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Polar coordinate

Recall: plane coordinate:

fix a point O, & two perpendicular lines intersectly at O

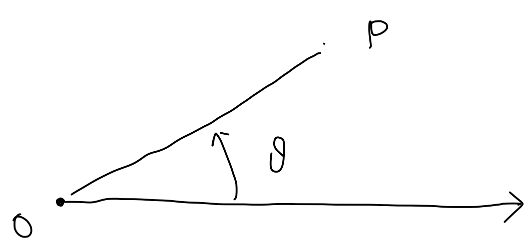
horizontal one: x-axi

vertical one: y-axi

then every point in the plane $\rightsquigarrow$ a pair of real numbers (x, y)

Polar coordinate system: distance & direction

- A fixed point O
- A ray starting from O: polar axis



For any point P

1. connect OP
2. $r = |OP|$
3. θ = the angle between polar axis & OP

counterclockwise: $\theta > 0$

clockwise: $\theta < 0$

θ is only well-defined up to multiples of 2π

In this way:

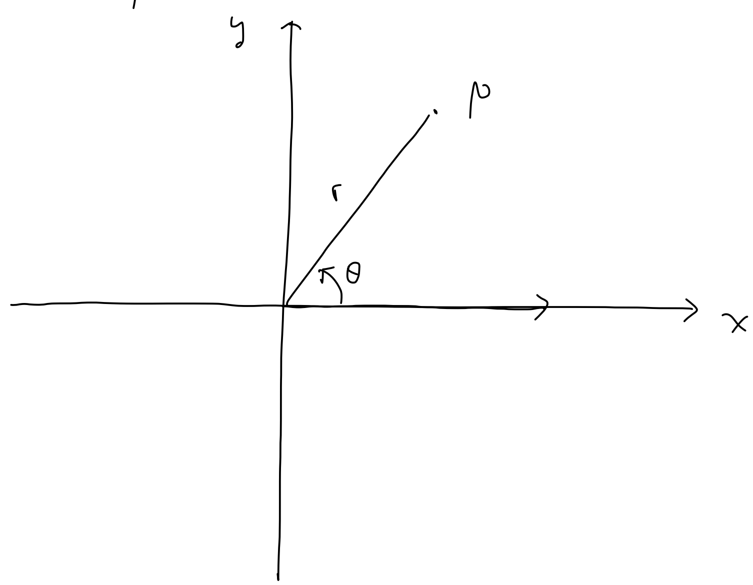
point on the plane $\rightsquigarrow$ (r, θ), $r \geq 0$, $\theta \in \mathbb{R}$ (up to 2π)

this "correspondence" is not 1-1 !

<u>How to get the reverse map?</u>

i.e. if we are given (r, θ), how do we know the original pt ?

Let's set up the xy-coordinate, s.t. positive x-axis = polar axis



$(r, \theta) \rightsquigarrow (x, y)$?

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \Rightarrow \text{not one-to-one}$$

$$(r, \theta) = (r, \theta + 2\pi)$$

Originally, we require $r > 0$, now we can extend the definition to all $r \in \mathbb{R}$:

$r < 0$: first find $(-r, \theta)$, the reflex about O

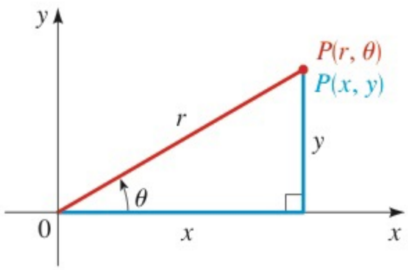


FIGURE 6

RELATIONSHIP BETWEEN POLAR AND RECTANGULAR COORDINATES

1. To change from polar to rectangular coordinates, use the formulas

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta$$

2. To change from rectangular to polar coordinates, use the formulas

$$r^2 = x^2 + y^2 \qquad \text{and} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

FIGURE 2

EXAMPLE 1 ■ Plotting Points in Polar Coordinates

Plot the points whose polar coordinates are given.

(a) $(1, 3\pi/4)$ **(b)** $(3, -\pi/6)$ **(c)** $(3, 3\pi)$ **(d)** $(-4, \pi/4)$

SOLUTION The points are plotted in Figure 3. Note that the point in part (d) lies 4 units from the origin along the angle $5\pi/4$, because the given value of r is negative.

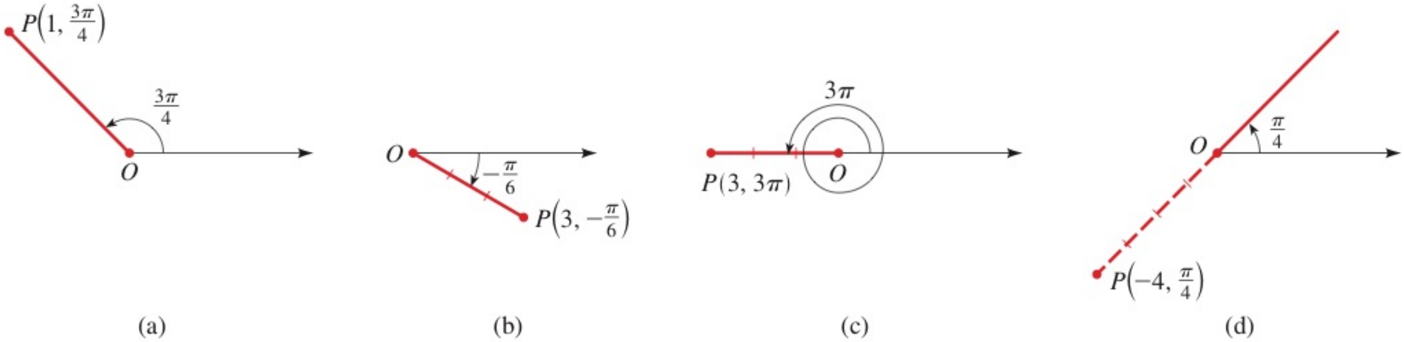


FIGURE 3

EXAMPLE 4 ■ Converting Rectangular Coordinates to Polar Coordinates

Find polar coordinates for the point that has rectangular coordinates $(2, -2)$.

SOLUTION Using $x = 2$, $y = -2$, we get

$$r^2 = x^2 + y^2 = 2^2 + (-2)^2 = 8$$

so $r = 2\sqrt{2}$ or $-2\sqrt{2}$. Also

$$\tan\theta = \frac{y}{x} = \frac{-2}{2} = -1$$

so $\theta = 3\pi/4$ or $-\pi/4$. Since the point $(2, -2)$ lies in Quadrant IV (see Figure 7), we can represent it in polar coordinates as $(2\sqrt{2}, -\pi/4)$ or $(-2\sqrt{2}, 3\pi/4)$.

Polar equations: (equations involving r, θ)

by the relationship with

rectangular coordinate $\leftrightarrow$ polar coordinate

we can convert

rectangular equations $\leftrightarrow$ polar equations

$$x^2 + y^2 = 1 \leftrightarrow r = 1$$

$$x^2 = 4y \leftrightarrow r^2 \cos^2\theta = 4r\sin\theta$$

$$r = \frac{4\sin\theta}{\cos^2\theta} = 4\tan\theta\sec\theta$$

EXAMPLE 6 ■ Converting Equations from Polar to Rectangular Coordinates

Express the polar equation in rectangular coordinates. If possible, determine the graph of the equation from its rectangular form.

(a) $r = 5 \sec \theta$ **(b)** $r = 2 \sin \theta$ **(c)** $r = 2 + 2 \cos \theta$

SOLUTION

(a) Since $\sec \theta = 1/\cos \theta$, we multiply both sides by $\cos \theta$.

$r = 5 \sec \theta$	Polar equation
$r \cos \theta = 5$	Multiply by $\cos \theta$
$x = 5$	Substitute $x = r \cos \theta$

The graph of $x = 5$ is the vertical line in Figure 8.

(b) We multiply both sides of the equation by r, because then we can use the formulas $r^2 = x^2 + y^2$ and $r \sin \theta = y$.

$r = 2 \sin \theta$	Polar equation
$r^2 = 2r \sin \theta$	Multiply by r
$x^2 + y^2 = 2y$	$r^2 = x^2 + y^2$ and $r \sin \theta = y$
$x^2 + y^2 - 2y = 0$	Subtract $2y$
$x^2 + (y - 1)^2 = 1$	Complete the square in y

This is the equation of a circle of radius 1 centered at the point $(0, 1)$. It is graphed in Figure 9.

(c) $r = 2 + 2\cos\theta$

$r^2 = 2r + 2r\cos\theta$

$x^2 + y^2 = \pm 2\sqrt{x^2+y^2} + 2x \Rightarrow (x^2+y^2-2x)^2 = 4(x^2+y^2)$

Distance formula:

71. DISCUSS ■ PROVE: The Distance Formula in Polar Coordinates

(a) Use the Law of Cosines to prove that the distance between the polar points (r_1, θ_1) and (r_2, θ_2) is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)}$$

(b) Find the distance between the points whose polar coordinates are $(3, 3\pi/4)$ and $(1, 7\pi/6)$, using the formula from part (a).

(c) Now convert the points in part (b) to rectangular coordinates. Find the distance between them, using the usual Distance Formula. Do you get the same answer?

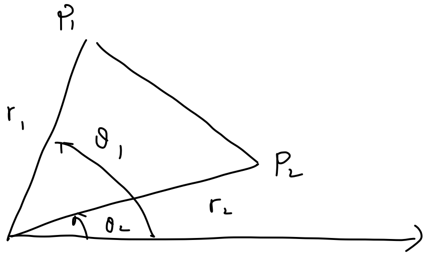
① $P_1 = (r_1\cos\theta_1, r_1\sin\theta_1)$, $P_2 = (r_2\cos\theta_2, r_2\sin\theta_2)$

$$d = \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

② Law of cosines



$$d = |P_1P_2| = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

Graph of polar equations

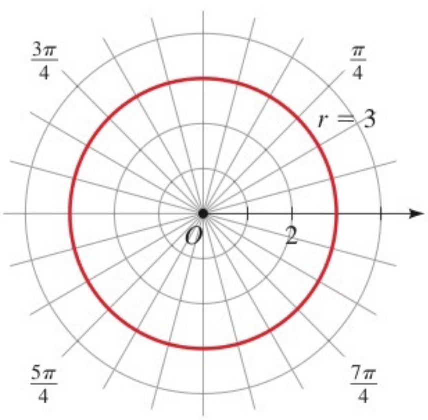


FIGURE 2

EXAMPLE 1 ■ Sketching the Graph of a Polar Equation

Sketch a graph of the equation $r = 3$, and express the equation in rectangular coordinates.

SOLUTION The graph consists of all points whose r-coordinate is 3, that is, all points that are 3 units away from the origin. So the graph is a circle of radius 3 centered at the origin, as shown in Figure 2.

Squaring both sides of the equation, we get

$$r^2 = 3^2 \qquad \text{Square both sides}$$

$$x^2 + y^2 = 9 \qquad \text{Substitute } r^2 = x^2 + y^2$$

So the equivalent equation in rectangular coordinates is $x^2 + y^2 = 9$.

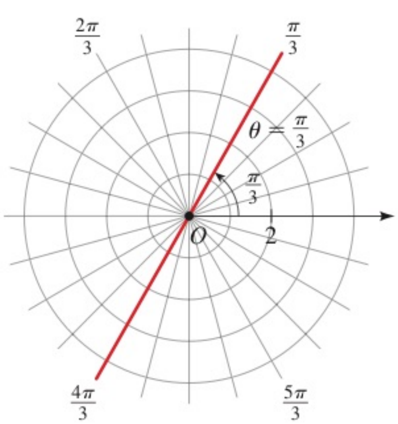


FIGURE 3

EXAMPLE 2 ■ Sketching the Graph of a Polar Equation

Sketch a graph of the equation $\theta = \pi/3$, and express the equation in rectangular coordinates.

SOLUTION The graph consists of all points whose θ-coordinate is $\pi/3$. This is the straight line that passes through the origin and makes an angle of $\pi/3$ with the polar axis (see Figure 3). Note that the points $(r, \pi/3)$ on the line with $r > 0$ lie in Quadrant I, whereas those with $r < 0$ lie in Quadrant III. If the point (x, y) lies on this line, then

$$\frac{y}{x} = \tan\theta = \tan\frac{\pi}{3} = \sqrt{3}$$

Thus the rectangular equation of this line is $y = \sqrt{3}x$.

Now Try Exercise 19 ■

EXAMPLE 3 ■ Sketching the Graph of a Polar Equation

Sketch a graph of the polar equation $r = 2\sin\theta$.

SOLUTION We first use the equation to determine the polar coordinates of several points on the curve. The results are shown in the following table.

$\boldsymbol{\theta}$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\boldsymbol{r = 2\sin\theta}$	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{2}$	1	0

We plot these points in Figure 4 and then join them to sketch the curve. The graph appears to be a circle. We have used values of θ only between 0 and π, since the same points (this time expressed with negative r-coordinates) would be obtained if we allowed θ to range from π to 2π.

The polar equation $r = 2\sin\theta$ in rectangular coordinates is

$$x^2 + (y - 1)^2 = 1$$

(see Section 8.1, Example 6(b)). From the rectangular form of the equation we see that the graph is a circle of radius 1 centered at $(0, 1)$.

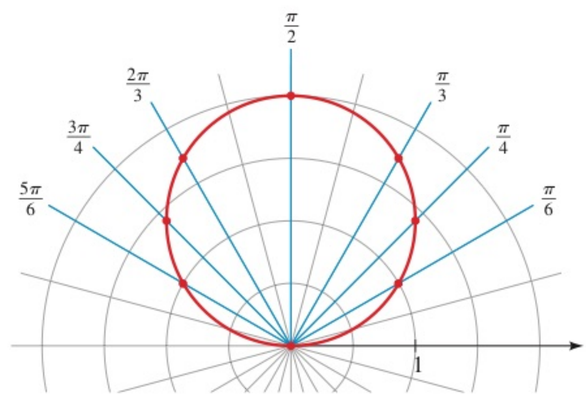


FIGURE 4 $r = 2\sin\theta$

Now Try Exercise 21 ■

EXAMPLE 4 ■ Sketching the Graph of a Cardioid

Sketch a graph of $r = 2 + 2\cos\theta$.

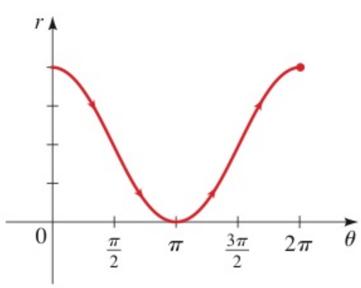


FIGURE 5 $r = 2 + 2\cos\theta$

SOLUTION Instead of plotting points as in Example 3, we first sketch the graph of $r = 2 + 2\cos\theta$ in *rectangular* coordinates in Figure 5. We can think of this graph as a table of values that enables us to read at a glance the values of r that correspond to increasing values of θ. For instance, we see that as θ increases from 0 to $\pi/2$, r (the distance from O) decreases from 4 to 2, so we sketch the corresponding part of the polar graph in Figure 6(a). As θ increases from $\pi/2$ to π, Figure 5 shows that r decreases from 2 to 0, so we sketch the next part of the graph as in Figure 6(b). As θ increases from π to $3\pi/2$, r increases from 0 to 2, as shown in part (c). Finally, as θ increases from $3\pi/2$ to 2π, r increases from 2 to 4, as shown in part (d). If we let θ increase beyond 2π or decrease beyond 0, we would simply retrace our path. Combining the portions of the graph from parts (a) through (d) of Figure 6, we sketch the complete graph in part (e).

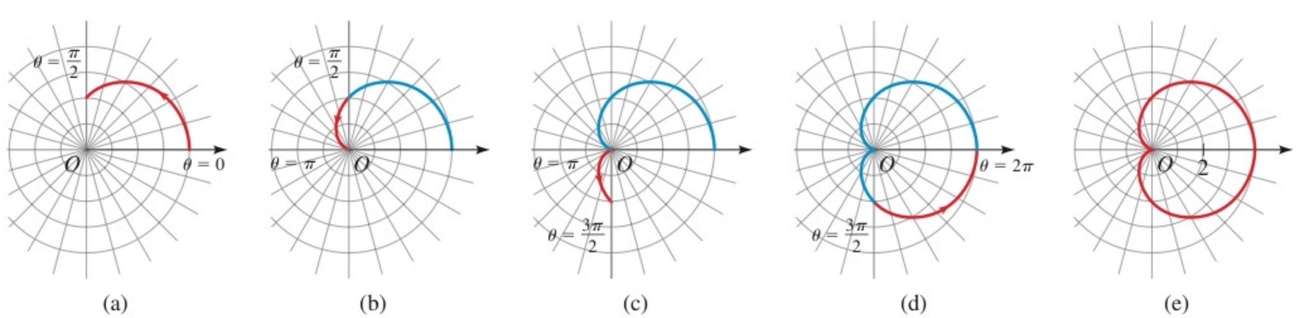


FIGURE 6 Steps in sketching $r = 2 + 2\cos\theta$

EXAMPLE 5 ■ Sketching the Graph of a Four-Leaved Rose

Sketch the curve $r = \cos 2\theta$.

SOLUTION As in Example 4, we first sketch the graph of $r = \cos 2\theta$ in *rectangular* coordinates, as shown in Figure 7. As θ increases from 0 to $\pi/4$, Figure 7 shows that r decreases from 1 to 0, so we draw the corresponding portion of the polar curve in Figure 8 (indicated by ①). As θ increases from $\pi/4$ to $\pi/2$, the value of r goes from 0 to -1. This means that the distance from the origin increases from 0 to 1, but instead of being in Quadrant I, this portion of the polar curve (indicated by ②) lies on the opposite side of the origin in Quadrant III. The remainder of the curve is drawn in a similar fashion, with the arrows and numbers indicating the order in which the portions are traced out. The resulting curve has four petals and is called a **four-leaved rose**.

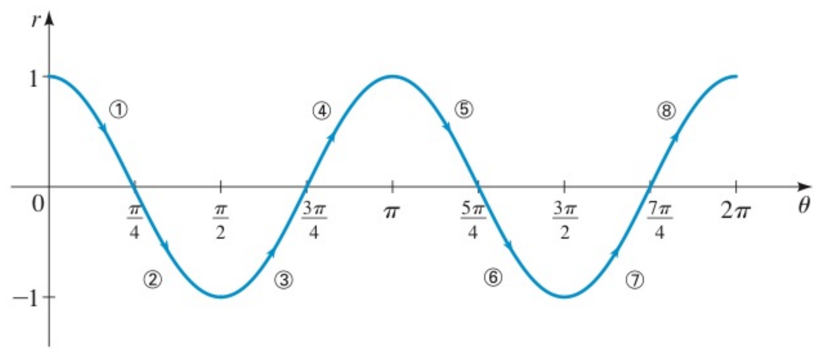


FIGURE 7 Graph of $r = \cos 2\theta$ sketched in rectangular coordinates

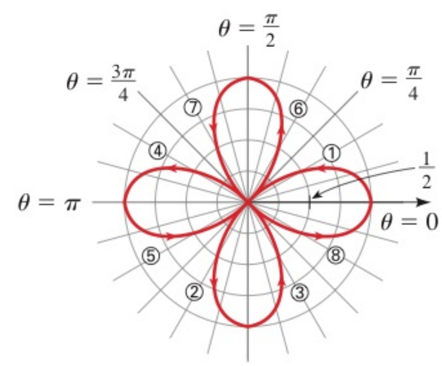


FIGURE 8 Four-leaved rose $r = \cos 2\theta$ sketched in polar coordinates

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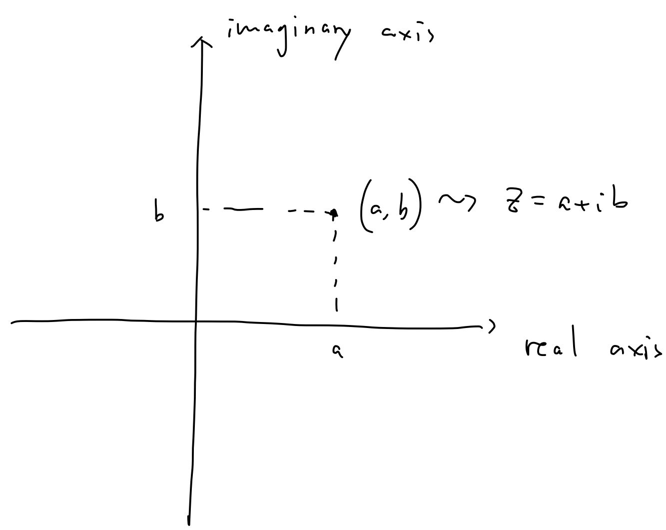
Recall: complex numbers

$$z = a + ib, \quad a, b \in \mathbb{R}, \quad i: \text{ imaginary unit}$$

Graph complex number



(a, b) : rectangular coordinate

$\Updownarrow$

(r, θ) : polar coordinate

EXAMPLE 1 ■ Graphing Complex Numbers

Graph the complex numbers $z_1 = 2 + 3i$, $z_2 = 3 - 2i$, and $z_1 + z_2$.

SOLUTION We have $z_1 + z_2 = (2 + 3i) + (3 - 2i) = 5 + i$. The graph is shown in Figure 2.

Relation to Complex numbers

POLAR FORM OF COMPLEX NUMBERS

A complex number $z = a + bi$ has the **polar form** (or **trigonometric form**)

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta} \qquad (r,\theta) \sim (a,b)$$

where $r = |z| = \sqrt{a^2 + b^2}$ and $\tan\theta = b/a$. The number r is the **modulus** of z, and θ is an **argument** of z.

EXAMPLE 5 ■ Writing Complex Numbers in Polar Form

Write each complex number in polar form.

(a) $1 + i$ **(b)** $-1 + \sqrt{3}i$ **(c)** $-4\sqrt{3} - 4i$ **(d)** $3 + 4i$

SOLUTION These complex numbers are graphed in Figure 8, which helps us find their arguments.

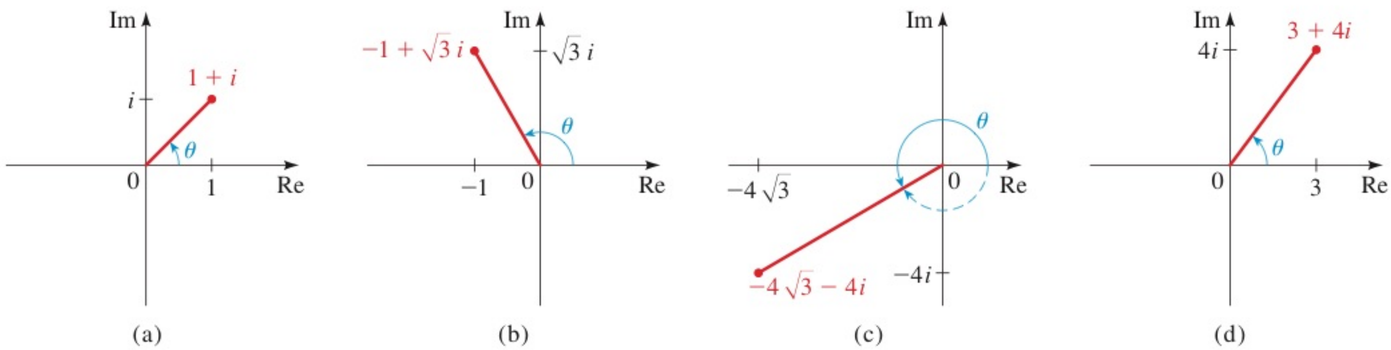


FIGURE 8

$\tan\theta = \frac{1}{1} = 1$

$\theta = \frac{\pi}{4}$

(a) An argument is $\theta = \pi/4$ and $r = \sqrt{1+1} = \sqrt{2}$. Thus

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$\tan\theta = \dfrac{\sqrt{3}}{-1} = -\sqrt{3}$

$\theta = \frac{2\pi}{3}$

(b) An argument is $\theta = 2\pi/3$ and $r = \sqrt{1+3} = 2$. Thus

$$-1 + \sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$\tan\theta = \dfrac{-4}{-4\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\theta = \frac{7\pi}{6}$

(c) An argument is $\theta = 7\pi/6$ (or we could use $\theta = -5\pi/6$), and $r = \sqrt{48 + 16} = 8$. Thus

$$-4\sqrt{3} - 4i = 8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

Rmk: Polar form is useful for calculating multiplication & division of complex numbers

rectangular form is useful for c- - -. addition & substraction

MULTIPLICATION AND DIVISION OF COMPLEX NUMBERS

If the two complex numbers z_1 and z_2 have the polar forms

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

then

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad \text{Multiplication}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \qquad z_2 \neq 0 \qquad \text{Division}$$

EXAMPLE 6 ■ Multiplying and Dividing Complex Numbers

Let

$$z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \qquad \text{and} \qquad z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Find **(a)** $z_1 z_2$ and **(b)** z_1/z_2.

■ De Moivre's Theorem

Repeated use of the Multiplication Formula gives the following useful formula for raising a complex number to a power n for any positive integer n.

DE MOIVRE'S THEOREM

If $z = r(\cos\theta + i\sin\theta)$, then for any integer n

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

EXAMPLE 7 ■ Finding a Power Using De Moivre's Theorem

Find $\left(\frac{1}{2} + \frac{1}{2}i\right)^{10}$.

SOLUTION Since $\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1 + i)$, it follows from Example 5(a) that

$$\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

65. $(-\sqrt{3} + i)^6$

66. $(1 - i)^{10}$

nth Roots of Complex Numbers

An ***n*th root** of a complex number z is any complex number w such that $w^n = z$. De Moivre's Theorem gives us a method for calculating the nth roots of any complex number.

nth ROOTS OF COMPLEX NUMBERS

If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then z has the n distinct nth roots

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, n - 1$.

$x^2 = -1$

$x^3 = -1$, $x^3 = 1$

EXAMPLE 8 ■ Finding Roots of a Complex Number

Find the six sixth roots of $z = -64$, and graph these roots in the complex plane.

SOLUTION In polar form, $z = 64(\cos\pi + i\sin\pi)$. Applying the formula for nth roots with $n = 6$, we get

$$w_k = 64^{1/6}\left[\cos\left(\frac{\pi + 2k\pi}{6}\right) + i\sin\left(\frac{\pi + 2k\pi}{6}\right)\right]$$

for $k = 0, 1, 2, 3, 4, 5$. Using $64^{1/6} = 2$, we find that the six sixth roots of -64 are

$$w_0 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + i$$

$$w_1 = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2i$$

$$w_2 = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -\sqrt{3} + i$$

$$w_3 = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = -\sqrt{3} - i$$

$$w_4 = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -2i$$

$$w_5 = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = \sqrt{3} - i$$

All these points lie on a circle of radius 2, as shown in Figure 9.

Let's focus on roots of unity

Def: $z \in \mathbb{C}$ is called a root of unity, if $\exists\, n > 0$, integer, s.t.

$$z^n = 1$$

97–98 ■ Finding *n*th roots of a Complex Number Let $w = \cos(2\pi/n) + i\sin(2\pi/n)$, where n is a positive integer.

97. Show that the n distinct roots of 1 are

$$1, w, w^2, w^3, \ldots, w^{n-1}$$

98. If $z \neq 0$ and s is any nth root of z, show that the n distinct roots of z are

$$s, sw, sw^2, sw^3, \ldots, sw^{n-1}$$

99. DISCUSS: Sums of Roots of Unity Find the exact values of all three cube roots of 1 (see Exercise 97), and then add them. Do the same for the fourth, fifth, sixth, and eighth roots of 1. What do you think is the sum of the nth roots of 1 for any n?

100. DISCUSS: Products of Roots of Unity Find the product of the three cube roots of 1 (see Exercise 97). Do the same for the fourth, fifth, sixth, and eighth roots of 1. What do you think is the product of the nth roots of 1 for any n?

Thm: $1 + \omega + \omega^2 + \omega^3 + \dots + \omega^{n-1} = 0$

$n=2$: $1, \ -1 \implies 0$

$n=3$: $1, \ -\frac{1}{2}+i\frac{\sqrt{3}}{2}, \ -\frac{1}{2}-i\frac{\sqrt{3}}{2} \implies 0$

$n=4$: $1, \ -1, \ i, \ -i \implies 0$

pf in general:

Let $S = 1 + \omega + \omega^2 + \dots + \omega^{n-1} \in \mathbb{C}$

$$\omega \cdot S = \omega + \omega^2 + \omega^3 + \dots + \omega^{n-1} + \underset{\overset{\shortparallel}{1}}{\omega^n} = S$$

$$\implies (\omega - 1) \cdot S = 0$$

$$\omega - 1 \neq 0 \implies S = 0$$

Thm: $1 \cdot \omega \cdot \omega^2 \cdot \dots \cdot \omega^{n-1} = \omega^{\frac{n(n-1)}{2}} = (-1)^{n-1} = \begin{cases} -1, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}$

n odd: $\implies \frac{n-1}{2}$ is an integer

$$\omega^{\frac{n(n-1)}{2}} = \left(\omega^n\right)^{\frac{n-1}{2}} = 1$$

n even: $\implies \left(\omega^{\frac{n}{2}}\right)^{\cdot(n-1)} = (-1)^{n-1} = -1$

Thm : $1 + \omega + \omega^2 + \omega^3 + \cdots + \omega^{n-1} = 0$

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

$$\Rightarrow \omega^k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$$

$$\Rightarrow \quad 1 + \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cos\frac{6\pi}{n} + \cdots + \cos\frac{2(n-1)\pi}{n} = 0 \qquad ①$$

$$\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \sin\frac{6\pi}{n} + \cdots + \sin\frac{2(n-1)\pi}{n} = 0$$

Question: Is $\cos\frac{2\pi}{n}$ an algebraic number?

i.e. $\exists\, f(x) \in \mathbb{Q}[x]$, s.t. $f\left(\cos\frac{2\pi}{n}\right) = 0$

Ans: Yes: recall: $\cos nx = f_n(\cos x)$, $f_n \in \mathbb{Z}[x]$

①: $1 + f_1\left(\cos\frac{2\pi}{n}\right) + f_2\left(\cos\frac{2\pi}{n}\right) + \cdots + f_{n-1}\left(\cos\frac{2\pi}{n}\right) = 0$

Let $f(x) = 1 + f_1(x) + f_2(x) + \cdots + f_{n-1}(x)$