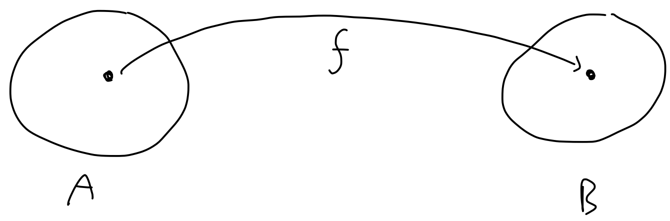
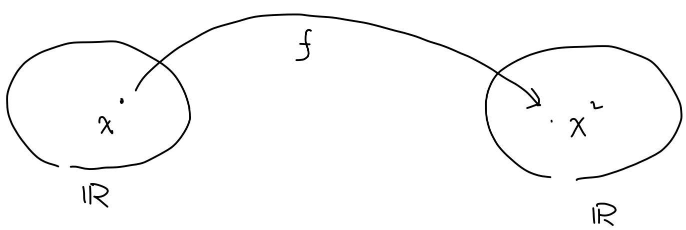
§ 2.1. Functions

DEFINITION OF A FUNCTION

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.



e.g. f : square the number, $A = B = \mathbb{R}$

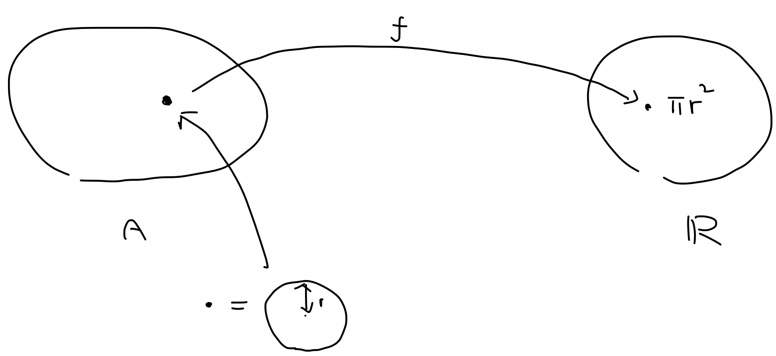


i.e. $1 \longmapsto 1^2 = 1$

$2 \longmapsto 2^2 = 4$

e.g. A : set of circles , $B = \mathbb{R}$

f : the area of the circle



Machine / Blackbox

It is helpful to think of a function as a **machine** (see Figure 3). If x is in the domain of the function f, then when x enters the machine, it is accepted as an **input** and the machine produces an **output** $f(x)$ according to the rule of the function. Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

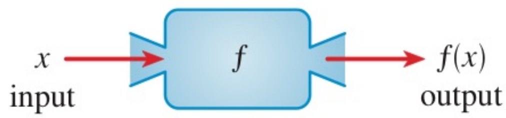


FIGURE 3 Machine diagram of f

Graphically:

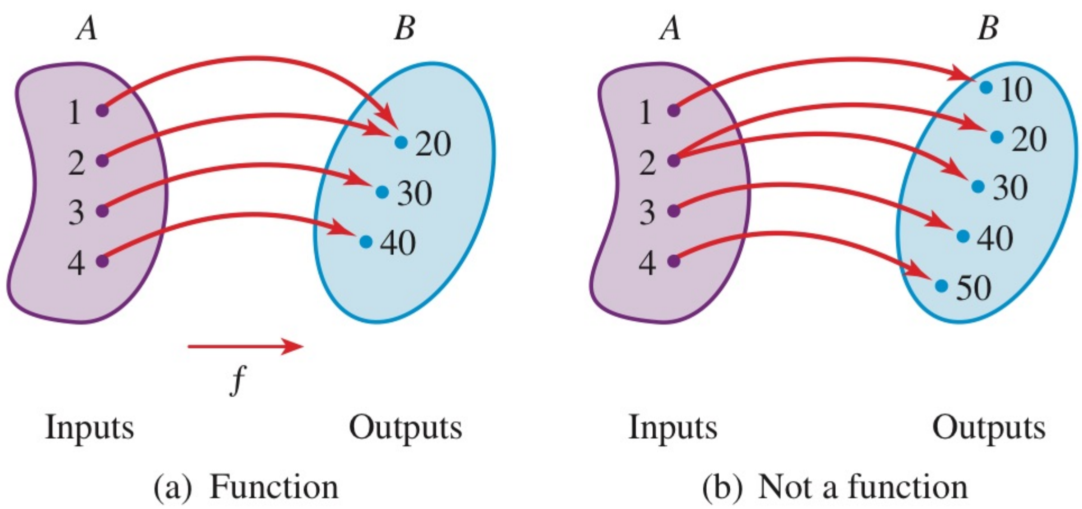


FIGURE 4 Arrow diagrams

Some basic notations / definitions

We usually consider functions for which the sets A and B are sets of real numbers. The symbol $f(x)$ is read "f of x" or "f at x" and is called the **value of f at x**, or the **image of x under f**. The set A is called the **domain** of the function. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain, that is,

$$\text{range of } f = \{f(x) \mid x \in A\}$$

e.g.

f : square the number and plus 4

A function f is defined by the formula

$$f(x) = x^2 + 4$$

(a) Express in words how f acts on the input x to produce the output $f(x)$.

(b) Evaluate $f(3)$, $f(-2)$, and $f(\sqrt{5})$.

(c) Find the domain and range of f.

(d) Draw a machine diagram for f.

SOLUTION

(a) The formula tells us that f first squares the input x and then adds 4 to the result. So f is the function

"square, then add 4"

(b) The values of f are found by substituting for x in the formula $f(x) = x^2 + 4$.

$f(3) = 3^2 + 4 = 13$ Replace x by 3

$f(-2) = (-2)^2 + 4 = 8$ Replace x by -2

$f(\sqrt{5}) = (\sqrt{5})^2 + 4 = 9$ Replace x by $\sqrt{5}$

(c) The domain of f consists of all possible inputs for f. Since we can evaluate the formula $f(x) = x^2 + 4$ for every real number x, the domain of f is the set $\mathbb{R}$ of all real numbers.

The range of f consists of all possible outputs of f. Because $x^2 \geq 0$ for all real numbers x, we have $x^2 + 4 \geq 4$, so for every output of f we have $f(x) \geq 4$. Thus the range of f is $\{y \mid y \geq 4\} = [4, \infty)$.

(d) A machine diagram for f is shown in Figure 5.

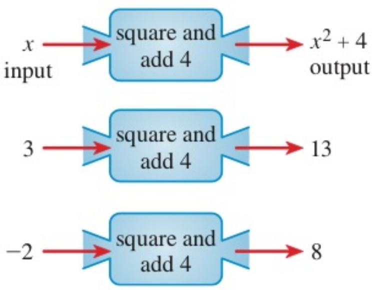


FIGURE 5 Machine diagram

EXAMPLE 2 ■ Evaluating a Function

Let $f(x) = 3x^2 + x - 5$. Evaluate each function value.

(a) $f(-2)$ **(b)** $f(0)$ **(c)** $f(4)$ **(d)** $f(\frac{1}{2})$

SOLUTION To evaluate f at a number, we substitute the number for x in the definition of f.

(a) $f(-2) = 3 \cdot (-2)^2 + (-2) - 5 = 5$

(b) $f(0) = 3 \cdot 0^2 + 0 - 5 = -5$

(c) $f(4) = 3 \cdot (4)^2 + 4 - 5 = 47$

(d) $f(\frac{1}{2}) = 3 \cdot (\frac{1}{2})^2 + \frac{1}{2} - 5 = -\frac{15}{4}$

EXAMPLE 3 ■ A Piecewise Defined Function

A cell phone plan costs \$39 a month. The plan includes 2 gigabytes (GB) of free data and charges \$15 per gigabyte for any additional data used. The monthly charges are a function of the number of gigabytes of data used, given by

$$C(x) = \begin{cases} 39 & \text{if } 0 \le x \le 2 \\ 39 + 15(x - 2) & \text{if } x > 2 \end{cases}$$

Find $C(0.5)$, $C(2)$, and $C(4)$.

SOLUTION Remember that a function is a rule. Here is how we apply the rule for this function. First we look at the value of the input, x. If $0 \le x \le 2$, then the value of $C(x)$ is 39. On the other hand, if $x > 2$, then the value of $C(x)$ is $39 + 15(x - 2)$.

Since $0.5 \le 2$, we have $C(0.5) = 39$.

Since $2 \le 2$, we have $C(2) = 39$.

Since $4 > 2$, we have $C(4) = 39 + 15(4 - 2) = 69$.

Thus the plan charges \$39 for 0.5 GB, \$39 for 2 GB, and \$69 for 4 GB.

Domain of a function (Recall: domain of a rational expression)

Recall that the *domain* of a function is the set of all inputs for the function. The domain of a function may be stated explicitly. For example, if we write

$$f(x) = x^2 \qquad 0 \le x \le 5$$

then the domain is the set of all real numbers x for which $0 \le x \le 5$. If the function is given by an algebraic expression and the domain is not stated explicitly, then by convention *the domain of the function is the domain of the algebraic expression—that is, the set of all real numbers for which the expression is defined as a real number*. For example, consider the functions

$$f(x) = \frac{1}{x-4} \qquad g(x) = \sqrt{x}$$

The function f is not defined at $x = 4$, so its domain is $\{x \mid x \neq 4\}$. The function g is not defined for negative x, so its domain is $\{x \mid x \geq 0\}$.

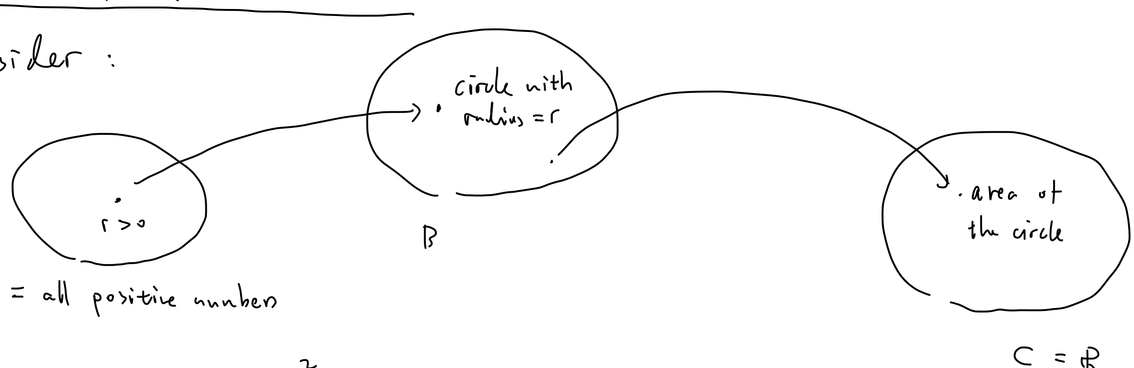
e.g.

Find the domain of each function.

(a) $f(x) = \dfrac{1}{x^2 - x}$ **(b)** $g(x) = \sqrt{9 - x^2}$ **(c)** $h(t) = \dfrac{t}{\sqrt{t+1}}$

How to understand a function:

Consider:

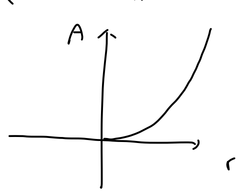


total: $r \rightsquigarrow \pi r^2$

verbally: • take the square of the radius, then times π

algebraically • $A(r) = \pi r^2$

graphically:



§. Graphs of functions

To graph a function f, we plot the points $(x, f(x))$ in a coordinate plane. In other words, we plot the points (x, y) whose x-coordinate is an input and whose y-coordinate is the corresponding output of the function.

THE GRAPH OF A FUNCTION

If f is a function with domain A, then the **graph** of f is the set of ordered pairs

$$\{(x, f(x)) \mid x \in A\}$$

plotted in a coordinate plane. In other words, the graph of f is the set of all points (x, y) such that $y = f(x)$; that is, the graph of f is the graph of the equation $y = f(x)$.

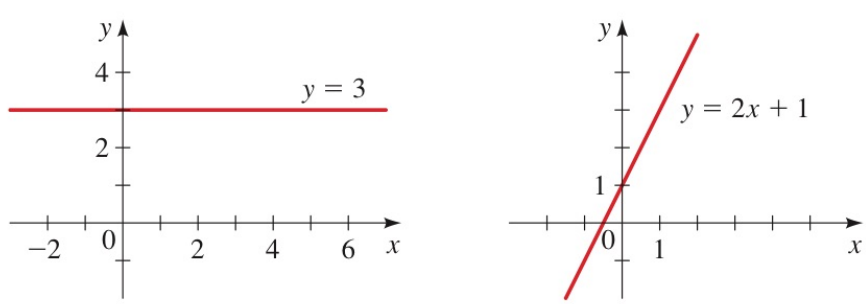
- $A(r) = \pi r^2$
- $f(x) = x^2$, $f(x) = x+1$

quadratic   lin / linear

e.g.

- linear: $f(x) = mx+b$
- constant $f(x) = b$
- power: $f(x) = x^n$
- root: $f(x) = x^{\frac{1}{n}}$

A function f of the form $f(x) = mx + b$ is called a **linear function** because its graph is the graph of the equation $y = mx + b$, which represents a line with slope m and y-intercept b. A special case of a linear function occurs when the slope is $m = 0$. The function $f(x) = b$, where b is a given number, is called a **constant function** because all its values are the same number, namely, b. Its graph is the horizontal line $y = b$. Figure 2 shows the graphs of the constant function $f(x) = 3$ and the linear function $f(x) = 2x + 1$.



The constant function $f(x) = 3$ The linear function $f(x) = 2x + 1$

Functions of the form $f(x) = x^n$ are called **power functions**, and functions of the form $f(x) = x^{1/n}$ are called **root functions**. In the next example we graph two power functions and a root function.

Examples: power functions / root functions

EXAMPLE 1 ■ **Graphing Functions by Plotting Points**

Sketch graphs of the following functions.

(a) $f(x) = x^2$ **(b)** $g(x) = x^3$ **(c)** $h(x) = \sqrt{x}$

SOLUTION We first make a table of values. Then we plot the points given by the table and join them by a smooth curve to obtain the graph. The graphs are sketched in Figure 3.

x	$f(x) = x^2$
0	0
$\pm\frac{1}{2}$	$\frac{1}{4}$
± 1	1
± 2	4
± 3	9

x	$g(x) = x^3$
0	0
$\frac{1}{2}$	$\frac{1}{8}$
1	1
2	8
$-\frac{1}{2}$	$-\frac{1}{8}$
-1	-1
-2	-8

x	$h(x) = \sqrt{x}$
0	0
1	1
2	$\sqrt{2}$
3	$\sqrt{3}$
4	2
5	$\sqrt{5}$

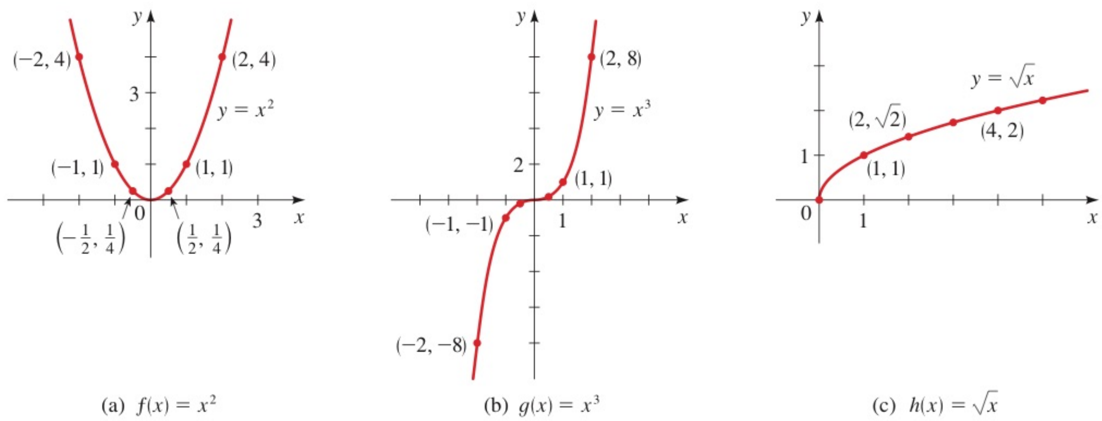
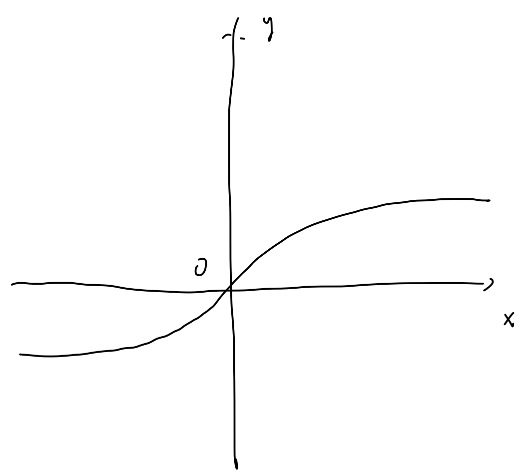


FIGURE 3

$f(x) = x^{\frac{1}{3}}$



Piecewise functions

EXAMPLE 4 ■ Graph of a Piecewise Defined Function

Sketch the graph of the function

$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$$

SOLUTION If $x \le 1$, then $f(x) = x^2$, so the part of the graph to the left of $x = 1$ coincides with the graph of $y = x^2$, which we sketched in Figure 3. If $x > 1$, then $f(x) = 2x + 1$, so the part of the graph to the right of $x = 1$ coincides with the line $y = 2x + 1$, which we graphed in Figure 2. This enables us to sketch the graph in Figure 6.

The solid dot at $(1, 1)$ indicates that this point is included in the graph; the open dot at $(1, 3)$ indicates that this point is excluded from the graph.

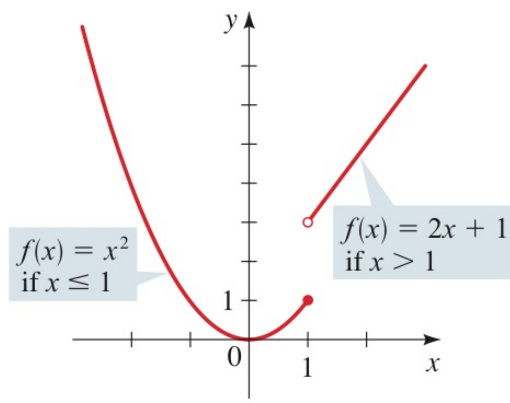


FIGURE 6
$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 2x + 1 & \text{if } x > 1 \end{cases}$$

Now Try Exercise 35

EXAMPLE 5 ■ Graph of the Absolute Value Function

Sketch a graph of the absolute value function $f(x) = |x|$.

SOLUTION Recall that

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

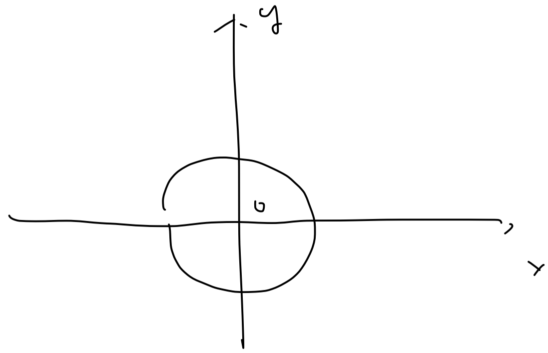
Using the same method as in Example 4, we note that the graph of f coincides with the line $y = x$ to the right of the y-axis and coincides with the line $y = -x$ to the left of the y-axis (see Figure 7).

Up to now: we go from a function to a graph on the coordinate plane

functions $\longrightarrow$ graph

Q: Can we do the inverse?

Example: graph of a circle:



Can we find a function whose graph is the circle?

Ans: No, because for $x=0$, we have two values 1 & -1

General: Vertical line test

For a graph to be a graph of some functions, it suffices to

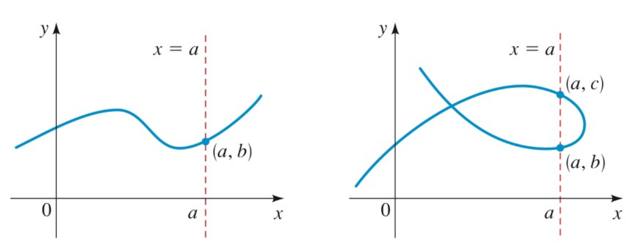
The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions? This is answered by the following test.

THE VERTICAL LINE TEST

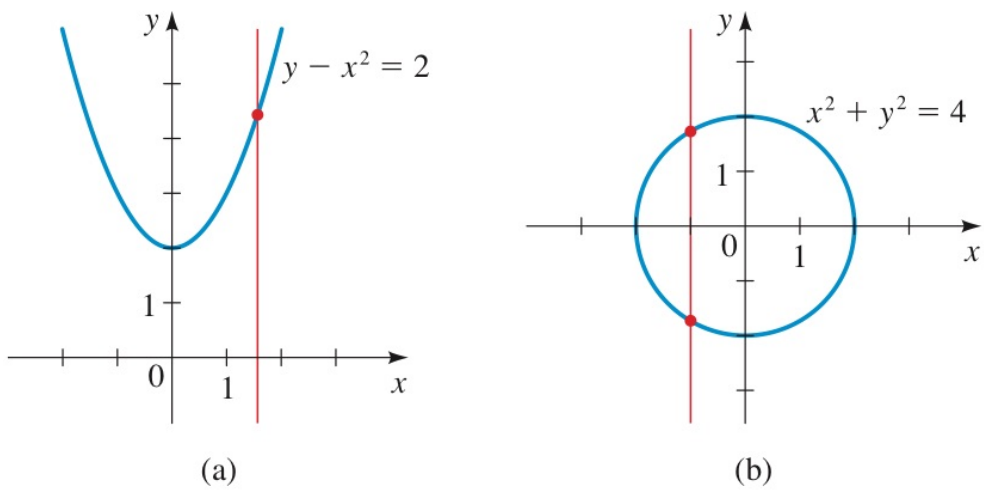
A curve in the coordinate plane is the graph of a function if and only if no vertical line intersects the curve more than once.

e.g.

We can see from Figure 10 why the Vertical Line Test is true. If each vertical line $x = a$ intersects a curve only once at (a, b), then exactly one functional value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and at (a, c), then the curve cannot represent a function because a function cannot assign two different values to a.



The graphs of the equations in Example 9 are shown in Figure 12. The Vertical Line Test shows graphically that the equation in Example 9(a) defines a function but the equation in Example 9(b) does not.

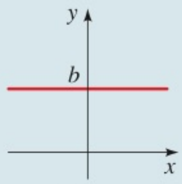


The following box shows the graphs of some functions that you will see frequently in this book.

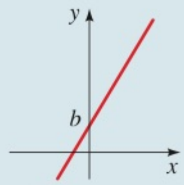
SOME FUNCTIONS AND THEIR GRAPHS

Linear functions

$f(x) = mx + b$



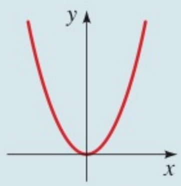
$f(x) = b$



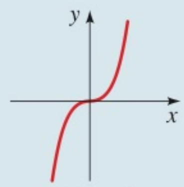
$f(x) = mx + b$

Power functions

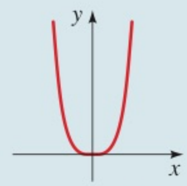
$f(x) = x^n$



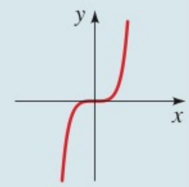
$f(x) = x^2$



$f(x) = x^3$



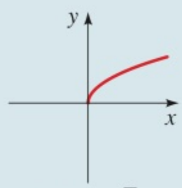
$f(x) = x^4$



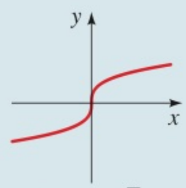
$f(x) = x^5$

Root functions

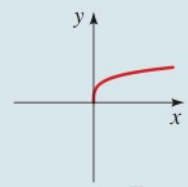
$f(x) = \sqrt[n]{x}$



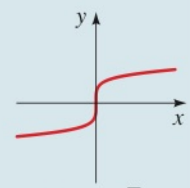
$f(x) = \sqrt{x}$



$f(x) = \sqrt[3]{x}$



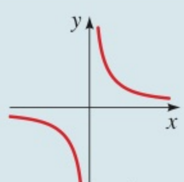
$f(x) = \sqrt[4]{x}$



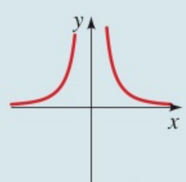
$f(x) = \sqrt[5]{x}$

Reciprocal functions

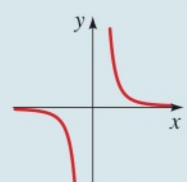
$f(x) = \dfrac{1}{x^n}$



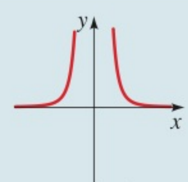
$f(x) = \dfrac{1}{x}$



$f(x) = \dfrac{1}{x^2}$



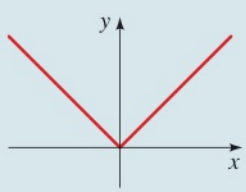
$f(x) = \dfrac{1}{x^3}$



$f(x) = \dfrac{1}{x^4}$

Absolute value function

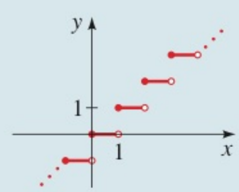
$f(x) = |x|$



$f(x) = |x|$

Greatest integer function

$f(x) = [\![x]\!]$



$f(x) = [\![x]\!]$

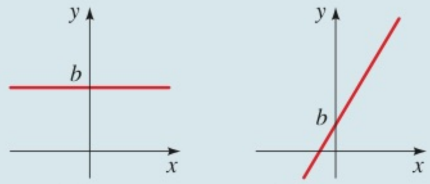
Graph of functions : Meaning: at x, value is $f(x)$

The following box shows the graphs of some functions that you will see frequently in this book.

SOME FUNCTIONS AND THEIR GRAPHS

Linear functions

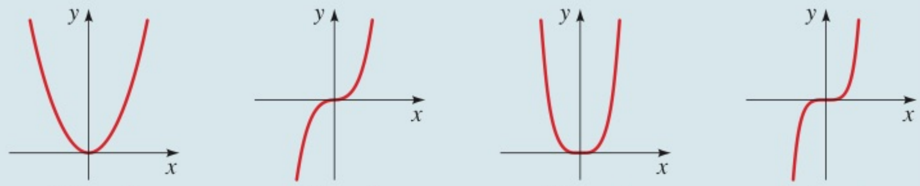
$f(x) = mx + b$



$f(x) = b$ $\quad$ $f(x) = mx + b$

Power functions

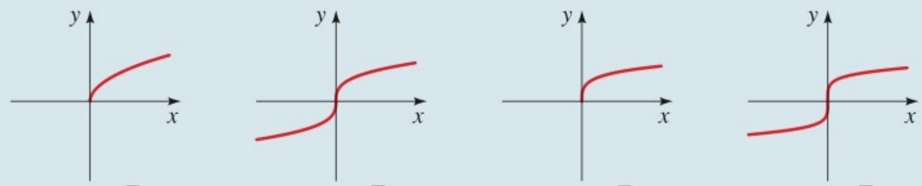
$f(x) = x^n$



$f(x) = x^2$ $\quad$ $f(x) = x^3$ $\quad$ $f(x) = x^4$ $\quad$ $f(x) = x^5$

Root functions

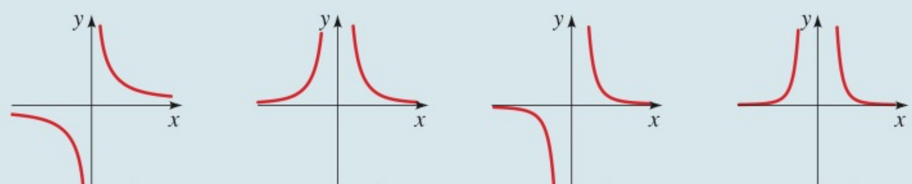
$f(x) = \sqrt[n]{x}$



$f(x) = \sqrt{x}$ $\quad$ $f(x) = \sqrt[3]{x}$ $\quad$ $f(x) = \sqrt[4]{x}$ $\quad$ $f(x) = \sqrt[5]{x}$

Reciprocal functions

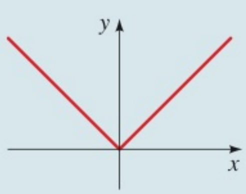
$f(x) = \dfrac{1}{x^n}$



$f(x) = \dfrac{1}{x}$ $\quad$ $f(x) = \dfrac{1}{x^2}$ $\quad$ $f(x) = \dfrac{1}{x^3}$ $\quad$ $f(x) = \dfrac{1}{x^4}$

Absolute value function

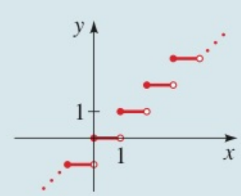
$f(x) = |x|$



$f(x) = |x|$

Greatest integer function

$f(x) = [\![x]\!]$



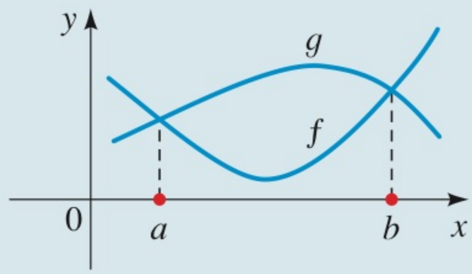
$f(x) = [\![x]\!]$

Solving equations / inequalities graphically

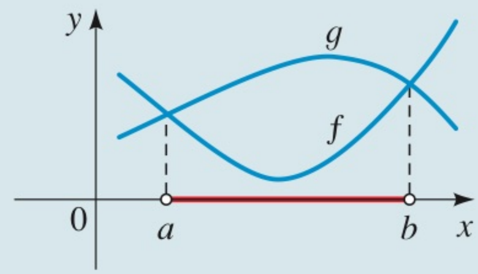
SOLVING EQUATIONS AND INEQUALITIES GRAPHICALLY

The **solution(s) of the equation** $f(x) = g(x)$ are the values of x where the graphs of f and g intersect.

The **solution(s) of the inequality** $f(x) < g(x)$ are the values of x where the graph of g is higher than the graph of f.



The solutions of $f(x) = g(x)$ are the values a and b.



The solution of $f(x) < g(x)$ is the interval (a, b).

e.g.

EXAMPLE 3 ■ Solving Graphically

Solve the given equation or inequality graphically.

(a) $2x^2 + 3 = 5x + 6$

(b) $2x^2 + 3 \le 5x + 6$

(c) $2x^2 + 3 > 5x + 6$

You can also solve the equations and inequalities algebraically. Check that your solutions match the solutions we obtained graphically.

SOLUTION We first define functions f and g that correspond to the left-hand side and to the right-hand side of the equation or inequality. So we define

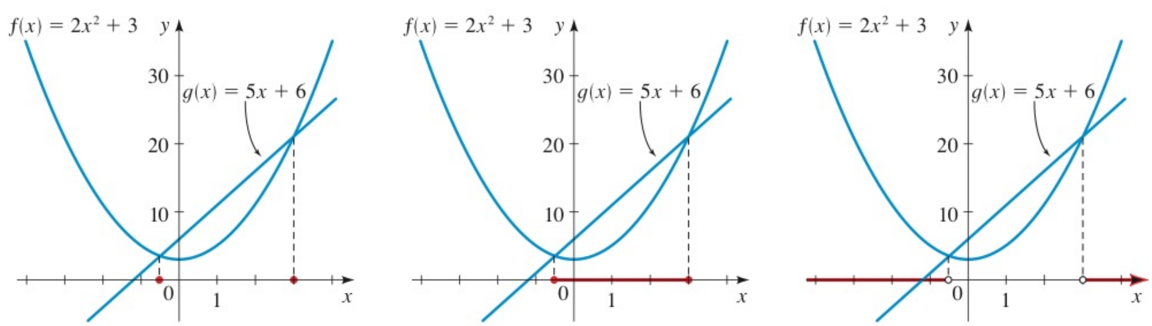
$$f(x) = 2x^2 + 3 \qquad \text{and} \qquad g(x) = 5x + 6$$

Next, we sketch graphs of f and g on the same set of axes.

(a) The given equation is equivalent to $f(x) = g(x)$. From the graph in Figure 3(a) we see that the solutions of the equation are $x = -0.5$ and $x = 3$.

(b) The given inequality is equivalent to $f(x) \le g(x)$. From the graph in Figure 3(b) we see that the solution is the interval $[-0.5, 3]$.

(c) The given inequality is equivalent to $f(x) > g(x)$. From the graph in Figure 3(c) we see that the solution is $(-\infty, -0.5) \cup (3, \infty)$.



(a) Solution: $x = -0.5, 3$ (b) Solution: $[-0.5, 3]$ (c) Solution: $(-\infty, -0.5) \cup (3, \infty)$

FIGURE 3 Graphs of $f(x) = 2x^2 + 3$ and $g(x) = 5x + 6$

Two ways: $2x^2+3 \geq 5x+6$

$\Rightarrow 2x^2 - 5x - 3 \geq 0$

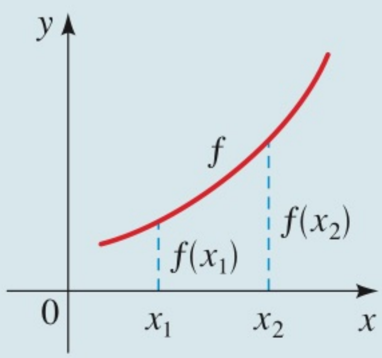
$(2x+1)(x-3) \geq 0$

· Increasing / Decreasing functions

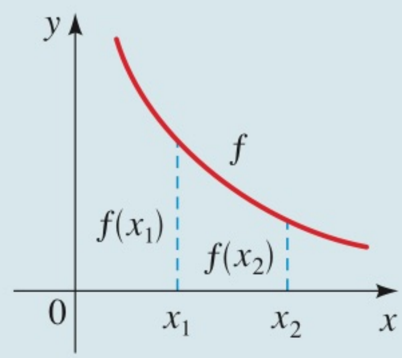
DEFINITION OF INCREASING AND DECREASING FUNCTIONS

f is **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

f is **decreasing** on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.



f is increasing



f is decreasing

e.g. $f(x) = x^2$

$f(x) = x^3$

$f(x) = \frac{1}{x}$

EXAMPLE 7 ■ Finding Intervals Where a Function Increases and Decreases

(a) Sketch the graph of the function $f(x) = x^{2/3}$.

(b) Find the domain and range of the function.

(c) Find the intervals on which f is increasing and on which f is decreasing.

SOLUTION

(a) We use a graphing calculator to sketch the graph in Figure 8.

(b) From the graph we observe that the domain of f is $\mathbb{R}$ and the range is $[0, \infty)$.

(c) From the graph we see that f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

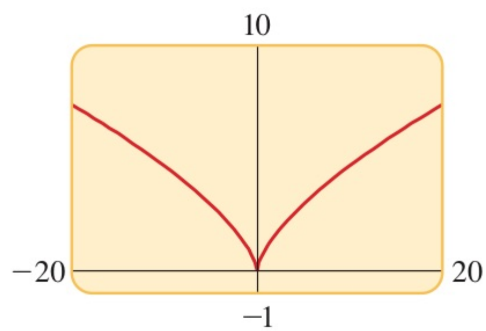
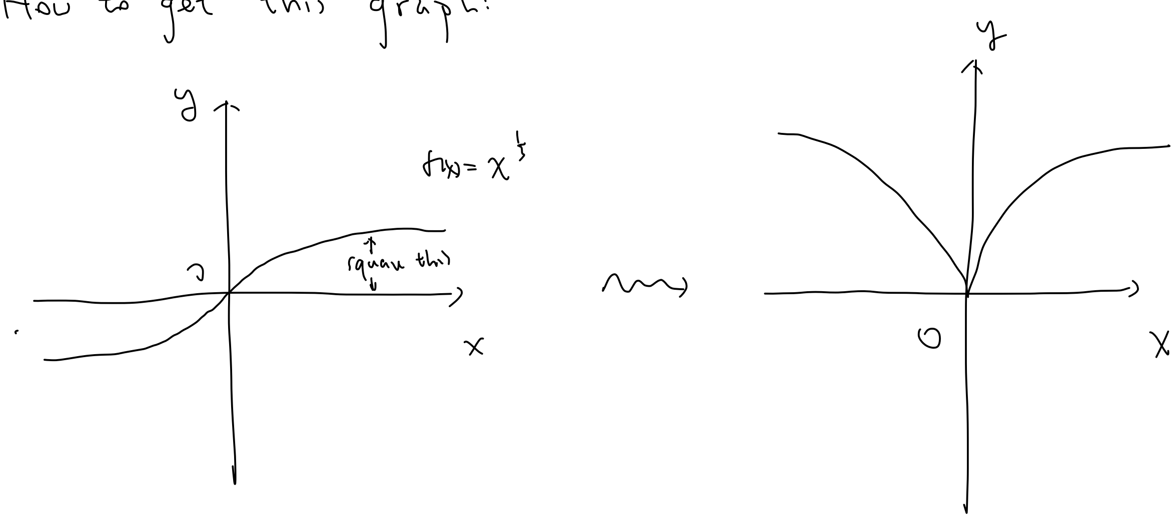


FIGURE 8 Graph of $f(x) = x^{2/3}$

How to get this graph:



Local Maximum / Local minimum

Def:

LOCAL MAXIMA AND MINIMA OF A FUNCTION

1. The function value $f(a)$ is a **local maximum value** of f if

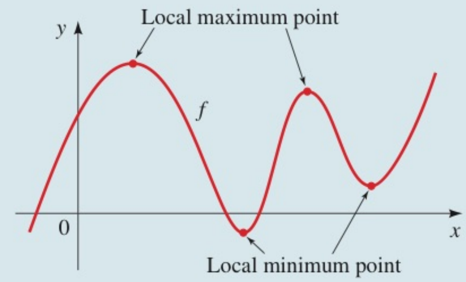
$$f(a) \geq f(x) \quad \text{when } x \text{ is near } a$$

(This means that $f(a) \geq f(x)$ for all x in some open interval containing a.) In this case we say that f has a **local maximum** at $x = a$.

2. The function value $f(a)$ is a **local minimum value** of f if

$$f(a) \leq f(x) \quad \text{when } x \text{ is near } a$$

(This means that $f(a) \leq f(x)$ for all x in some open interval containing a.) In this case we say that f has a **local minimum** at $x = a$.



Note: Don't need to be a global maximum / global minimum

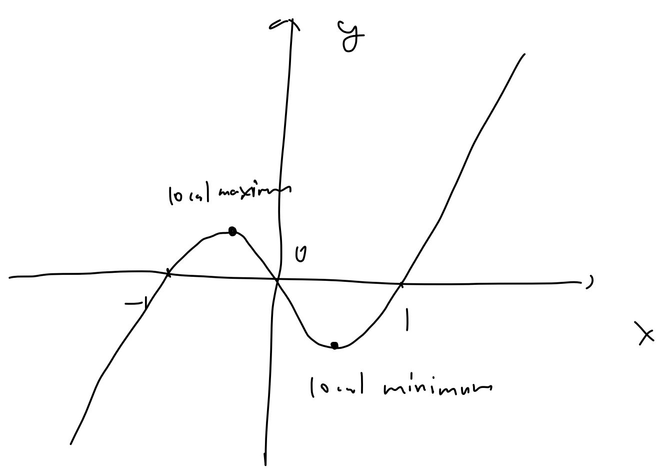
$f(a) \geq f(x), \forall x$ $\qquad$ $f(a) \leq f(x), \forall x$

may exists more than one points!

e.g. $f(x) = x^2$

$x = 0$. both global & local minimum

$$f(x) = x^3 - x = x(x^2 - 1)$$



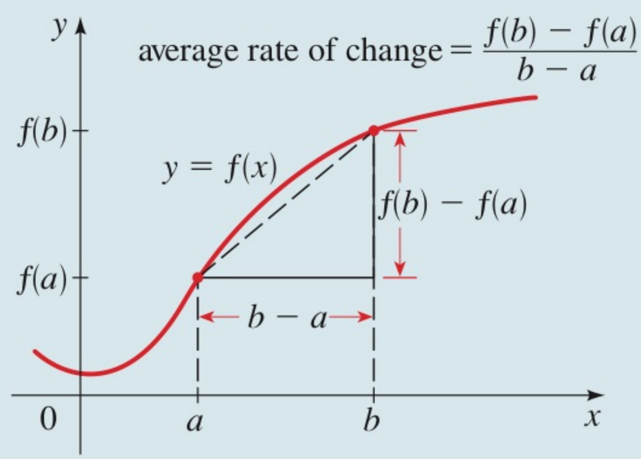
Average Rate of Change

AVERAGE RATE OF CHANGE

The **average rate of change** of the function $y = f(x)$ between $x = a$ and $x = b$ is

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

The average rate of change is the slope of the **secant line** between $x = a$ and $x = b$ on the graph of f, that is, the line that passes through $(a, f(a))$ and $(b, f(b))$.



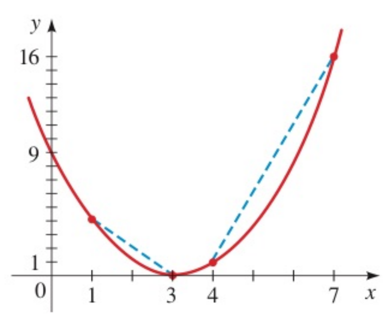


FIGURE 2 $f(x) = (x - 3)^2$

For the function $f(x) = (x - 3)^2$, whose graph is shown in Figure 2, find the net change and the average rate of change between the following points:

(a) $x = 1$ and $x = 3$ **(b)** $x = 4$ and $x = 7$

SOLUTION

(a) Net change $= f(3) - f(1)$ — Definition

$= (3 - 3)^2 - (1 - 3)^2$ — Use $f(x) = (x - 3)^2$

$= -4$ — Calculate

Average rate of change $= \dfrac{f(3) - f(1)}{3 - 1}$ — Definition

$= \dfrac{-4}{2} = -2$ — Calculate

(b) Net change $= f(7) - f(4)$ — Definition

$= (7 - 3)^2 - (4 - 3)^2$ — Use $f(x) = (x - 3)^2$

$= 15$ — Calculate

Average rate of change $= \dfrac{f(7) - f(4)}{7 - 4}$ — Definition

$= \dfrac{15}{3} = 5$ — Calculate

Linear Functions Have Constant Rate of Change

Recall that a function of the form $f(x) = mx + b$ is a linear function (see page 160) Its graph is a line with slope m. On the other hand, if a function f has constant rate o change, then it must be a linear function. (You are asked to prove these facts in Exer cises 51 and 52 in Section 2.5.) In general, the average rate of change of a linear func tion between any two points is the constant m. In the next example we find the averag rate of change for a particular linear function.

EXAMPLE 4 ■ Linear Functions Have Constant Rate of Change

Let $f(x) = 3x - 5$. Find the average rate of change of f between the following points.

(a) $x = 0$ and $x = 1$

(b) $x = 3$ and $x = 7$

(c) $x = a$ and $x = a + h$

What conclusion can you draw from your answers?

SOLUTION

(a) Average rate of change $= \dfrac{f(1) - f(0)}{1 - 0} = \dfrac{(3 \cdot 1 - 5) - (3 \cdot 0 - 5)}{1}$

$$= \frac{(-2) - (-5)}{1} = 3$$

(b) Average rate of change $= \dfrac{f(7) - f(3)}{7 - 3} = \dfrac{(3 \cdot 7 - 5) - (3 \cdot 3 - 5)}{4}$

$$= \frac{16 - 4}{4} = 3$$

(c) Average rate of change $= \dfrac{f(a + h) - f(a)}{(a + h) - a} = \dfrac{[3(a + h) - 5] - [3a - 5]}{h}$

$$= \frac{3a + 3h - 5 - 3a + 5}{h} = \frac{3h}{h} = 3$$

EXAMPLE 2 ■ Average Speed of a Falling Object

If an object is dropped from a high cliff or a tall building, then the distance it has fallen after t seconds is given by the function $d(t) = 16t^2$. Find its average speed (average rate of change) over the following intervals:

(a) Between 1 s and 5 s **(b)** Between $t = a$ and $t = a + h$

SOLUTION

(a) Average rate of change

$$= \frac{d(5) - d(1)}{5 - 1} \qquad \text{Definition}$$

$$= \frac{16(5)^2 - 16(1)^2}{5 - 1} \qquad \text{Use } d(t) = 16t^2$$

$$= \frac{400 - 16}{4} \qquad \text{Calculate}$$

$$= 96 \text{ ft/s} \qquad \text{Calculate}$$

(b) Average rate of change

$$= \frac{d(a + h) - d(a)}{(a + h) - a} \qquad \text{Definition}$$

$$= \frac{16(a + h)^2 - 16(a)^2}{(a + h) - a} \qquad \text{Use } d(t) = 16t^2$$

$$= \frac{16(a^2 + 2ah + h^2 - a^2)}{h} \qquad \text{Expand and factor 16}$$

$$= \frac{16(2ah + h^2)}{h} \qquad \text{Simplify numerator}$$

$$= \frac{16h(2a + h)}{h} \qquad \text{Factor } h$$

$$= \underline{16(2a + h)} \qquad \text{Simplify}$$

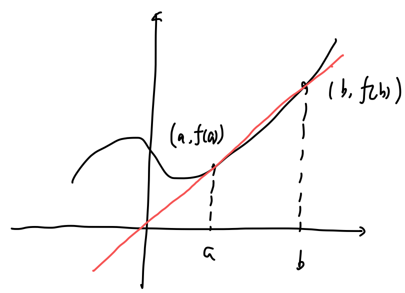
when h is very very small, the "average speed" will just be the speed at the time $\underline{t=a}$, which is $32a$

Recall last time: average rate of change

Def: f is a function, the a.r.c between a & b is:

$$\frac{f(b)-f(a)}{b-a}$$

i.e. slope of the secant line



Ex.: linear function has constant a.r.c = slope

Transformations of functions

Vertical Shifting

Adding a constant to a function shifts its graph vertically: upward if the constant is positive and downward if it is negative.

In general, suppose we know the graph of $y = f(x)$. How do we obtain from it the graphs of

$$y = f(x) + c \qquad \text{and} \qquad y = f(x) - c \qquad (c > 0)$$

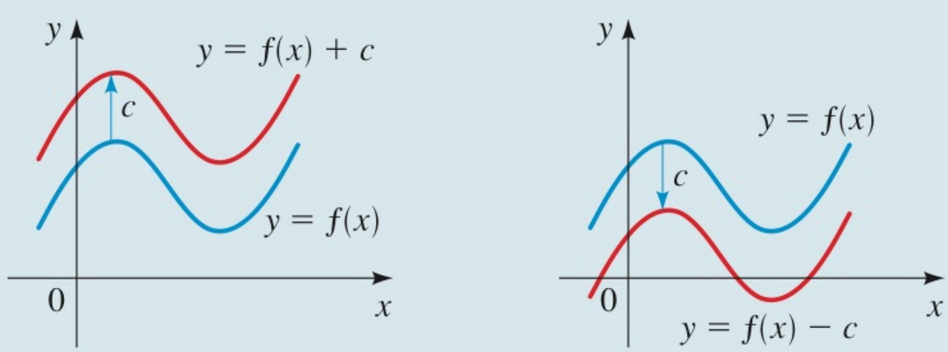
The y-coordinate of each point on the graph of $y = f(x) + c$ is c units above the y-coordinate of the corresponding point on the graph of $y = f(x)$. So we obtain the graph of $y = f(x) + c$ simply by shifting the graph of $y = f(x)$ upward c units. Similarly, we obtain the graph of $y = f(x) - c$ by shifting the graph of $y = f(x)$ downward c units.

VERTICAL SHIFTS OF GRAPHS

Suppose $c > 0$.

To graph $y = f(x) + c$, shift the graph of $y = f(x)$ upward c units.

To graph $y = f(x) - c$, shift the graph of $y = f(x)$ downward c units.



e.x. $y = x^2$ & $x^2 + 1$

$y = x$ & $x - 1$

e.g.

EXAMPLE 1 ■ Vertical Shifts of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = x^2 + 3$ **(b)** $h(x) = x^2 - 2$

(b) Similarly, to graph h we shift the graph of f downward 2 units, as shown in Figure 1.

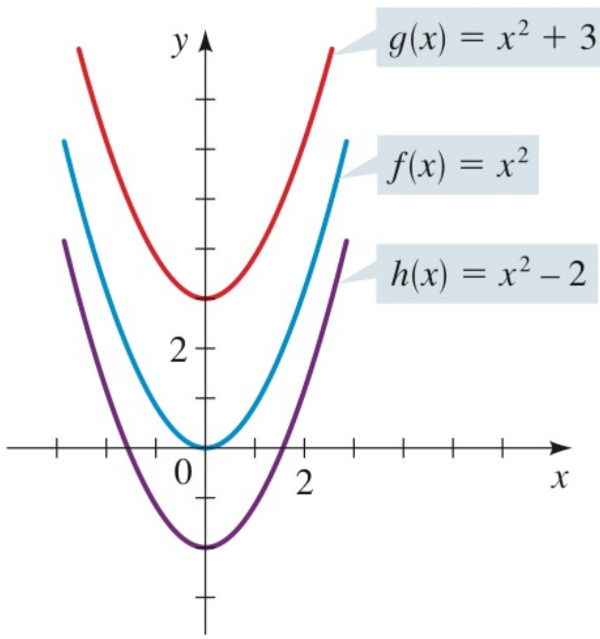


FIGURE 1

Horizontal Shifting

Suppose that we know the graph of $y = f(x)$. How do we use it to obtain the graphs of

$$y = f(x + c) \qquad \text{and} \qquad y = f(x - c) \qquad (c > 0)$$

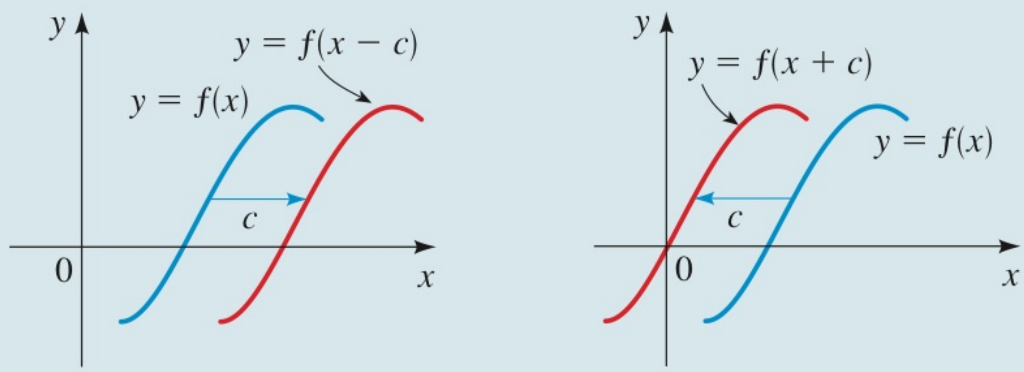
The value of $f(x - c)$ at x is the same as the value of $f(x)$ at $x - c$. Since $x - c$ is c units to the left of x, it follows that the graph of $y = f(x - c)$ is just the graph of $y = f(x)$ shifted to the right c units. Similar reasoning shows that the graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted to the left c units. The following box summarizes these facts.

HORIZONTAL SHIFTS OF GRAPHS

Suppose $c > 0$.

To graph $y = f(x - c)$, shift the graph of $y = f(x)$ to the right c units.

To graph $y = f(x + c)$, shift the graph of $y = f(x)$ to the left c units.



e.x. $f(x) = x \xrightarrow{\text{graph}} y = x$

$\downarrow$

$f(x+1) = x+1 \qquad y = x+1$

e.g.

EXAMPLE 2 ■ Horizontal Shifts of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = (x + 4)^2$ **(b)** $h(x) = (x - 2)^2$

The graphs of g and h are sketched in Figure 2.

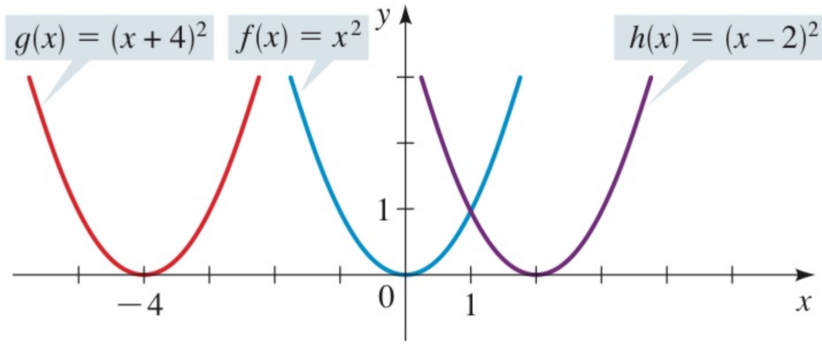
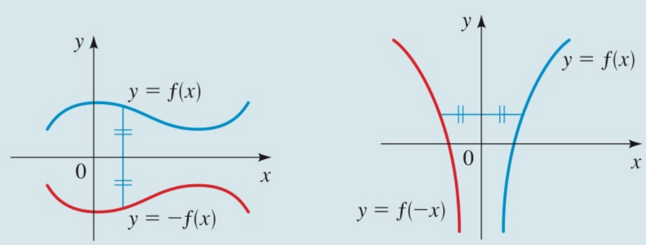


FIGURE 2

· Reflecting:

REFLECTING GRAPHS

To graph $y = -f(x)$, reflect the graph of $y = f(x)$ in the x-axis.
To graph $y = f(-x)$, reflect the graph of $y = f(x)$ in the y-axis.



EXAMPLE 4 ■ Reflecting Graphs

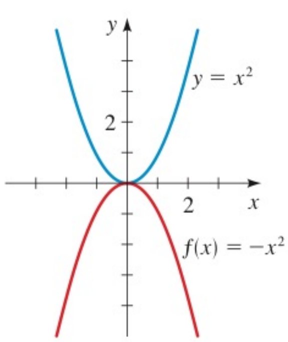
Sketch the graph of each function.

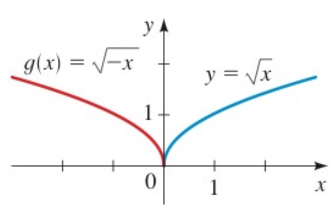
(a) $f(x) = -x^2$ **(b)** $g(x) = \sqrt{-x}$

SOLUTION

(a) We start with the graph of $y = x^2$. The graph of $f(x) = -x^2$ is the graph of $y = x^2$ reflected in the x-axis (see Figure 4).

(b) We start with the graph of $y = \sqrt{x}$ (Example 1(c) in Section 2.2). The graph of $g(x) = \sqrt{-x}$ is the graph of $y = \sqrt{x}$ reflected in the y-axis (see Figure 5). Note that the domain of the function $g(x) = \sqrt{-x}$ is $\{x \mid x \leq 0\}$.





Vertical Stretching and Shrinking

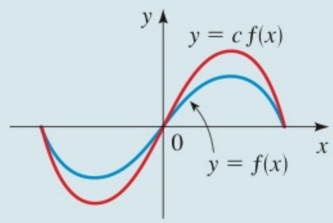
Suppose we know the graph of $y = f(x)$. How do we use it to obtain the graph of $y = cf(x)$? The y-coordinate of $y = cf(x)$ at x is the same as the corresponding y-coordinate of $y = f(x)$ multiplied by c. Multiplying the y-coordinates by c has the effect of vertically stretching or shrinking the graph by a factor of c (if $c > 0$).

VERTICAL STRETCHING AND SHRINKING OF GRAPHS

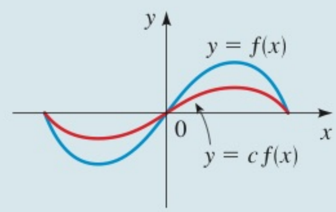
To graph $y = cf(x)$:

If $c > 1$, stretch the graph of $y = f(x)$ vertically by a factor of c.

If $0 < c < 1$, shrink the graph of $y = f(x)$ vertically by a factor of c.



$c > 1$



$0 < c < 1$

EXAMPLE 5 ■ Vertical Stretching and Shrinking of Graphs

Use the graph of $f(x) = x^2$ to sketch the graph of each function.

(a) $g(x) = 3x^2$ **(b)** $h(x) = \frac{1}{3}x^2$

SOLUTION

(a) The graph of g is obtained by multiplying the y-coordinate of each point on the graph of f by 3. That is, to obtain the graph of g, we stretch the graph of f vertically by a factor of 3. The result is the narrowest parabola in Figure 6.

(b) The graph of h is obtained by multiplying the y-coordinate of each point on the graph of f by $\frac{1}{3}$. That is, to obtain the graph of h, we shrink the graph of f vertically by a factor of $\frac{1}{3}$. The result is the widest parabola in Figure 6.

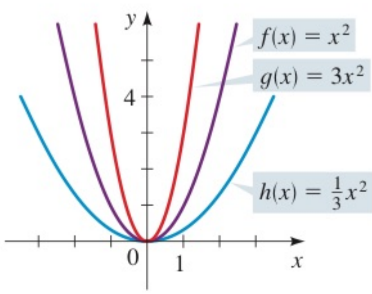


FIGURE 6

Now Try Exercises 41 and 43

EXAMPLE 6 ■ Combining Shifting, Stretching, and Reflecting

Sketch the graph of the function $f(x) = 1 - 2(x - 3)^2$.

SOLUTION Starting with the graph of $y = x^2$, we first shift to the right 3 units to get the graph of $y = (x - 3)^2$. Then we reflect in the x-axis and stretch by a factor of 2 to get the graph of $y = -2(x - 3)^2$. Finally, we shift upward 1 unit to get the graph of $f(x) = 1 - 2(x - 3)^2$ shown in Figure 7.

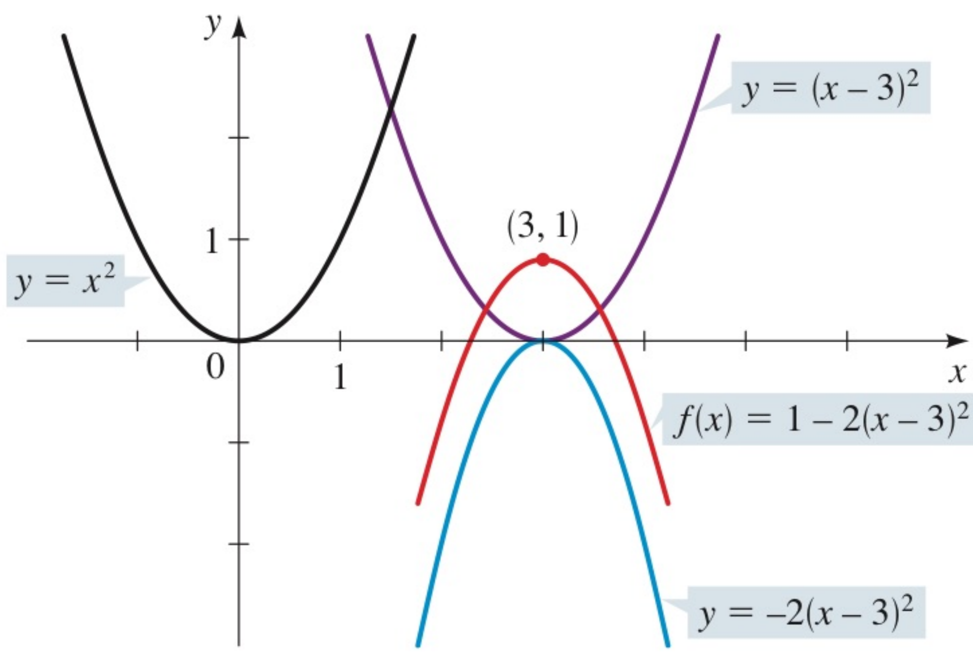


FIGURE 7

Horizontal Stretching and Shrinking

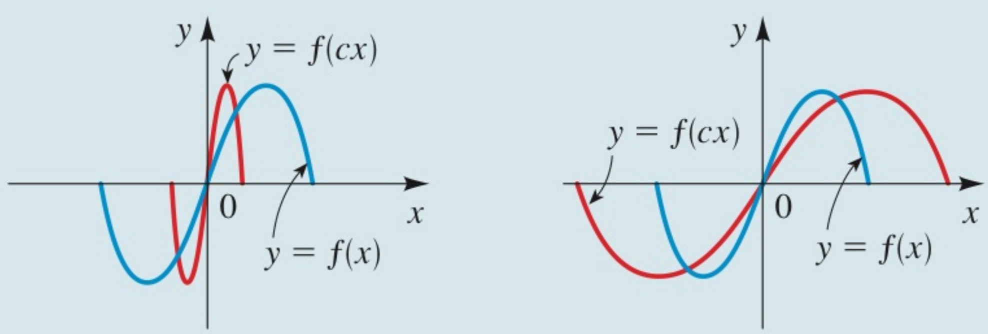
Now we consider horizontal shrinking and stretching of graphs. If we know the graph of $y = f(x)$, then how is the graph of $y = f(cx)$ related to it? The y-coordinate of $y = f(cx)$ at x is the same as the y-coordinate of $y = f(x)$ at cx. Thus the x-coordinates in the graph of $y = f(x)$ correspond to the x-coordinates in the graph of $y = f(cx)$ multiplied by c. Looking at this the other way around, we see that the x-coordinates in the graph of $y = f(cx)$ are the x-coordinates in the graph of $y = f(x)$ multiplied by $1/c$. In other words, to change the graph of $y = f(x)$ to the graph of $y = f(cx)$, we must shrink (or stretch) the graph horizontally by a factor of $1/c$ (if $c > 0$), as summarized in the following box.

HORIZONTAL SHRINKING AND STRETCHING OF GRAPHS

To graph $y = f(cx)$:

If $c > 1$, shrink the graph of $y = f(x)$ horizontally by a factor of $1/c$.

If $0 < c < 1$, stretch the graph of $y = f(x)$ horizontally by a factor of $1/c$.

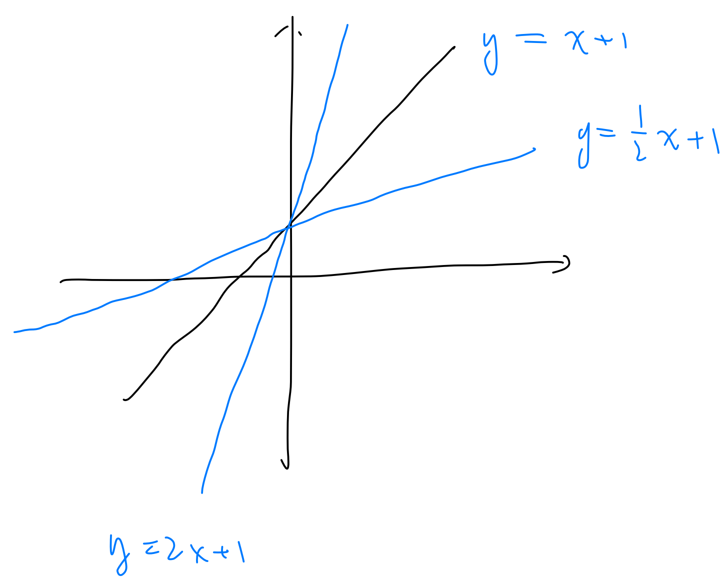


$c > 1$ $\qquad$ $0 < c < 1$

e.g. $\quad f(x) = x + 1$

$c = 2. \quad y = 2x + 1$

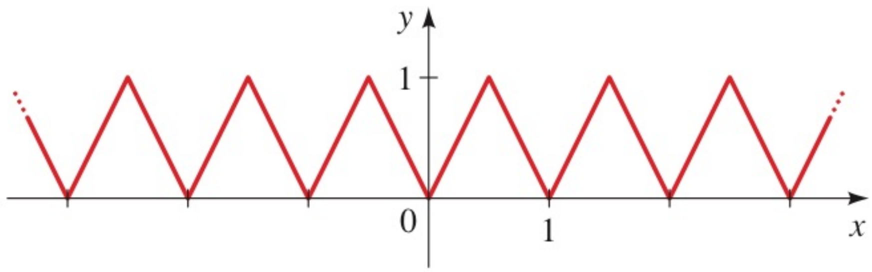
$c = \frac{1}{2}, \quad y = \frac{1}{2}x + 1$



EXAMPLE 7 ■ Horizontal Stretching and Shrinking of Graphs

The graph of $y = f(x)$ is shown in Figure 8. Sketch the graph of each function.

(a) $y = f(2x)$ **(b)** $y = f\left(\frac{1}{2}x\right)$



SOLUTION Using the principles described on page 203, we **(a)** *shrink* the graph horizontally by the factor $\frac{1}{2}$ to obtain the graph in Figure 9, and **(b)** *stretch* the graph horizontally by the factor 2 to obtain the graph in Figure 10.

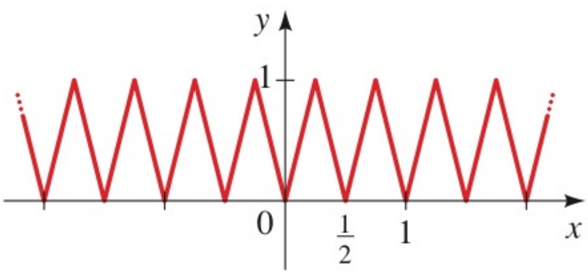


FIGURE 9 $y = f(2x)$

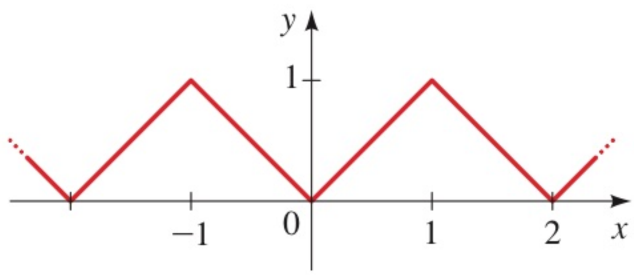


FIGURE 10 $y = f\left(\frac{1}{2}x\right)$

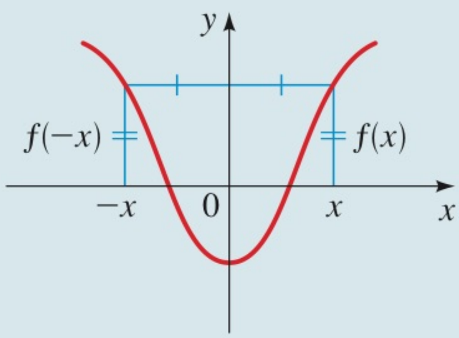
Even & Odd functions

EVEN AND ODD FUNCTIONS

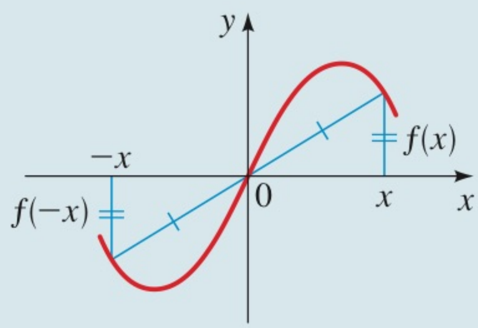
Let f be a function.

f is **even** if $f(-x) = f(x)$ for all x in the domain of f.

f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.



The graph of an even function is symmetric with respect to the y-axis.



The graph of an odd function is symmetric with respect to the origin.

EXAMPLE 8 ■ Even and Odd Functions

Determine whether the functions are even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$

(b) $g(x) = 1 - x^4$

(c) $h(x) = 2x - x^2$

SOLUTION

(a)
$$\begin{aligned} f(-x) &= (-x)^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is an odd function.

(b) $g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$

So g is even.

(c) $h(-x) = 2(-x) - (-x)^2 = -2x - x^2$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd.

Combining functions

Sums, Differences, Products, and Quotients

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers. For example, we define the function $f + g$ by

$$(f + g)(x) = f(x) + g(x)$$

The new function $f + g$ is called the **sum** of the functions f and g; its value at x is $f(x) + g(x)$. Of course, the sum on the right-hand side makes sense only if both $f(x)$ and $g(x)$ are defined, that is, if x belongs to the domain of f and also to the domain of g. So if the domain of f is A and the domain of g is B, then the domain of $f + g$ is the intersection of these domains, that is, $A \cap B$. Similarly, we can define the **difference** $f - g$, the **product** fg, and the **quotient** f/g of the functions f and g. Their domains are $A \cap B$, but in the case of the quotient we must remember not to divide by 0.

ALGEBRA OF FUNCTIONS

Let f and g be functions with domains A and B. Then the functions $f + g$, $f - g$, fg, and f/g are defined as follows.

$(f + g)(x) = f(x) + g(x)$	Domain $A \cap B$
$(f - g)(x) = f(x) - g(x)$	Domain $A \cap B$
$(fg)(x) = f(x)g(x)$	Domain $A \cap B$
$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$	Domain $\{x \in A \cap B \mid g(x) \neq 0\}$

Note: domain will change!

EXAMPLE 1 ■ Combinations of Functions and Their Domains

Let $f(x) = \dfrac{1}{x - 2}$ and $g(x) = \sqrt{x}$.

(a) Find the functions $f + g$, $f - g$, fg, and f/g and their domains.

(b) Find $(f + g)(4)$, $(f - g)(4)$, $(fg)(4)$, and $(f/g)(4)$.

Recall: Composition of functions, Note: $f \circ g \neq f \times g = fg$

COMPOSITION OF FUNCTIONS

Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. We can picture $f \circ g$ using an arrow diagram (Figure 4).

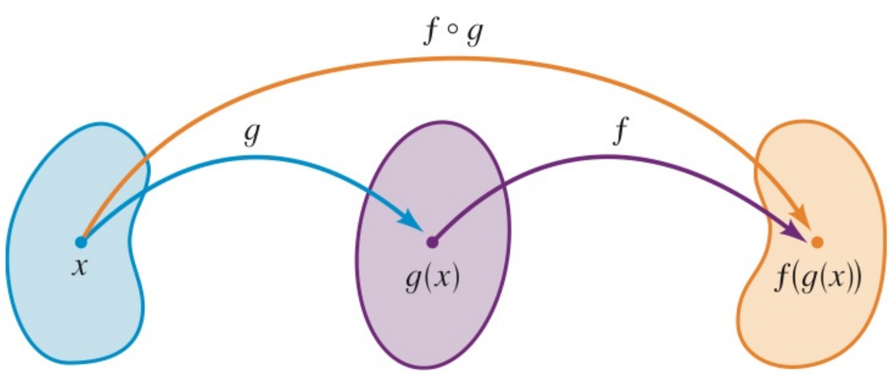


FIGURE 4 Arrow diagram for $f \circ g$

In general:

$f \circ g \neq g \circ f$

EXAMPLE 3 ■ Finding the Composition of Functions

Let $f(x) = x^2$ and $g(x) = x - 3$.

(a) Find the functions $f \circ g$ and $g \circ f$ and their domains.

(b) Find $(f \circ g)(5)$ and $(g \circ f)(7)$.

SOLUTION

(a) We have

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g \\ &= f(x-3) && \text{Definition of } g \\ &= (x-3)^2 && \text{Definition of } f \end{aligned}$$

and

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f \\ &= g(x^2) && \text{Definition of } f \\ &= x^2 - 3 && \text{Definition of } g \end{aligned}$$

The domains of both $f \circ g$ and $g \circ f$ are $\mathbb{R}$.

(b) We have

$$(f \circ g)(5) = f(g(5)) = f(2) = 2^2 = 4$$

$$(g \circ f)(7) = g(f(7)) = g(49) = 49 - 3 = 46$$