§ 1.6. Complex numbers

Question from last section:

How to solve: $x^2 + 1 = 0$

- There doesn't exist real numbers x s.t. $x^2+1=0$

  Why? because $\forall$ real number x, $x^2 \geqslant 0 \Rightarrow x^2+1 \geqslant 1$, no way $=0$

- To "solve" this equation, we have to extend the number system $\mathbb{R}$

Def: Complex numbers

A complex number is an expression

$$a + bi$$

where a & b are real numbers, i is a symbol (imaginary unit)

Real part: $Re(a+bi) = a$

$Im(a+bi) = b$

An expression doesn't make any sense, we need to define various "operations" on these expressions:

- Addition

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

$$1+2i + 3+4i = 4+6i$$

- Substraction

$$(a+bi) - (c+di) = (a-c) + (b-d)i$$

$$(1+2i) - (3+4i) = -2-2i$$

☆ · Multiplication:

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

e.g. $(1+2i)(3+4i) = -5+10i$

$$i^2 = i \cdot i = -1$$

· Division:

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2}$$

c & d are not all 0

$$= \frac{ac+bd+(-ad+bc)i}{c^2+d^2}$$

Basic properties:

We use symbol x, y to represent complex numbers:

- Commutativity:

$$x + y = y + x, \qquad x \cdot y = y \cdot x$$

- associavity:

$$x(y_1 + y_2) = x \cdot y_1 + x \cdot y_2$$

- existence of "negative" & "reciprocal"

$$x + (-x) = 0$$

$$\text{if } x \neq 0, \quad \exists\, x^{-1}, \quad x^{-1} \cdot x = x \cdot x^{-1} = 1$$

$$\text{e.g.} \quad 1+2i \rightsquigarrow \frac{1-2i}{5} = (1+2i)^{-1}$$

Synol: $\mathbb{C}$

- complex conjugate: $z = a + bi \longrightarrow \bar{z} = a - bi$

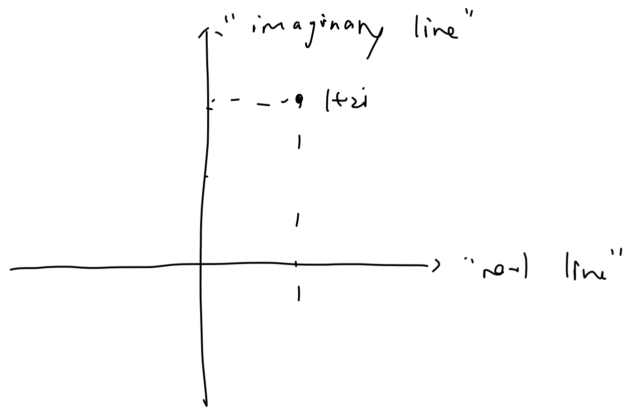
i.e.

$$\overline{1+2i} = 1 - 2i$$

$$\bar{i} = -i$$

$$\bar{a} = a, \quad \text{for } a \in \mathbb{R}$$

- $\mathbb{C}$: two-dimensional number system:



Let's come back to the original equation

$$x^2 + 1 = 0$$

We are seeking solution in complex numbers:

Suppose: $x = a + bi \Rightarrow (a+bi)^2 + 1 = 0$

$$a^2 - b^2 + 1 + 2abi = 0$$

$\Rightarrow a^2 - b^2 + 1 = 0$ & $2ab = 0$

① if $b = 0 \Rightarrow a^2 + 1 = 0$ ×

② if $a = 0 \Rightarrow -b^2 + 1 = 0 \Rightarrow b = \pm 1$

$\Rightarrow x = i$ or $-i$ "2 solutions"

· Extendly square root symbol:

principal square root of $-r$: $\sqrt{-r} = \sqrt{r}\, i$ or $i\sqrt{r}$

· $\sqrt{-1} = i$, $\sqrt{-4} = 2i$,

<u>Note</u>: $\sqrt{-a} \cdot \sqrt{-b} \neq \sqrt{(-a)\cdot(-b)}$!

· Complex solutions of quadratic equations:

$$ax^2 + bx + c = 0 \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\Delta = b^2 - 4ac > 0 \Rightarrow 2$ real solutions

$\Delta = b^2 - 4ac < 0 \Rightarrow 2$ complex solutions (conjugate to each other)

$\Delta = b^2 - 4ac = 0 \Rightarrow$ "2" same solutions

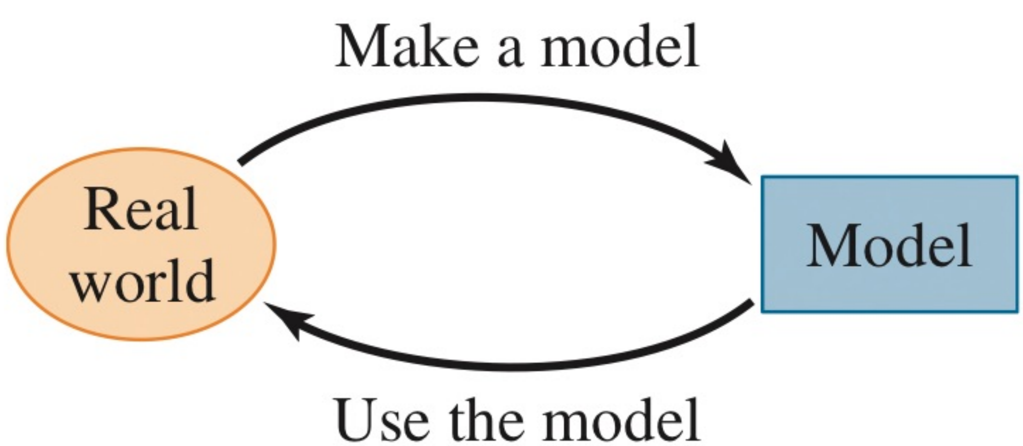
Solve each equation.

(a) $x^2 + 9 = 0$ **(b)** $x^2 + 4x + 5 = 0$

§ 1.7. Modeling with equations

GUIDELINES FOR MODELING WITH EQUATIONS

1. **Identify the Variable.** Identify the quantity that the problem asks you to find. This quantity can usually be determined by a careful reading of the question that is posed at the end of the problem. Then **introduce notation** for the variable (call it x or some other letter).
2. **Translate from Words to Algebra.** Read each sentence in the problem again, and express all the quantities mentioned in the problem in terms of the variable you defined in Step 1. To organize this information, it is sometimes helpful to **draw a diagram** or **make a table**.
3. **Set Up the Model.** Find the crucial fact in the problem that gives a relationship between the expressions you listed in Step 2. **Set up an equation** (or **model**) that expresses this relationship.
4. **Solve the Equation and Check Your Answer.** Solve the equation, check your answer, and express it as a sentence that answers the question posed in the problem.



EXAMPLE 1 ■ Renting a Car

A car rental company charges \$30 a day and 15¢ a mile for renting a car. Helen rents a car for two days, and her bill comes to \$108. How many miles did she drive?

SOLUTION **Identify the variable.** We are asked to find the number of miles Helen has driven. So we let

$$x = \text{number of miles driven}$$

Translate from words to algebra. Now we translate all the information given in the problem into the language of algebra.

In Words	**In Algebra**
Number of miles driven	x
Mileage cost (at \$0.15 per mile)	$0.15x$
Daily cost (at \$30 per day)	$2(30)$

Set up the model. Now we set up the model.

$$\boxed{\text{mileage cost}} + \boxed{\text{daily cost}} = \boxed{\text{total cost}}$$

$$0.15x + 2(30) = 108$$

Solve. Now we solve for x.

$$0.15x = 48 \qquad \text{Subtract 60}$$

$$x = \frac{48}{0.15} \qquad \text{Divide by 0.15}$$

$$x = 320 \qquad \text{Calculator}$$

Helen drove her rental car 320 miles.

A rectangular building lot is 8 ft longer than it is wide and has an area of 2900 ft^2. Find the dimensions of the lot.

SOLUTION **Identify the variable.** We are asked to find the width and length of the lot. So let

$$w = \text{width of lot}$$

Translate from words to algebra. Then we translate the information given in the problem into the language of algebra (see Figure 2).

In Words	In Algebra
Width of lot	w
Length of Lot	$w + 8$

Set up the model. Now we set up the model.

$$\boxed{\begin{matrix}\text{width}\\ \text{of lot}\end{matrix}} \cdot \boxed{\begin{matrix}\text{length}\\ \text{of lot}\end{matrix}} = \boxed{\begin{matrix}\text{area}\\ \text{of lot}\end{matrix}}$$

$$w(w + 8) = 2900$$

Solve. Now we solve for w.

$$\begin{aligned} w^2 + 8w &= 2900 && \text{Expand} \\ w^2 + 8w - 2900 &= 0 && \text{Subtract 2900} \\ (w - 50)(w + 58) &= 0 && \text{Factor} \\ w = 50 \quad \text{or} \quad w &= -58 && \text{Zero-Product Property} \end{aligned}$$

Since the width of the lot must be a positive number, we conclude that $w = 50$ ft. The length of the lot is $w + 8 = 50 + 8 = 58$ ft.

§1.8. Inequalities

Inequalities take the following form:

mathematical expressions 1 $\overset{\text{"}\leq\text{"}}{\geq}$ mathematical expression 2

e.g. $x \geq 0$

$4x + 7 \leq 19$

$x^2 - 2x - 3 \leq 0$

$5x^3 - x^2 + 6x - 9 \geq 5x^3 - 2x^2 + 8x - 6$

General rule:

RULES FOR INEQUALITIES

Rule

1. $A \leq B \quad \Leftrightarrow \quad A + C \leq B + C$

2. $A \leq B \quad \Leftrightarrow \quad A - C \leq B - C$

3. If $C > 0$, then $A \leq B \quad \Leftrightarrow \quad CA \leq CB$

4. If $C < 0$, then $A \leq B \quad \Leftrightarrow \quad CA \geq CB$

5. If $A > 0$ and $B > 0$,
then $A \leq B \quad \Leftrightarrow \quad \dfrac{1}{A} \geq \dfrac{1}{B}$

6. If $A \leq B$ and $C \leq D$,
then $A + C \leq B + D$

7. If $A \leq B$ and $B \leq C$, then $A \leq C$

Solving linear inequalities:

$$ax + b \geq c \quad (\text{or } \leq c) \qquad a \neq 0$$

- $a > 0$:

$$x \geq \frac{c-b}{a} \quad \left(\text{or } \leq \frac{c-b}{a}\right) \qquad 4x+7 \geq 19,\ x \geq 3$$

- $a < 0$

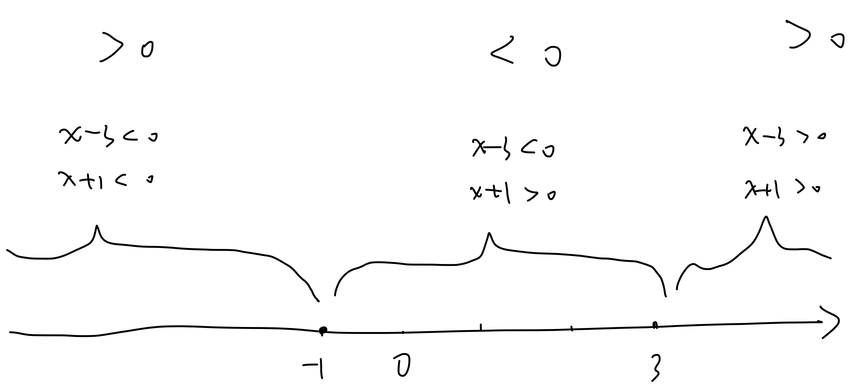
$$x \leq \frac{c-b}{a} \quad \left(\text{or } x \geq \frac{c-b}{a}\right) \qquad -4x+7 \geq 19,\ x \leq -3$$

Solving quadratic inequalities:

$$ax^2 + bx + c \geq d \quad (\leq d)$$
$$\Updownarrow$$
$$ax^2 + bx + c - d \geq 0 \quad (\leq 0)$$

factorization first! of $\underline{ax^2 + bx + c - d}$

e.g.
$$x^2 - 2x - 3 \geq 0$$
$$(x-3)(x+1) \geq 0$$



· inequality with quotient:

Solve the inequality $\dfrac{1+x}{1-x} \geq 1$.

SOLUTION **Move all terms to one side.** We move the terms to the left-hand side and simplify using a common denominator.

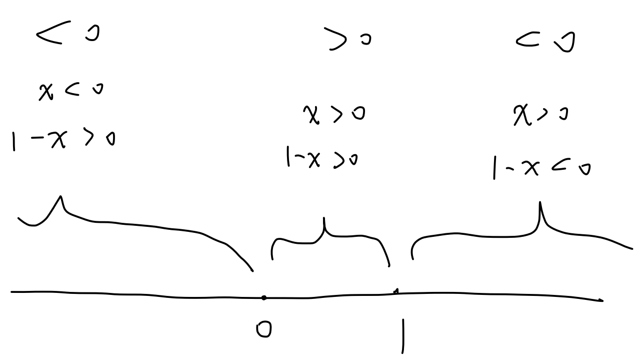
$$\frac{1+x}{1-x} \geq 1 \qquad \text{Given inequality}$$

$$\frac{1+x}{1-x} - 1 \geq 0 \qquad \text{Subtract 1}$$

$$\frac{1+x}{1-x} - \frac{1-x}{1-x} \geq 0 \qquad \text{Common denominator } 1-x$$

$$\frac{1+x-1+x}{1-x} \geq 0 \qquad \text{Combine the fractions}$$

$$\frac{2x}{1-x} \geq 0 \qquad \text{Simplify}$$



General principal: factor to product of "linear ones"

original $\Longrightarrow$ product/quotient of linear ≥ 0 (≤ 0)

eg.

$$x + \frac{5}{x-1} \geqslant -3$$

$$\Updownarrow$$

$$\frac{x^2 + 2x + 2}{x - 1} > 0$$

$$\Updownarrow$$

$$\frac{(x+1)^2+1}{x-1} > 0 \quad \Leftrightarrow \quad x-1 > 0$$

EXAMPLE 6 ■ Solving an Absolute Value Inequality

Solve the inequality $|x - 5| < 2$.

SOLUTION 1 The inequality $|x - 5| < 2$ is equivalent to

$$-2 < x - 5 < 2 \qquad \text{Property 1}$$

$$3 < x < 7 \qquad \text{Add 5}$$

The solution set is the open interval $(3, 7)$.

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Quadratic inequalities (continued)

General shape: $ax^2+bx+c \overset{(\leq)}{\geq} dx^2+ex+f$

e.g. $2x^2-3x+4 \geq x^2+2x-2$

$x^2-6x+7 \geq 2x^2+15$

Step 1: substract both sides by RHS:

$(a-d)x^2+(b-e)x+(c-f) \overset{(\leq)}{\geq} 0$ $\quad \left(a'x^2+b'x+c' \overset{(\leq)}{\geq} 0\right)$

e.g. $(2-1)x^2+(-3-2)x+(4-(-2)) \geq 0$

$$\|$$

$x^2-5x+6 \geq 0$

if $a'<0$, multiply both sides by -1, which reverse the direction of the inequality:

e.g. $x^2-6x+7 \geq 2x^2+15 \Rightarrow -x^2-6x-8 \geq 0$

$\Leftrightarrow x^2+6x+8 \leq 0$

Step 2: Calculating disc:

$$\Delta = b'^2-4a'c'$$

- If $\Delta \leq 0$, solution set is either whole real line / $\emptyset$

≥ 0 $\quad$ $a'>0$: $\mathbb{R}$ $\qquad\qquad$ ≤ 0 $\quad$ $a'>0$: $\emptyset$

- If $\Delta > 0$, continue to the next step

__Step 3__: Factorization by quadratic root formula

$$\text{root} = \frac{-b' \pm \sqrt{(b')^2 - 4a'c'}}{2a'}$$

$$\Rightarrow \quad a'x^2 + b'x + c' = a'\left(x - \frac{-b' + \sqrt{b'^2 - 4a'c'}}{2a'}\right)\left(x - \frac{-b' - \sqrt{b'^2 - 4a'c'}}{2a'}\right)$$

$$\left(\text{notice that if } \Delta = 0 \Rightarrow a'x^2 + b'x + c' = a'\left(x - \frac{-b}{2a}\right)^2\right)$$

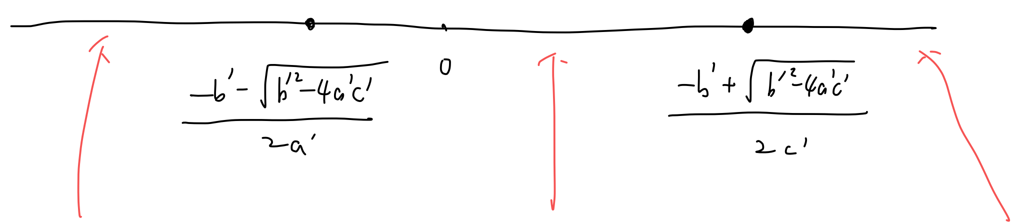
e.g. $\quad (x-2)(x-3) \geq 0$

$\qquad (x-2)(x-4) \leq 0$

__Step 4__: Determine the sign on the real line

$+ \qquad\qquad - \qquad\qquad +$

sign $= <0 \cdot <0 \; > 0$ $\qquad$ sign $= <0 \cdot >0 \; < 0$ $\qquad$ sign $= >0 \cdot >0 \; > 0$



$x - \frac{-b' - \sqrt{b'^2 - 4a'c'}}{2a'} < 0$ $\qquad$ $x - \frac{-b' - \sqrt{b'^2 - 4a'c'}}{2a'} > 0$ $\qquad$ $x - \frac{-b' - \sqrt{b'^2 - 4a'c'}}{2a'} > 0$

$x - \frac{-b' + \sqrt{b'^2 - 4a'c'}}{2a'} < 0$ $\qquad$ $x - \frac{-b' + \sqrt{b'^2 - 4a'c'}}{2a'} < 0$ $\qquad$ $x - \frac{-b' + \sqrt{b'^2 - 4a'c'}}{2a'} > 0$

__Think__: What will happen if we replace $\geq$ (resp. $\leq$) by $>$ (resp. $<$)

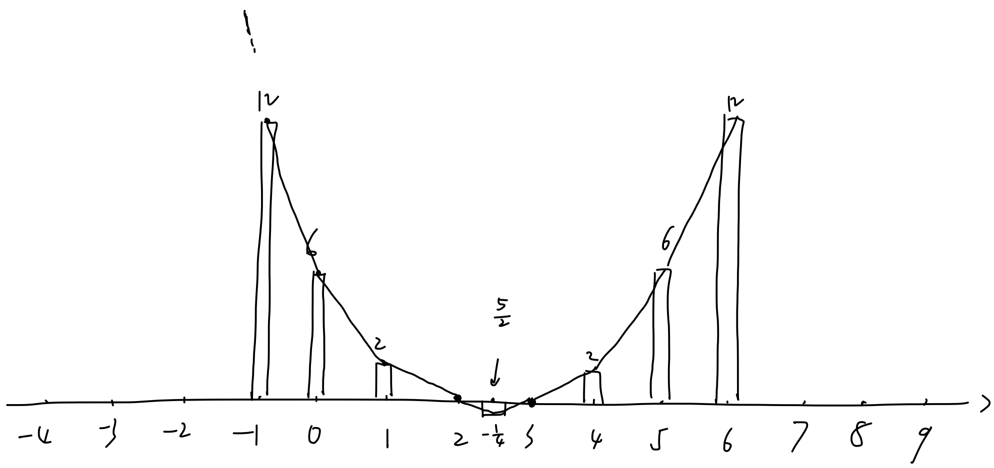
Another way: more geometric

Suppose we are given inequalities:

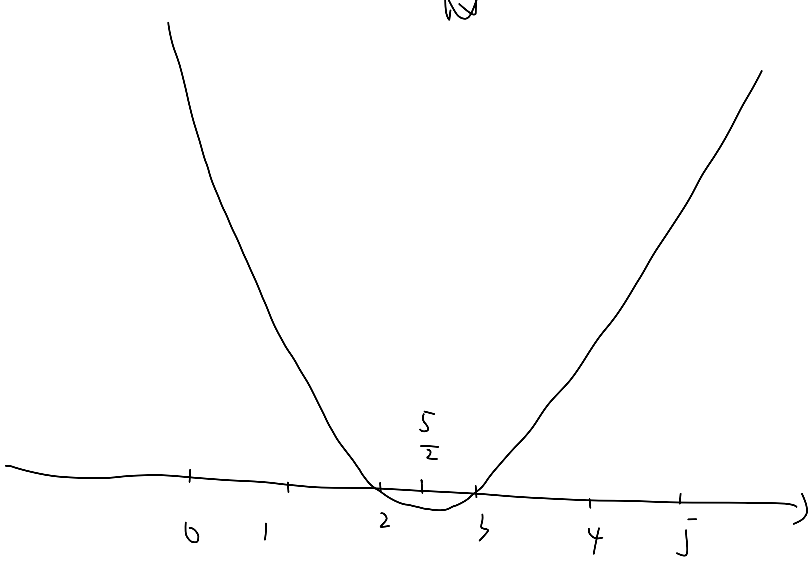
$$x^2-5x+6 \geq 0$$

try to "draw" the algebraic expression x^2-5x+6

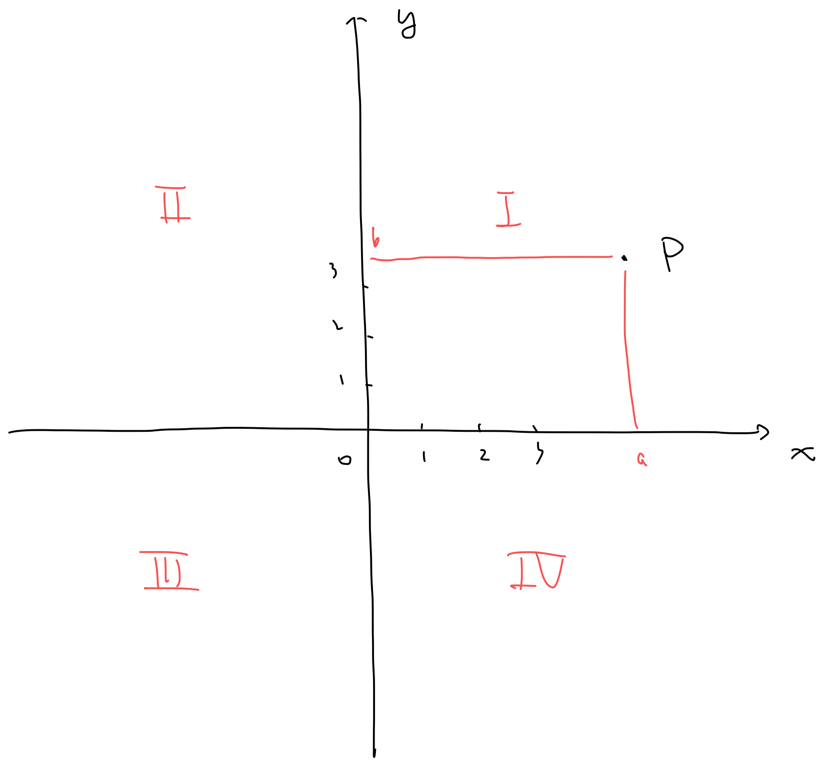
$$6-\frac{25}{4}=\frac{\quad}{4}$$



take more and more points



§1.9. Coordinate plane (generalization of "real lines")



two perpendicular real lines intersecting at the "original point"

horizontal one: x-axis

vertical one: y-axis

<u>Key point</u>: For every point P on this plane, we can use a pair of real numbers (a, b) to represent it

Conversely, for every pair of real numbers (x,y) we can locate a unique point P on the plane

points on the plane $\longleftrightarrow$ pairs of real numbers (x,y)

geometry algebra

EXAMPLE 1 ■ **Graphing Regions in the Coordinate Plane**

Describe and sketch the regions given by each set.

(a) $\{(x, y) \mid x \geq 0\}$ **(b)** $\{(x, y) \mid y = 1\}$ **(c)** $\{(x, y) \mid |y| < 1\}$

Inequalities involving absolute value

basic cases: $c > 0$

$$|x| \geq c \iff x \leq -c \text{ or } x \geq c$$

$$|x| \leq c \iff -c \leq x \leq c$$

$$|x| > c \iff x < -c \text{ or } x > c$$

$$|x| < c \iff -c < x < c$$

example:

- $|x-5| < 2$

  view "$x-5$" as a whole

  $$-2 < x-5 < 2 \iff \underbrace{-2+5}_{=3} < x < \underbrace{2+5}_{=7}$$

- $|3x+2| \geq 4$

  either $3x+2 \geq 4$ or $3x+2 \leq -4$

  $$\Downarrow \qquad\qquad \Downarrow$$

  $$x \geq \frac{2}{3} \qquad\qquad x \leq -2$$

Graphing polynomials:

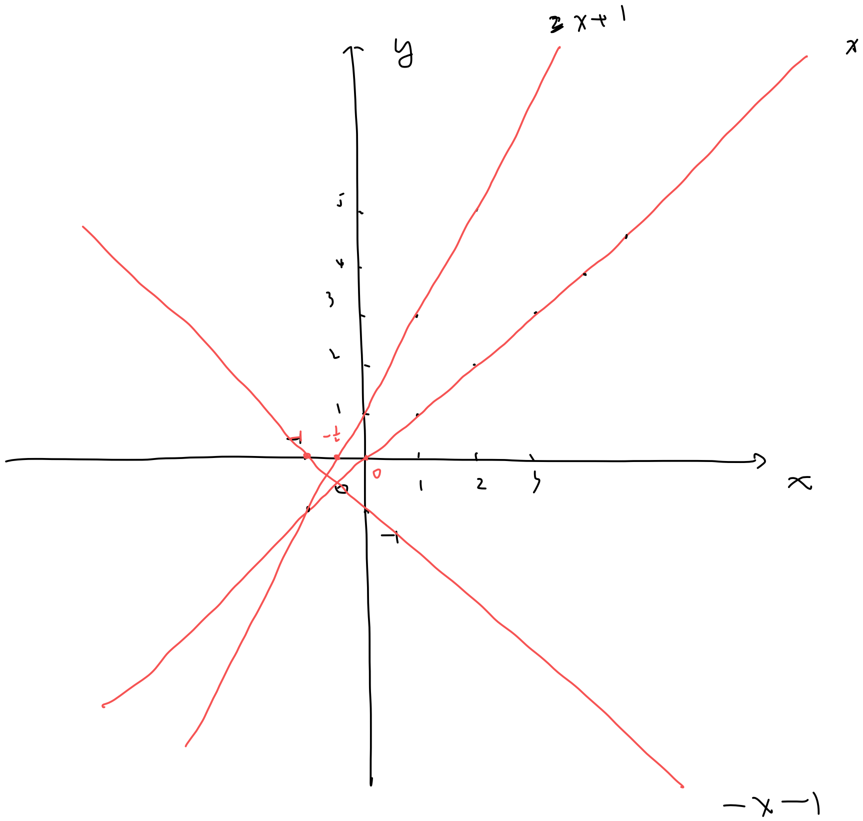
Linear polynomial: $ax + b$

e.g. x

$2x + 1$

$-x - 1$

· for a given value x compute $ax+b$ consider the point $(x, ax+b)$



· graphs are lines: intersection of the line with x axis is the solution of the equation

$$x = 0 \Rightarrow 0$$

$$2x + 1 = 0 \Rightarrow -\tfrac{1}{2}$$

$$-x - 1 = 0 \Rightarrow -1$$

Algebra $\leftrightsquigarrow$ Geometry

polynomial $\Longrightarrow$ graphs

solutions $\Longleftarrow$ intersection

philosophy : Geometry is drawn Algebra

Algebra is writen Geometry