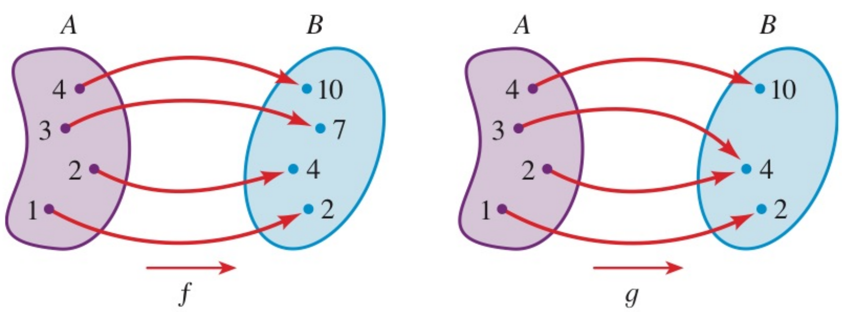
One-to-One functions and their inverses

One-to-One Functions

Let's compare the functions f and g whose arrow diagrams are shown in Figure 1. Note that f never takes on the same value twice (any two numbers in A have different images), whereas g does take on the same value twice (both 2 and 3 have the same image, 4). In symbols, $g(2) = g(3)$ but $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. Functions that have this latter property are called *one-to-one*.



f is one-to-one　　g is not one-to-one

FIGURE 1

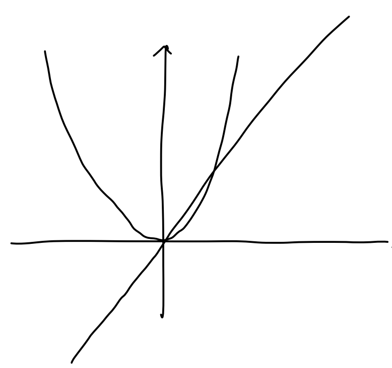
DEFINITION OF A ONE-TO-ONE FUNCTION

A function with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2$$

Ex / Counter-example:

- $f(x) = x$
- $f(x) = x^2$



horizontal line test:

HORIZONTAL LINE TEST

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Inverse function:

The Inverse of a Function

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

DEFINITION OF THE INVERSE OF A FUNCTION

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any y in B.

Example: n-th power & principal n-th root

$$f(x) = x^n$$

$$f^{-1}(y) = x, \text{ s.t. } f(x) = y, \text{ i.e. } x^n = y$$

$$= y^{\frac{1}{n}} = \sqrt[n]{y}$$

This definition says that if f takes x to y, then f^{-1} takes y back to x. (If f were not one-to-one, then f^{-1} would not be defined uniquely.) The arrow diagram in Figure 6 indicates that f^{-1} reverses the effect of f. From the definition we have

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

Note: $f^{-1}(x)$ doesn't mean $\dfrac{1}{f(x)}$

e.x.

EXAMPLE 4 ■ Finding f^{-1} for Specific Values

If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(5)$, $f^{-1}(7)$, and $f^{-1}(-10)$.

SOLUTION From the definition of f^{-1} we have

$$f^{-1}(5) = 1 \quad \text{because} \quad f(1) = 5$$
$$f^{-1}(7) = 3 \quad \text{because} \quad f(3) = 7$$
$$f^{-1}(-10) = 8 \quad \text{because} \quad f(8) = -10$$

Figure 7 shows how f^{-1} reverses the effect of f in this case.

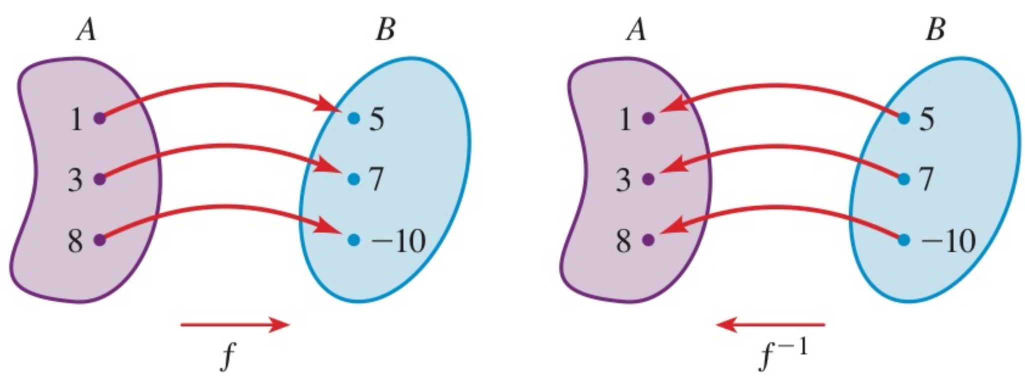


FIGURE 7

Basic property:

By definition the inverse function f^{-1} undoes what f does: If we start with x, apply f, and then apply f^{-1}, we arrive back at x, where we started. Similarly, f undoes what f^{-1} does. In general, any function that reverses the effect of f in this way must be the inverse of f. These observations are expressed precisely as follows.

INVERSE FUNCTION PROPERTY

Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies the following cancellation properties:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$
$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Conversely, any function f^{-1} satisfying these equations is the inverse of f.

These properties indicate that f is the inverse function of f^{-1}, so we say that f and f^{-1} are *inverses of each other*.

How to find the inverse (if exists)?

Finding the Inverse of a Function

Now let's examine how we compute inverse functions. We first observe from the definition of f^{-1} that

$$y = f(x) \quad \Leftrightarrow \quad f^{-1}(y) = x$$

So if $y = f(x)$ and if we are able to solve this equation for x in terms of y, then we must have $x = f^{-1}(y)$. If we then interchange x and y, we have $y = f^{-1}(x)$, which is the desired equation.

HOW TO FIND THE INVERSE OF A ONE-TO-ONE FUNCTION

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

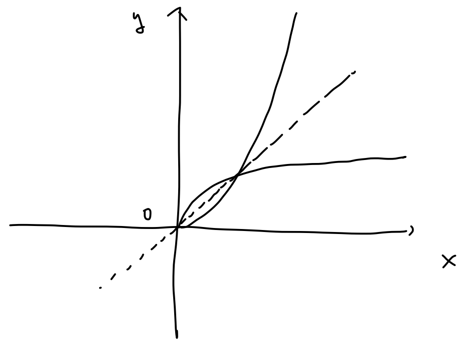
EXAMPLE 7 ■ Finding the Inverse of a Function

Find the inverse of the function $f(x) = 3x - 2$.

e.g. x^n & $\sqrt[n]{x}$

Graph of the inverse function

e.g. $f(x) = x^2$ & $f^{-1}(x) = \sqrt{x}$, $x \geq 0$



The principle of interchanging x and y to find the inverse function also gives us a method for obtaining the graph of f^{-1} from the graph of f. If $f(a) = b$, then $f^{-1}(b) = a$. Thus the point (a, b) is on the graph of f if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from the point (a, b) by reflecting in the line $y = x$ (see Figure 9). Therefore, as Figure 10 illustrates, the following is true.

> The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.

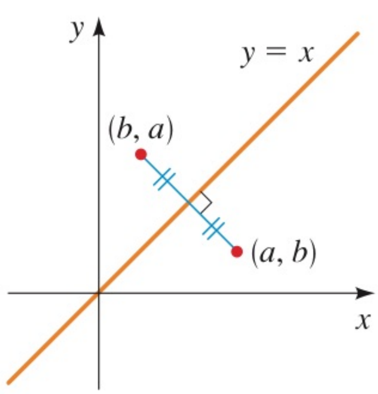


FIGURE 9

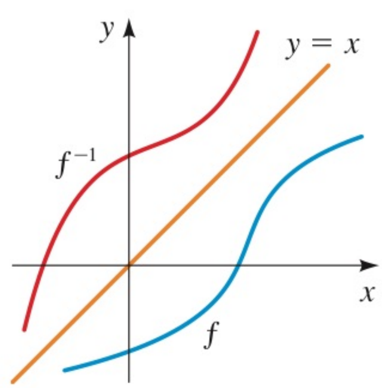


FIGURE 10

EXAMPLE 10 ■ Graphing the Inverse of a Function

(a) Sketch the graph of $f(x) = \sqrt{x - 2}$.

(b) Use the graph of f to sketch the graph of f^{-1}.

(c) Find an equation for f^{-1}.

SOLUTION

(a) Using the transformations from Section 2.6, we sketch the graph of $y = \sqrt{x - 2}$ by plotting the graph of the function $y = \sqrt{x}$ (Example 1(c) in Section 2.2) and shifting it to the right 2 units.

(b) The graph of f^{-1} is obtained from the graph of f in part (a) by reflecting it in the line $y = x$, as shown in Figure 11.

(c) Solve $y = \sqrt{x - 2}$ for x, noting that $y \geq 0$.

$$\sqrt{x - 2} = y$$

$$x - 2 = y^2 \qquad \text{Square each side}$$

$$x = y^2 + 2 \qquad y \geq 0 \qquad \text{Add 2}$$

Interchange x and y, as follows:

$$y = x^2 + 2 \qquad x \geq 0$$

Thus $$f^{-1}(x) = x^2 + 2 \qquad x \geq 0$$

This expression shows that the graph of f^{-1} is the right half of the parabola $y = x^2 + 2$, and from the graph shown in Figure 11 this seems reasonable.

Polynomial functions

A polynomial function is a function that is defined by a polynomial expression. So a **polynomial function of degree *n*** is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad a_n \neq 0$$

We have already studied polynomial functions of degree 0 and 1. These are functions of the form $P(x) = a_0$ and $P(x) = a_1 x + a_0$, respectively, whose graphs are lines. In this

e.g. Quadratic functions:

> **QUADRATIC FUNCTIONS**
>
> A **quadratic function** is a polynomial function of degree 2. So a quadratic function is a function of the form
>
> $$f(x) = ax^2 + bx + c \qquad a \neq 0$$

We see in this section how quadratic functions model many real-world phenomena. We begin by analyzing the graphs of quadratic functions.

for deg $n = 0, 1$, very-well understood

Let's first consider $n=2$. we want to know:

① graph of f

② maximum/minimum

Recall: How to graph a quadratic function?

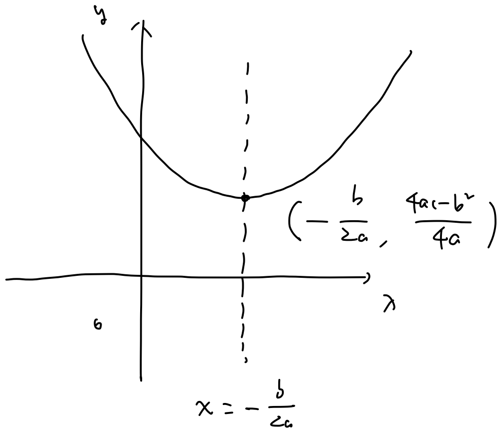
$$f(x) = ax^2 + bx + c$$

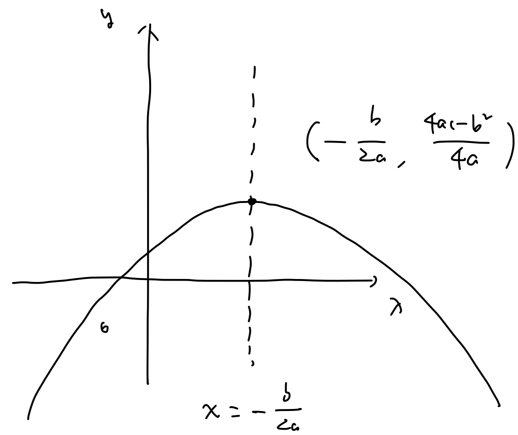
$$= a(x-h)^2 + k. \quad (\text{standard form})$$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

$$h = -\frac{b}{2a}, \quad k = \frac{4ac-b^2}{4a}$$

then





$a > 0$

$a < 0$

f has a minimum

$$f\left(-\frac{b}{2a}\right) = \frac{4ac-b^2}{4a}$$

f has a maximum

$$f\left(-\frac{b}{2a}\right) = \frac{4ac-b^2}{4a}$$

e.g.

EXAMPLE 2 ■ **Minimum Value of a Quadratic Function**

Consider the quadratic function $f(x) = 5x^2 - 30x + 49$.

(a) Express f in standard form.

(b) Sketch a graph of f.

(c) Find the minimum value of f.

SOLUTION

(a) To express this quadratic function in standard form, we complete the square.

$$\begin{aligned} f(x) &= 5x^2 - 30x + 49 \\ &= 5(x^2 - 6x) + 49 && \text{Factor 5 from the } x\text{-terms} \\ &= 5(x^2 - 6x + 9) + 49 - 5\cdot 9 && \text{Complete the square: Add 9 inside parentheses, subtract } 5\cdot 9 \text{ outside} \\ &= 5(x-3)^2 + 4 && \text{Factor and simplify} \end{aligned}$$

(b) The graph is a parabola that has its vertex at $(3, 4)$ and opens upward, as sketched in Figure 2.

(c) Since the coefficient of x^2 is positive, f has a minimum value. The minimum value is $f(3) = 4$.

EXAMPLE 3 ■ **Maximum Value of a Quadratic Function**

Consider the quadratic function $f(x) = -x^2 + x + 2$.

(a) Express f in standard form.

(b) Sketch a graph of f.

(c) Find the maximum value of f.

EXAMPLE 4 ■ Finding Maximum and Minimum Values of Quadratic Functions

Find the maximum or minimum value of each quadratic function.

(a) $f(x) = x^2 + 4x$

(b) $g(x) = -2x^2 + 4x - 5$

SOLUTION

(a) This is a quadratic function with $a = 1$ and $b = 4$. Thus the maximum or minimum value occurs at

$$x = -\frac{b}{2a} = -\frac{4}{2 \cdot 1} = -2$$

Since $a > 0$, the function has the *minimum* value

$$f(-2) = (-2)^2 + 4(-2) = -4$$

(b) This is a quadratic function with $a = -2$ and $b = 4$. Thus the maximum or minimum value occurs at

$$x = -\frac{b}{2a} = -\frac{4}{2 \cdot (-2)} = 1$$

Since $a < 0$, the function has the *maximum* value

$$f(1) = -2(1)^2 + 4(1) - 5 = -3$$

Let's now consider general degree n:

POLYNOMIAL FUNCTIONS

A **polynomial function of degree n** is a function of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a nonnegative integer and $a_n \neq 0$.

The numbers $a_0, a_1, a_2, \ldots, a_n$ are called the **coefficients** of the polynomial.

The number a_0 is the **constant coefficient** or **constant term**.

The number a_n, the coefficient of the highest power, is the **leading coefficient**, and the term $a_n x^n$ is the **leading term**.

- For a $P(x)$, we ask the following questions:
  1. Graph of $P(x)$
  2. (local) Maximum / minimum
- For general n, very difficult, we need some techniques.

Long division of polynomials

DIVISION ALGORITHM

If $P(x)$ and $D(x)$ are polynomials, with $D(x) \neq 0$, then there exist unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or of degree less than the degree of $D(x)$, such that

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)} \qquad \text{or} \qquad \underbrace{P(x)}_{\text{Dividend}} = \underbrace{D(x)}_{\text{Divisor}} \cdot \underbrace{Q(x)}_{\text{Quotient}} + \underbrace{R(x)}_{\text{Remainder}}$$

The polynomials $P(x)$ and $D(x)$ are called the **dividend** and **divisor**, respectively, $Q(x)$ is the **quotient**, and $R(x)$ is the **remainder**.

EXAMPLE 1 ■ Long Division of Polynomials

Divide $6x^2 - 26x + 12$ by $x - 4$. Express the result in each of the two forms shown in the above box.

$$
\begin{array}{r}
6x - 2 \\
x - 4 \;\overline{\big)\, 6x^2 - 26x + 12} \\
\underline{6x^2 - 24x \phantom{{}+12}} \\
-2x + 12 \\
\underline{-2x + 8} \\
4
\end{array}
$$

$$6x^2 - 26x + 12 = (6x-2)(x-4) + 4$$

Let $P(x) = 8x^4 + 6x^2 - 3x + 1$ and $D(x) = 2x^2 - x + 2$. Find polynomials $Q(x)$ and $R(x)$ such that $P(x) = D(x) \cdot Q(x) + R(x)$.

$$
\begin{array}{r}
4x^2 + 2x \\
2x^2 - x + 2 \;\overline{\big)\, 8x^4 + 6x^2 - 3x + 1} \\
\underline{8x^4 - 4x^3 + 8x^2 \phantom{{}-3x+1}} \\
4x^3 - 2x^2 - 3x + 1 \\
\underline{4x^3 - 2x^2 + 4x \phantom{{}+1}} \\
-7x + 1
\end{array}
$$

$$8x^4 + 6x^2 - 3x + 1 = (2x^2 - x + 2)(4x^2 + 2x) + (-7x + 1)$$

REMAINDER THEOREM

If the polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

If $P(x) = (x-c)\,Q(x)$, the $P(c) = 0$ $\Rightarrow$ Does the converse hold?

Ans: Yes!

FACTOR THEOREM

c is a zero of P if and only if $x - c$ is a factor of $P(x)$.

Application: factorize polynomial

EXAMPLE 5 ■ Factoring a Polynomial Using the Factor Theorem

Let $P(x) = x^3 - 7x + 6$. Show that $P(1) = 0$, and use this fact to factor $P(x)$ completely.

SOLUTION Substituting, we see that $P(1) = 1^3 - 7 \cdot 1 + 6 = 0$. By the Factor Theorem this means that $x - 1$ is a factor of $P(x)$. Using synthetic or long division (shown in the margin), we see that

$$\begin{aligned} P(x) &= x^3 - 7x + 6 && \text{Given polynomial} \\ &= (x-1)(x^2 + x - 6) && \text{See margin} \\ &= (x-1)(x-2)(x+3) && \text{Factor quadratic } x^2 + x - 6 \end{aligned}$$

Now Try Exercises 53 and 57 ■

EXAMPLE 6 ■ Finding a Polynomial with Specified Zeros

Find a polynomial of degree four that has zeros -3, 0, 1, and 5, and the coefficient of x^3 is -6.

Graphing polynomials:

Some ingredients

- End behavior
- Find zeroes,
- Analyze behaviours between & near zeroes

End behaviour:

The **end behavior** of a polynomial is a description of what happens as x becomes large in the positive or negative direction. To describe end behavior, we use the following **arrow notation**.

Symbol	Meaning
$x \to \infty$	x goes to infinity; that is, x increases without bound
$x \to -\infty$	x goes to negative infinity; that is, x decreases without bound

For example, the monomial $y = x^2$ in Figure 1(b) has the following end behavior.

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to \infty \quad \text{as} \quad x \to -\infty$$

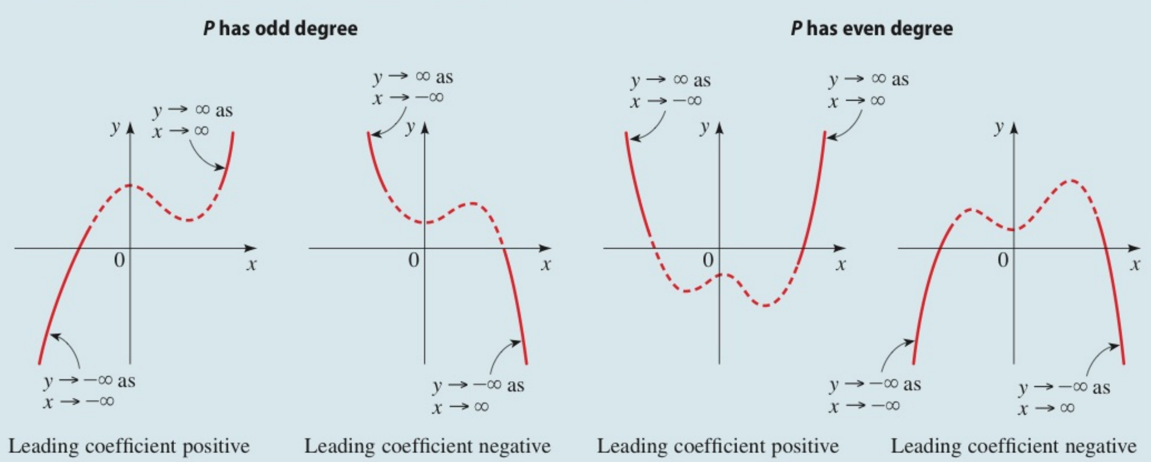
The monomial $y = x^3$ in Figure 1(c) has the following end behavior.

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

For any polynomial *the end behavior is determined by the term that contains the highest power of* x, because when x is large, the other terms are relatively insignificant in size. The following box shows the four possible types of end behavior, based on the highest power and the sign of its coefficient.

END BEHAVIOR OF POLYNOMIALS

The end behavior of the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is determined by the degree n and the sign of the leading coefficient a_n, as indicated in the following graphs.



EXAMPLE 2 ■ End Behavior of a Polynomial

Determine the end behavior of the polynomial

$$P(x) = -2x^4 + 5x^3 + 4x - 7$$

SOLUTION The polynomial P has degree 4 and leading coefficient -2. Thus P has *even* degree and *negative* leading coefficient, so it has the following end behavior.

$$y \to -\infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

The graph in Figure 4 illustrates the end behavior of P.

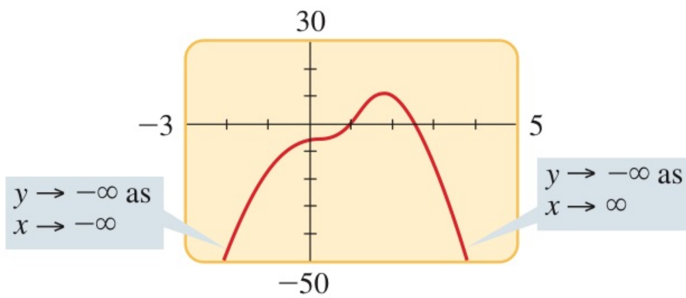


FIGURE 4 $P(x) = -2x^4 + 5x^3 + 4x - 7$