ln e^{2x} &= \ln 2.5 && \text{Take ln of each side} \\
2x &= \ln 2.5 && \text{Property of ln} \\
x &= \frac{\ln 2.5}{2} && \text{Divide by 2 (exact solution)} \\
&\approx 0.458 && \text{Calculator (approximate solution)}
\end{aligned}
$$

EXAMPLE 5 ■ An Exponential Equation of Quadratic Type

Solve the equation $e^{2x} - e^x - 6 = 0$.

SOLUTION To isolate the exponential term, we factor.

$$
\begin{aligned}
e^{2x} - e^x - 6 &= 0 && \text{Given equation} \\
(e^x)^2 - e^x - 6 &= 0 && \text{Law of Exponents} \\
(e^x - 3)(e^x + 2) &= 0 && \text{Factor (a quadratic in } e^x\text{)} \\
e^x - 3 = 0 \quad \text{or} \quad e^x + 2 &= 0 && \text{Zero-Product Property} \\
e^x = 3 \qquad\qquad e^x &= -2
\end{aligned}
$$

The equation $e^x = 3$ leads to $x = \ln 3$. But the equation $e^x = -2$ has no solution because $e^x > 0$ for all x. Thus $x = \ln 3 \approx 1.0986$ is the only solution. You should check that this answer satisfies the original equation.

logarithmic functions

$$\log_a x = \log_a y \quad \Rightarrow \quad x = y$$

GUIDELINES FOR SOLVING LOGARITHMIC EQUATIONS

1. Isolate the logarithmic term on one side of the equation; you might first need to combine the logarithmic terms.
2. Write the equation in exponential form (or raise the base to each side of the equation).
3. Solve for the variable.

EXAMPLE 7 ■ Solving a Logarithmic Equation

Solve the equation $\log(x^2 + 1) = \log(x - 2) + \log(x + 3)$.

EXAMPLE 8 ■ Solving Logarithmic Equations

Solve each equation for x.

(a) $\ln x = 8$

(b) $\log_2(25 - x) = 3$

EXAMPLE 9 ■ Solving a Logarithmic Equation

Solve the equation $4 + 3\log(2x) = 16$.

SOLUTION We first isolate the logarithmic term. This allows us to write the equation in exponential form.

$4 + 3\log(2x) = 16$	Given equation
$3\log(2x) = 12$	Subtract 4
$\log(2x) = 4$	Divide by 3
$2x = 10^4$	Exponential form (or raise 10 to each side)
$x = 5000$	Divide by 2

Review for Chapter 3 & 4

Chapter 3: polynomial / rational functions

Chapter 4. exponential / logarithmic functions

- Polynomial functions
  - zeroes $\rightsquigarrow$ the fundamental thm of algebra (complex / real)

    rational zeroes : $\frac{p}{q}$

    Long division

  - graphs : end behaviour (at $\pm\infty$)

    zero behaviour

    $\downarrow$

    $c_1 \quad c_2 \quad \cdots \quad c_r$

- Rational functions $\quad r(x) = \frac{P(x)}{Q(x)}$
  - Cut points
  - graphs : Asymptotes
    - vertical $\rightsquigarrow$ $Q(x) = 0$
    - horizontal $\rightsquigarrow$ $dg\, P(x) \leq dg\, Q(x)$
    - slant $\rightsquigarrow$ $dg\, P(x) = dg\, Q(x) + 1$
  - end behaviour $\quad r(x) = D(x) + \frac{R(x)}{Q(x)}$
  - cut point behaviour

Chapter 4. exponential & logarithmic functions

a^x $\log_a x$

1. Express the quadratic function $f(x) = x^2 - x - 6$ in standard form, and sketch its graph.

2. Find the maximum or minimum value of the quadratic function $g(x) = 2x^2 + 6x + 3$.

4. Graph the polynomial $P(x) = -(x + 2)^3 + 27$, showing clearly all x- and y-intercepts.

(b) Use long division to find the quotient and remainder when $2x^5 + 4x^4 - x^3 - x^2 + 7$ is divided by $2x^2 - 1$.

6. Let $P(x) = 2x^3 - 5x^2 - 4x + 3$.

(a) List all possible rational zeros of P.

(b) Find the complete factorization of P.

(c) Find the zeros of P.

(d) Sketch the graph of P.

11. Consider the following rational functions:

$$r(x) = \frac{2x - 1}{x^2 - x - 2} \qquad s(x) = \frac{x^3 + 27}{x^2 + 4} \qquad t(x) = \frac{x^3 - 9x}{x + 2} \qquad u(x) = \frac{x^2 + x - 6}{x^2 - 25} \qquad w(x) = \frac{x^3 + 6x^2 + 9x}{x + 3}$$

(a) Which of these rational functions has a horizontal asymptote?

(b) Which of these functions has a slant asymptote?

(c) Which of these functions has no vertical asymptote?

(d) Which of these functions has a "hole"?

(e) What are the asymptotes of the function $r(x)$?

(f) Graph $y = u(x)$, showing clearly any asymptotes and x- and y-intercepts the function may have.

1. Sketch the graph of each function, and state its domain, range, and asymptote. Show the x- and y-intercepts on the graph.

(a) $f(x) = 2^{-x} + 4$ **(b)** $g(x) = \log_3(x + 3)$

4. Find the exact value of the expression.

(a) $10^{\log 36}$ **(b)** $\ln e^3$ **(c)** $\log_3 \sqrt{27}$

(d) $\log_2 80 - \log_2 10$ **(e)** $\log_8 4$ **(f)** $\log_6 4 + \log_6 9$

8. Solve the logarithmic equation for x.

(a) $\log(2x) = 3$ **(b)** $\log(x + 1) + \log 2 = \log(5x)$

(c) $5 \ln(3 - x) = 4$ **(d)** $\log_2(x + 2) + \log_2(x - 1) = 2$