Graphing polynomial

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

- End behaviour

4 cases

n odd		n even	
$a_n > 0$	$a_n < 0$	$a_n > 0$	$a_n < 0$

- Find zeroes
- Analyze behavior near zeroes

Zeroes:

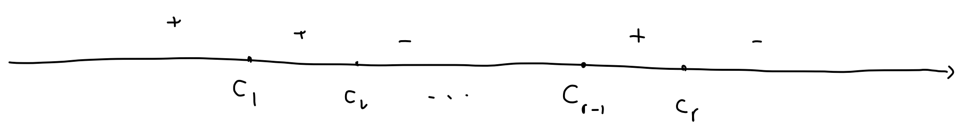
Suppose we can find all the zeroes of $P(x)$:

$$c_1 < c_2 < \dots < c_r$$

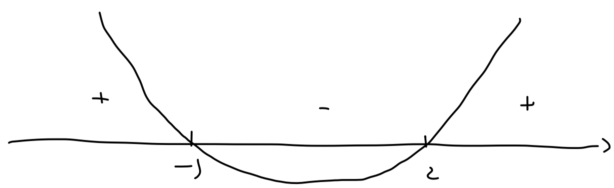
Fact 1: between 2 successive zeroes, the sign of $P(x)$ is fixed,

i.e. $P(x)$ is always positive or negative.

Pick test pts!



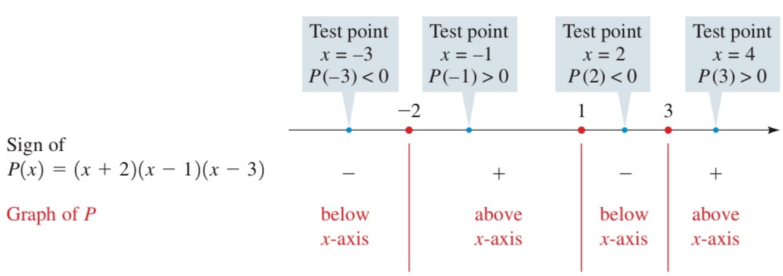
e.g. $P(x) = (x-2)(x+3)$



e.g.

Sketch the graph of the polynomial function $P(x) = (x + 2)(x - 1)(x - 3)$.

SOLUTION The zeros are $x = -2$, 1, and 3. These determine the intervals $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$, and $(3, \infty)$. Using test points in these intervals, we get the information in the following sign diagram (see Section 1.8).



Plotting a few additional points and connecting them with a smooth curve helps us to complete the graph in Figure 7.

	x	$P(x)$
Test point →	−3	−24
	−2	0
Test point →	−1	8
	0	6
	1	0
Test point →	2	−4
	3	0
Test point →	4	18

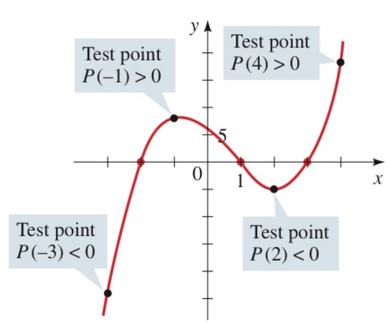


FIGURE 7 $P(x) = (x + 2)(x - 1)(x - 3)$

Question: Is it always the case that $+$ & $-$ appear alternatively?

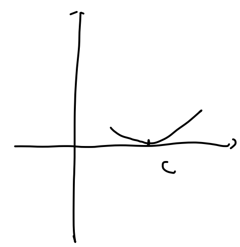
EXAMPLE 5 ■ Finding Zeros and Graphing a Polynomial Function

Let $P(x) = x^3 - 2x^2 - 3x$.

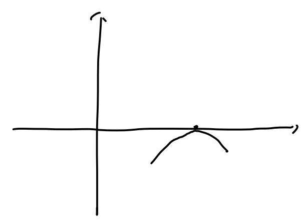
(a) Find the zeros of P. **(b)** Sketch a graph of P.

Fact 2: If for the zero $x = c$

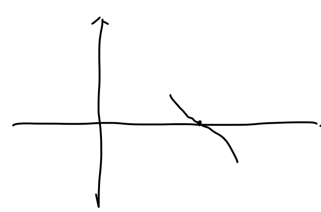
both left & right are $+$



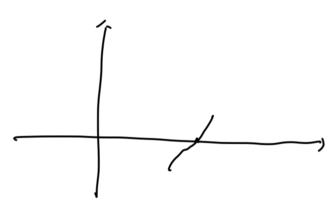
left & right are $-$



left > 0, right < 0



left < 0, right > 0



EXAMPLE 6 ■ Finding Zeros and Graphing a Polynomial Function

Let $P(x) = -2x^4 - x^3 + 3x^2$.

(a) Find the zeros of P. **(b)** Sketch a graph of P.

SOLUTION

(a) To find the zeros, we factor completely.

$$\begin{aligned} P(x) &= -2x^4 - x^3 + 3x^2 \\ &= -x^2(2x^2 + x - 3) && \text{Factor } -x^2 \\ &= -x^2(2x + 3)(x - 1) && \text{Factor quadratic} \end{aligned}$$

Thus the zeros are $x = 0$, $x = -\frac{3}{2}$, and $x = 1$.

(b) The x-intercepts are $x = 0$, $x = -\frac{3}{2}$, and $x = 1$. The y-intercept is $P(0) = 0$. W make a table of values of $P(x)$, making sure that we choose test points between (and to the right and left of) successive zeros.

Since P is of even degree and its leading coefficient is negative, it has the following end behavior.

$$y \to -\infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

We plot the points from the table and connect the points by a smooth curve to complete the graph in Figure 9.

x	$P(x)$
-2	-12
-1.5	0
-1	2
-0.5	0.75
0	0
0.5	0.5
1	0
1.5	-6.75

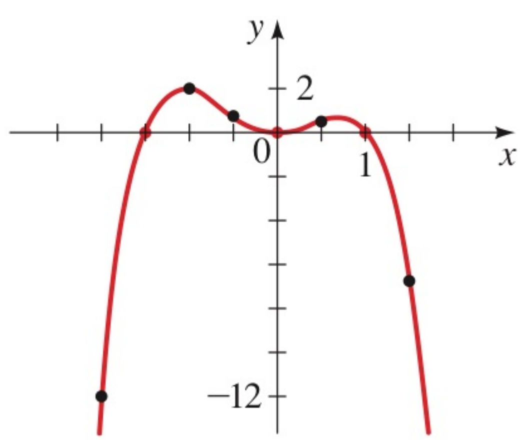


FIGURE 9 $P(x) = -2x^4 - x^3 + 3x^2$

Now Try Exercise 35

EXAMPLE 7 ■ Finding Zeros and Graphing a Polynomial Function

Let $P(x) = x^3 - 2x^2 - 4x + 8$.

(a) Find the zeros of P. **(b)** Sketch a graph of P.

SOLUTION

(a) To find the zeros, we factor completely.

$$\begin{aligned} P(x) &= x^3 - 2x^2 - 4x + 8 \\ &= x^2(x - 2) - 4(x - 2) && \text{Group and factor} \\ &= (x^2 - 4)(x - 2) && \text{Factor } x - 2 \\ &= (x + 2)(x - 2)(x - 2) && \text{Difference of squares} \\ &= (x + 2)(x - 2)^2 && \text{Simplify} \end{aligned}$$

Thus the zeros are $x = -2$ and $x = 2$.

(b) The x-intercepts are $x = -2$ and $x = 2$. The y-intercept is $P(0) = 8$. The table gives additional values of $P(x)$.

Since P is of odd degree and its leading coefficient is positive, it has the following end behavior.

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

We connect the points by a smooth curve to complete the graph in Figure 10.

x	$P(x)$
-3	-25
-2	0
-1	9
0	8
1	3
2	0
3	5

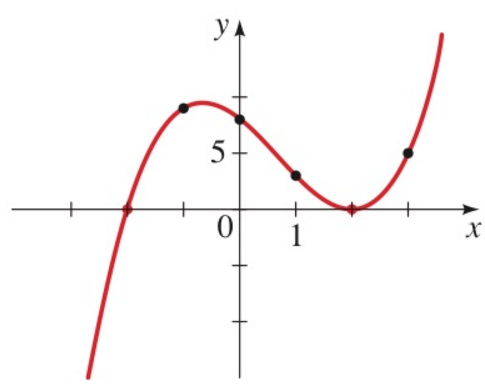


FIGURE 10
$P(x) = x^3 - 2x^2 - 4x + 8$

Real zeros of Polynomials

Natural Questions: How to find zeroes of polynomials

Today we focus on the special case of "rational polynomial"

RATIONAL ZEROS THEOREM

If the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients (where $a_n \neq 0$ and $a_0 \neq 0$), then every rational zero of P is of the form

$$\frac{p}{q}$$

where p and q are integers and

p is a factor of the constant coefficient a_0

q is a factor of the leading coefficient a_n

$$\begin{aligned} P(x) &= (x-2)(x-3)(x+4) && \text{Factored form} \\ &= x^3 - x^2 - 14x + 24 && \text{Expanded form} \end{aligned}$$

From the factored form we see that the zeros of P are 2, 3, and -4. When the polynomial is expanded, the constant 24 is obtained by multiplying $(-2) \times (-3) \times 4$. This means that the zeros of the polynomial are all factors of the constant term. The following generalizes this observation.

EXAMPLE 1 ■ Using the Rational Zeros Theorem

Find the rational zeros of $P(x) = x^3 - 3x + 2$.

SOLUTION Since the leading coefficient is 1, any rational zero must be a divisor of the constant term 2. So the possible rational zeros are ± 1 and ± 2. We test each of these possibilities.

$$\begin{aligned} P(1) &= (1)^3 - 3(1) + 2 = 0 \\ P(-1) &= (-1)^3 - 3(-1) + 2 = 4 \\ P(2) &= (2)^3 - 3(2) + 2 = 4 \\ P(-2) &= (-2)^3 - 3(-2) + 2 = 0 \end{aligned}$$

The rational zeros of P are 1 and -2.

The following box explains how we use the Rational Zeros Theorem with synthetic division to factor a polynomial.

FINDING THE RATIONAL ZEROS OF A POLYNOMIAL

1. **List Possible Zeros.** List all possible rational zeros, using the Rational Zeros Theorem.
2. **Divide.** Use synthetic division to evaluate the polynomial at each of the candidates for the rational zeros that you found in Step 1. When the remainder is 0, note the quotient you have obtained.
3. **Repeat.** Repeat Steps 1 and 2 for the quotient. Stop when you reach a quotient that is quadratic or factors easily, and use the quadratic formula or factor to find the remaining zeros.

EXAMPLE 2 ■ Finding Rational Zeros

Write the polynomial $P(x) = 2x^3 + x^2 - 13x + 6$ in factored form, and find all its zeros.

SOLUTION By the Rational Zeros Theorem the rational zeros of P are of the form

$$\text{possible rational zero of } P = \frac{\text{factor of constant term}}{\text{factor of leading coefficient}}$$

The constant term is 6 and the leading coefficient is 2, so

$$\text{possible rational zero of } P = \frac{\text{factor of 6}}{\text{factor of 2}}$$

The factors of 6 are $\pm1, \pm2, \pm3, \pm6$, and the factors of 2 are $\pm1, \pm2$. Thus the possible rational zeros of P are

$$\pm\frac{1}{1}, \quad \pm\frac{2}{1}, \quad \pm\frac{3}{1}, \quad \pm\frac{6}{1}, \quad \pm\frac{1}{2}, \quad \pm\frac{2}{2}, \quad \pm\frac{3}{2}, \quad \pm\frac{6}{2}$$

Simplifying the fractions and eliminating duplicates, we get the following list of possible rational zeros:

$$\pm1, \quad \pm2, \quad \pm3, \quad \pm6, \quad \pm\frac{1}{2}, \quad \pm\frac{3}{2}$$

Story: For quadratic one:

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

always 2 roots

counted with multiplicities

The Cubic formula

Here are the steps for finding the roots of a cubic polynomial of the form

$$x^3 + ax + b \quad \text{if} \quad -\left(\frac{a}{3}\right)^3 - \left(\frac{b}{2}\right)^2 > 0$$

Step 1. Let D be the complex number

$$D = -\frac{b}{2} + i\sqrt{-\left(\frac{a}{3}\right)^3 - \left(\frac{b}{2}\right)^2}$$

Step 2. Find a complex number $z \in \mathbb{C}$ such that $z^3 = D$.

Step 3. Let R be the real part of z, and let I be the imaginary part of z, so that R and I are real numbers with $z = R + iI$.

Step 4. The three roots of $x^3 + ax + b$ are the real numbers $2R$, $-R + \sqrt{3}I$, and $-R - \sqrt{3}I$.

observation: zeros for cubic ones can be expressed as algebraic operations of coefficients

$+, -, \times, \div$

$\sqrt[n]{\ }, (\ )^n$

For polynomial of deg = 4

zeroes can also be expressed into algebraic expressions of coefficients

Very explicit!

edit: I believe that the formula given below gives the correct solutions for x to $ax^4+bx^3+cx^2+dx+e=0$ for all complex a, b, c, d, and e, under the assumption that $w=\sqrt{z}$ is the complex number such that $w^2=z$ and $\arg(w)\in(-\frac{\pi}{2},\frac{\pi}{2}]$ and $w=\sqrt[3]{z}$ is the complex number such that $w^3=z$ and $\arg(w)\in(-\frac{\pi}{3},\frac{\pi}{3}]$ (these are typically how computer algebra systems and calculators define the principal roots). Some intermediate parameters p_k are defined to keep the formula simple and to help in keeping the choices of roots consistent.

Let:

$$p_1 = 2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace$$

$$p_2 = p_1 + \sqrt{-4(c^2-3bd+12ae)^3 + p_1^2}$$

$$p_3 = \frac{c^2-3bd+12ae}{3a\sqrt[3]{\frac{p_2}{2}}} + \frac{\sqrt[3]{\frac{p_2}{2}}}{3a}$$

$$p_4 = \sqrt{\frac{b^2}{4a^2} - \frac{2c}{3a} + p_3}$$

$$p_5 = \frac{b^2}{2a^2} - \frac{4c}{3a} - p_3$$

$$p_6 = \frac{-\frac{b^3}{a^3} + \frac{4bc}{a^2} - \frac{8d}{a}}{4p_4}$$

Then:

$$x = -\frac{b}{4a} - \frac{p_4}{2} - \frac{\sqrt{p_5-p_6}}{2}$$

$$\text{or } x = -\frac{b}{4a} - \frac{p_4}{2} + \frac{\sqrt{p_5-p_6}}{2}$$

$$\text{or } x = -\frac{b}{4a} + \frac{p_4}{2} - \frac{\sqrt{p_5+p_6}}{2}$$

$$\text{or } x = -\frac{b}{4a} + \frac{p_4}{2} + \frac{\sqrt{p_5+p_6}}{2}$$

What about higher degree?

Expectation: zeros = some algebraic expressions involving coefficients

Galois: No such formulas!

Complex zeroes & The Fundamental Thm of Algebra

Recall: complex numbers: $z = \underset{\text{real part}}{x} + i\,\underset{\text{imaginary part}}{y}$, x, y are real numbers

The Fundamental Theorem of Algebra and Complete Factorization

The following theorem is the basis for much of our work in factoring polynomials and solving polynomial equations.

FUNDAMENTAL THEOREM OF ALGEBRA

Every polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad (n \geq 1, a_n \neq 0)$$

with complex coefficients has at least one complex zero.

Because any real number is also a complex number, the theorem applies to polynomials with real coefficients as well.

The Fundamental Theorem of Algebra and the Factor Theorem together show that a polynomial can be factored completely into linear factors, as we now prove.

COMPLETE FACTORIZATION THEOREM

If $P(x)$ is a polynomial of degree $n \geq 1$, then there exist complex numbers $a, c_1, c_2, \ldots, c_n$ (with $a \neq 0$) such that

$$P(x) = a(x - c_1)(x - c_2)\cdots(x - c_n)$$

baby example: $P(x) = x^2+1$: has no solutions over real numbers

but over complex numbers, 2 solutions: $\pm i$

$P(x) = (x-i)(x+i)$

· Some polynomials are not factorizable over $\mathbb{R}$, but always factorizable over $\mathbb{C}$

Example: combining finding rational zeroes

EXAMPLE 1 ■ Factoring a Polynomial Completely

Let $P(x) = x^3 - 3x^2 + x - 3$.

(a) Find all the zeros of P.

(b) Find the complete factorization of P.

SOLUTION

(a) We first factor P as follows.

$$\begin{aligned} P(x) &= x^3 - 3x^2 + x - 3 && \text{Given} \\ &= x^2(x-3) + (x-3) && \text{Group terms} \\ &= (x-3)(x^2+1) && \text{Factor } x-3 \end{aligned}$$

We find the zeros of P by setting each factor equal to 0:

$$P(x) = \underbrace{(x-3)}_{\text{This factor is 0 when } x = 3}\underbrace{(x^2+1)}_{\text{This factor is 0 when } x = i \text{ or } -i}$$

Setting $x - 3 = 0$, we see that $x = 3$ is a zero. Setting $x^2 + 1 = 0$, we get $x^2 = -1$, so $x = \pm i$. So the zeros of P are 3, i, and $-i$.

(b) Since the zeros are 3, i, and $-i$, the complete factorization of P is

$$\begin{aligned} P(x) &= (x-3)(x-i)[x-(-i)] \\ &= (x-3)(x-i)(x+i) \end{aligned}$$

EXAMPLE 2 ■ Factoring a Polynomial Completely

Let $P(x) = x^3 - 2x + 4$.

(a) Find all the zeros of P.

(b) Find the complete factorization of P.

In the Complete Factorization Theorem the numbers $c_1, c_2, \ldots, c_n$ are the zeros of P. These zeros need not all be different. If the factor $x - c$ appears k times in the complete factorization of $P(x)$, then we say that c is a zero of **multiplicity *k*** (see page 263). For example, the polynomial

$$P(x) = (x - 1)^3(x + 2)^2(x + 3)^5$$

has the following zeros:

$$1 \text{ (multiplicity 3)} \qquad -2 \text{ (multiplicity 2)} \qquad -3 \text{ (multiplicity 5)}$$

The polynomial P has the same number of zeros as its degree: It has degree 10 and has 10 zeros, provided that we count multiplicities. This is true for all polynomials, as we prove in the following theorem.

ZEROS THEOREM

Every polynomial of degree $n \geq 1$ has exactly n zeros, provided that a zero of multiplicity k is counted k times.

eg.

EXAMPLE 3 ■ Factoring a Polynomial with Complex Zeros

Find the complete factorization and all five zeros of the polynomial

$$P(x) = 3x^5 + 24x^3 + 48x$$

SOLUTION Since $3x$ is a common factor, we have

$$\begin{aligned} P(x) &= 3x(x^4 + 8x^2 + 16) \\ &= 3x(x^2 + 4)^2 \end{aligned}$$

This factor is 0 when $x = 0$

This factor is 0 when $x = 2i$ or $x = -2i$

EXAMPLE 4 ■ Finding Polynomials with Specified Zeros

(a) Find a polynomial $P(x)$ of degree 4, with zeros i, $-i$, 2, and -2, and with $P(3) = 25$.

(b) Find a polynomial $Q(x)$ of degree 4, with zeros -2 and 0, where -2 is a zero of multiplicity 3.

Now let's come back to polynomials with real coefficients

<u>Observation</u>: For a polynomials with real coefficients

$z = a + bi$ is a 0 $\Rightarrow$ $\bar{z} = a - bi$ is also a 0

<u>Conjugation</u>: $z \mapsto \bar{z}$

CONJUGATE ZEROS THEOREM

If the polynomial P has real coefficients and if the complex number z is a zero of P, then its complex conjugate $\bar{z}$ is also a zero of P.

EXAMPLE 6 ■ A Polynomial with a Specified Complex Zero

Find a polynomial $P(x)$ of degree 3 that has integer coefficients and zeros $\frac{1}{2}$ and $3 - i$.

SOLUTION Since $3 - i$ is a zero, then so is $3 + i$ by the Conjugate Zeros Theorem. This means that $P(x)$ must have the following form.

$$\begin{aligned}
P(x) &= a\left(x - \tfrac{1}{2}\right)[x - (3 - i)][x - (3 + i)] \\
&= a\left(x - \tfrac{1}{2}\right)[(x - 3) + i][(x - 3) - i] && \text{Regroup} \\
&= a\left(x - \tfrac{1}{2}\right)[(x - 3)^2 - i^2] && \text{Difference of Squares Formula} \\
&= a\left(x - \tfrac{1}{2}\right)(x^2 - 6x + 10) && \text{Expand} \\
&= a\left(x^3 - \tfrac{13}{2}x^2 + 13x - 5\right) && \text{Expand}
\end{aligned}$$

To make all coefficients integers, we set $a = 2$ and get

$$P(x) = 2x^3 - 13x^2 + 26x - 10$$

Any other polynomial that satisfies the given requirements must be an integer multiple of this one.

Linear and Quadratic Factors

We have seen that a polynomial factors completely into linear factors if we use complex numbers. If we don't use complex numbers, then a polynomial with real coefficients can always be factored into linear and quadratic factors. We use this property in Section 10.7 when we study partial fractions. A quadratic polynomial with no real zeros is called **irreducible** over the real numbers. Such a polynomial cannot be factored without using complex numbers.

LINEAR AND QUADRATIC FACTORS THEOREM

Every polynomial with real coefficients can be factored into a product of linear and irreducible quadratic factors with real coefficients.

EXAMPLE 7 ■ Factoring a Polynomial into Linear and Quadratic Factors

Let $P(x) = x^4 + 2x^2 - 8$.

(a) Factor P into linear and irreducible quadratic factors with real coefficients.

(b) Factor P completely into linear factors with complex coefficients.

SOLUTION

(a)

$$\begin{aligned} P(x) &= x^4 + 2x^2 - 8 \\ &= (x^2 - 2)(x^2 + 4) \\ &= (x - \sqrt{2})(x + \sqrt{2})(x^2 + 4) \end{aligned}$$

The factor $x^2 + 4$ is irreducible, since it has no real zeros.

(b) To get the complete factorization, we factor the remaining quadratic factor:

$$\begin{aligned} P(x) &= (x - \sqrt{2})(x + \sqrt{2})(x^2 + 4) \\ &= (x - \sqrt{2})(x + \sqrt{2})(x - 2i)(x + 2i) \end{aligned}$$

Now Try Exercise 67 ■

Short review of Number systems:

some polynomials with coefficients in certain number systems don't have solutions in the original number system!

$$\begin{array}{ll} \mathbb{Z} & \\ \downarrow & 3x - 2 = 0 \quad \text{, solution may not in } \mathbb{Z} \\ \mathbb{Q} & \\ \downarrow & x^2 - 2 \\ \mathbb{R} & \\ \downarrow & x^2 + 1 \\ \mathbb{C} & \end{array}$$

until $\mathbb{C}$: every polynomial must have a zero in itself

we don't need to extend any more!

<u>Question</u>: Does every number in $\mathbb{R}$ or $\mathbb{C}$ is a zero for some polynomials with coefficients in $\mathbb{Q}$?

algebraic numbers

<u>No</u>! : π, e, ...

Actually, $\mathbb{C} / \mathbb{R}$ are very large

almost all number in $\mathbb{R} / \mathbb{C}$ is not a solution