4/9

Analytic Trigonometry

Trigonometric identities

FUNDAMENTAL TRIGONOMETRIC IDENTITIES

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Even-Odd Identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \sec\left(\frac{\pi}{2} - x\right) = \csc x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$$

Using these to prove some trigonometric identities

EXAMPLE 1 ■ Simplifying a Trigonometric Expression

Simplify the expression $\cos t + \tan t \sin t$.

SOLUTION We start by rewriting the expression in terms of sine and cosine.

$$\begin{aligned}
\cos t + \tan t \sin t &= \cos t + \left(\frac{\sin t}{\cos t}\right)\sin t && \text{Reciprocal identity} \\
&= \frac{\cos^2 t + \sin^2 t}{\cos t} && \text{Common denominator} \\
&= \frac{1}{\cos t} && \text{Pythagorean identity} \\
&= \sec t && \text{Reciprocal identity}
\end{aligned}$$

Simplify the expression $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1+\sin\theta}$.

SOLUTION We combine the fractions by using a common denominator.

$$\begin{aligned}
\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{1+\sin\theta} &= \frac{\sin\theta\,(1+\sin\theta) + \cos^2\theta}{\cos\theta\,(1+\sin\theta)} && \text{Common denominator} \\
&= \frac{\sin\theta + \sin^2\theta + \cos^2\theta}{\cos\theta\,(1+\sin\theta)} && \text{Distribute } \sin\theta \\
&= \frac{\sin\theta + 1}{\cos\theta\,(1+\sin\theta)} && \text{Pythagorean identity} \\
&= \frac{1}{\cos\theta} = \sec\theta && \text{Cancel, and use reciprocal identity}
\end{aligned}$$

General steps: use Sin & cos to express all th other terms
then use:

$$\sin^2(x) + \cos^2(x) = 1$$

EXAMPLE 3 ■ Proving an Identity by Rewriting in Terms of Sine and Cosine

Consider the equation $\cos\theta\,(\sec\theta - \cos\theta) = \sin^2\theta$.

(a) Verify algebraically that the equation is an identity.

(b) Confirm graphically that the equation is an identity.

Euler's formula:

$$e^{ix} = \cos x + i \sin x, \quad \forall x \in \mathbb{R} \quad (\text{even all } x \in \mathbb{C})$$

A "fake" proof: Using "series expansion"

$$e^{x} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \quad \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \right)$$

$$\cos x = 1 + \sum_{\substack{n \geq 2, \\ n \text{ even}}} \frac{(-1)^{\frac{n}{2}}}{n!} x^n \quad \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots \right)$$

$$\sin x = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{(-1)^{\frac{n-1}{2}}}{n!} x^n \quad \left(x - \frac{x^3}{6} + \cdots \right)$$

$\Rightarrow$ extend the definition to all complex numbers

$$e^{ix} = 1 + \sum_{n=1}^{\infty} \frac{(ix)^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{i^n x^n}{n!} = \left(1 + ix - \frac{x^2}{2} - \frac{ix^3}{6} + \frac{x^4}{24} + \cdots \right)$$

$$= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} \right)$$

$$+ i \left(x - \frac{x^3}{6} + \cdots \right.$$

$$= \cos x + i \sin x$$

Recall: $e^{a+b} = e^a \cdot e^b, \quad \forall a, b \in \mathbb{C}$

$$e^{ix+iy} = e^{ix} \cdot e^{iy}$$

$$\cos(x+y) + i \sin(x+y) = (\cos x + i \sin x) \cdot (\cos y + i \sin y)$$

$$= \cos x \cos y - \sin x \sin y + i (\sin x \cos y + \cos x \sin y)$$

ADDITION AND SUBTRACTION FORMULAS

Formulas for sine:
$$\sin(s + t) = \sin s \cos t + \cos s \sin t$$
$$\sin(s - t) = \sin s \cos t - \cos s \sin t$$

Formulas for cosine:
$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$
$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

Formulas for tangent:
$$\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$
$$\tan(s - t) = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

Another "Geometric proof"

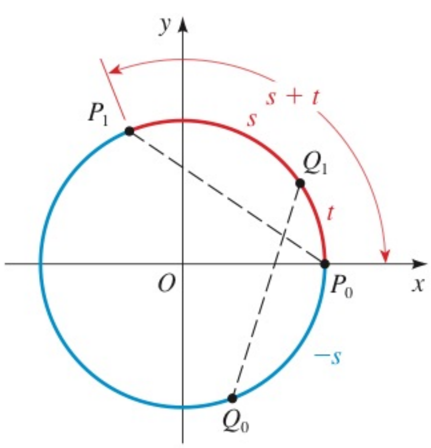


FIGURE 1

Proof of Addition Formula for Cosine To prove the formula

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

we use Figure 1. In the figure, the distances t, $s + t$, and $-s$ have been marked on the unit circle, starting at $P_0(1, 0)$ and terminating at Q_1, P_1, and Q_0, respectively. The coordinates of these points are as follows:

$$P_0(1, 0) \qquad Q_0(\cos(-s), \sin(-s))$$
$$P_1(\cos(s + t), \sin(s + t)) \qquad Q_1(\cos t, \sin t)$$

Since $\cos(-s) = \cos s$ and $\sin(-s) = -\sin s$, it follows that the point Q_0 has the coordinates $Q_0(\cos s, -\sin s)$. Notice that the distances between P_0 and P_1 and between Q_0 and Q_1 measured along the arc of the circle are equal. Since equal arcs are subtended by equal chords, it follows that $d(P_0, P_1) = d(Q_0, Q_1)$. Using the Distance Formula, we get

$$\sqrt{[\cos(s + t) - 1]^2 + [\sin(s + t) - 0]^2} = \sqrt{(\cos t - \cos s)^2 + (\sin t + \sin s)^2}$$

EXAMPLE 1 ■ Using the Addition and Subtraction Formulas

Find the exact value of each expression.

(a) $\cos 75°$ **(b)** $\cos \dfrac{\pi}{12}$

SOLUTION

(a) Notice that $75° = 45° + 30°$. Since we know the exact values of sine and cosine at $45°$ and $30°$, we use the Addition Formula for Cosine to get

$$\begin{aligned} \cos 75° &= \cos(45° + 30°) \\ &= \cos 45° \cos 30° - \sin 45° \sin 30° \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{2}\sqrt{3} - \sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

(b) Since $\dfrac{\pi}{12} = \dfrac{\pi}{4} - \dfrac{\pi}{6}$, the Subtraction Formula for Cosine gives

$$\begin{aligned} \cos \frac{\pi}{12} &= \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\ &= \cos \frac{\pi}{4} \cos \frac{\pi}{6} + \sin \frac{\pi}{4} \sin \frac{\pi}{6} \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4} \end{aligned}$$

EXAMPLE 2 ■ Using the Addition Formula for Sine

Find the exact value of the expression $\sin 20° \cos 40° + \cos 20° \sin 40°$.

SOLUTION We recognize the expression as the right-hand side of the Addition Formula for Sine with $s = 20°$ and $t = 40°$. So we have

$$\sin 20° \cos 40° + \cos 20° \sin 40° = \sin(20° + 40°) = \sin 60° = \frac{\sqrt{3}}{2}$$

EXAMPLE 4 ■ Proving an Identity

Verify the identity $\dfrac{1 + \tan x}{1 - \tan x} = \tan\left(\dfrac{\pi}{4} + x\right)$.

SOLUTION Starting with the right-hand side and using the Addition Formula for Tangent, we get

$$\text{RHS} = \tan\left(\frac{\pi}{4} + x\right) = \frac{\tan\dfrac{\pi}{4} + \tan x}{1 - \tan\dfrac{\pi}{4}\tan x}$$

$$= \frac{1 + \tan x}{1 - \tan x} = \text{LHS}$$

EXAMPLE 4 ■ Proving an Identity

Verify the identity $\dfrac{1 + \tan x}{1 - \tan x} = \tan\left(\dfrac{\pi}{4} + x\right)$.

SOLUTION Starting with the right-hand side and using the Addition Formula for Tangent, we get

$$\text{RHS} = \tan\left(\frac{\pi}{4} + x\right) = \frac{\tan\dfrac{\pi}{4} + \tan x}{1 - \tan\dfrac{\pi}{4}\tan x}$$

$$= \frac{1 + \tan x}{1 - \tan x} = \text{LHS}$$

EXAMPLE 6 ■ Simplifying an Expression Involving Inverse Trigonometric Functions

Write $\sin(\cos^{-1}x + \tan^{-1}y)$ as an algebraic expression in x and y, where $-1 \le x \le 1$ and y is any real number.

SOLUTION Let $\theta = \cos^{-1}x$ and $\phi = \tan^{-1}y$. Using the methods of Section 6.4, we sketch triangles with angles θ and ϕ such that $\cos\theta = x$ and $\tan\phi = y$ (see Figure 2). From the triangles we have

$$\sin\theta = \sqrt{1 - x^2} \qquad \cos\phi = \frac{1}{\sqrt{1 + y^2}} \qquad \sin\phi = \frac{y}{\sqrt{1 + y^2}}$$

From the Addition Formula for Sine we have

$$\begin{aligned}
\sin(\cos^{-1}x + \tan^{-1}y) &= \sin(\theta + \phi) \\
&= \sin\theta\cos\phi + \cos\theta\sin\phi && \text{Addition Formula for Sine} \\
&= \sqrt{1 - x^2}\,\frac{1}{\sqrt{1 + y^2}} + x\,\frac{y}{\sqrt{1 + y^2}} && \text{From triangles} \\
&= \frac{1}{\sqrt{1 + y^2}}\left(\sqrt{1 - x^2} + xy\right) && \text{Factor } \frac{1}{\sqrt{1 + y^2}}
\end{aligned}$$

Functions of the form $A \sin x + B \cos x$

SUMS OF SINES AND COSINES

If A and B are real numbers, then

$$A\sin x + B\cos x = k\sin(x + \phi)$$

where $k = \sqrt{A^2 + B^2}$ and ϕ satisfies

$$\cos\phi = \frac{A}{\sqrt{A^2 + B^2}} \qquad \text{and} \qquad \sin\phi = \frac{B}{\sqrt{A^2 + B^2}}$$

EXAMPLE 9 ■ Graphing a Trigonometric Function

Write the function $f(x) = -\sin 2x + \sqrt{3}\cos 2x$ in the form $k\sin(2x + \phi)$, and use the new form to graph the function.

SOLUTION Since $A = -1$ and $B = \sqrt{3}$, we have $k = \sqrt{A^2 + B^2} = \sqrt{1 + 3} = 2$. The angle ϕ satisfies $\cos\phi = -\frac{1}{2}$ and $\sin\phi = \sqrt{3}/2$. From the signs of these quantities we conclude that ϕ is in Quadrant II. Thus $\phi = 2\pi/3$. By the preceding theorem we can write

$$f(x) = -\sin 2x + \sqrt{3}\cos 2x = 2\sin\left(2x + \frac{2\pi}{3}\right)$$

Using the form

$$f(x) = 2\sin 2\left(x + \frac{\pi}{3}\right)$$

we see that the graph is a sine curve with amplitude 2, period $2\pi/2 = \pi$, and phase shift $-\pi/3$. The graph is shown in Figure 5.

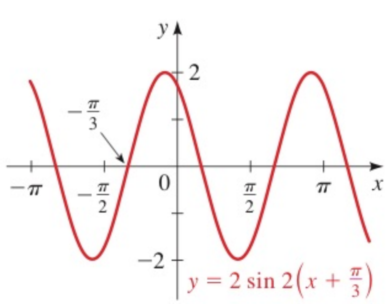


FIGURE 5

Double-Angle, Half-Angle, product-sum formulas

DOUBLE-ANGLE FORMULAS

Formula for sine: $\sin 2x = 2 \sin x \cos x$

Formulas for cosine:
$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1 \end{aligned}$$

Formula for tangent: $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

Proof of Double-Angle Formulas for Cosine

$$\begin{aligned} \cos 2x &= \cos(x + x) \\ &= \cos x \cos x - \sin x \sin x \\ &= \cos^2 x - \sin^2 x \end{aligned}$$

The second and third formulas for $\cos 2x$ are obtained from the formula we just proved and the Pythagorean identity. Substituting $\cos^2 x = 1 - \sin^2 x$ gives

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= (1 - \sin^2 x) - \sin^2 x \\ &= 1 - 2\sin^2 x \end{aligned}$$

The third formula is obtained in the same way, by substituting $\sin^2 x = 1 - \cos^2 x$. ■

e.g. ① $x = \frac{\pi}{6}$, $2x = \frac{\pi}{3}$

EXAMPLE 1 ■ Using the Double-Angle Formulas

② If $\cos x = -\frac{2}{3}$ and x is in Quadrant II, find $\cos 2x$ and $\sin 2x$.

EXAMPLE 2 ■ A Triple-Angle Formula

Write $\cos 3x$ in terms of $\cos x$.

SOLUTION

$$\begin{aligned}
\cos 3x &= \cos(2x + x) \\
&= \cos 2x \cos x - \sin 2x \sin x && \text{Addition formula} \\
&= (2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x && \text{Double-Angle Formulas} \\
&= 2\cos^3 x - \cos x - 2\sin^2 x \cos x && \text{Expand} \\
&= 2\cos^3 x - \cos x - 2\cos x\,(1 - \cos^2 x) && \text{Pythagorean identity} \\
&= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x && \text{Expand} \\
&= 4\cos^3 x - 3\cos x && \text{Simplify}
\end{aligned}$$

4/11

Review: $\sin(x+y) =$

$\cos(x+y) =$

$\sin(x-y) =$

$\cos(x-y) =$

$\tan(x+y) =$

$\tan(x-y) =$

Double-Angle

$\sin(2x) =$

$\cos(2x) =$

$\tan(2x) =$

Half-Angle Formula

FORMULAS FOR LOWERING POWERS

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

HALF-ANGLE FORMULAS

$$\sin \frac{u}{2} = \pm\sqrt{\frac{1 - \cos u}{2}} \qquad \cos \frac{u}{2} = \pm\sqrt{\frac{1 + \cos u}{2}}$$

$$\tan \frac{u}{2} = \frac{1 - \cos u}{\sin u} = \frac{\sin u}{1 + \cos u}$$

The choice of the $+$ or $-$ sign depends on the quadrant in which $u/2$ lies.

e.g.

EXAMPLE 5 ■ **Using a Half-Angle Formula**

Find the exact value of sin 22.5°.

SOLUTION Since 22.5° is half of 45°, we use the Half-Angle Formula for Sine with $u = 45°$. We choose the + sign because 22.5° is in the first quadrant.

$$\sin \frac{45°}{2} = \sqrt{\frac{1 - \cos 45°}{2}} \qquad \text{Half-Angle Formula}$$

$$= \sqrt{\frac{1 - \sqrt{2}/2}{2}} \qquad \cos 45° = \sqrt{2}/2$$

$$= \sqrt{\frac{2 - \sqrt{2}}{4}} \qquad \text{Common denominator}$$

$$= \tfrac{1}{2}\sqrt{2 - \sqrt{2}} \qquad \text{Simplify}$$

Univeral formula:

$$\sin x = \frac{2\tan\frac{x}{2}}{1 + \tan(\frac{x}{2})^2}$$

$$\cos x = \frac{1 - \tan(\frac{x}{2})^2}{1 + \tan(\frac{x}{2})^2}$$

PRODUCT-TO-SUM FORMULAS

$$\sin u \cos v = \tfrac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \tfrac{1}{2}[\sin(u + v) - \sin(u - v)]$$

$$\cos u \cos v = \tfrac{1}{2}[\cos(u + v) + \cos(u - v)]$$

$$\sin u \sin v = \tfrac{1}{2}[\cos(u - v) - \cos(u + v)]$$

EXAMPLE 9 ■ Expressing a Trigonometric Product as a Sum

Express $\sin 3x \sin 5x$ as a sum of trigonometric functions.

SOLUTION Using the fourth Product-to-Sum Formula with $u = 3x$ and $v = 5x$ and the fact that cosine is an even function, we get

$$\begin{aligned}
\sin 3x \sin 5x &= \tfrac{1}{2}[\cos(3x - 5x) - \cos(3x + 5x)] \\
&= \tfrac{1}{2}\cos(-2x) - \tfrac{1}{2}\cos 8x \\
&= \tfrac{1}{2}\cos 2x - \tfrac{1}{2}\cos 8x
\end{aligned}$$

SUM-TO-PRODUCT FORMULAS

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

EXAMPLE 11 ■ Proving an Identity

Verify the identity $\dfrac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x$.

SOLUTION We apply the second Sum-to-Product Formula to the numerator and the third formula to the denominator.

$$\text{LHS} = \frac{\sin 3x - \sin x}{\cos 3x + \cos x} = \frac{2 \cos \dfrac{3x+x}{2} \sin \dfrac{3x-x}{2}}{2 \cos \dfrac{3x+x}{2} \cos \dfrac{3x-x}{2}} \qquad \text{Sum-to-Product Formulas}$$

$$= \frac{2 \cos 2x \sin x}{2 \cos 2x \cos x} \qquad \text{Simplify}$$

$$= \frac{\sin x}{\cos x} = \tan x = \text{RHS} \qquad \text{Cancel}$$

Trigonometric equations

equations involving trigonometric functions

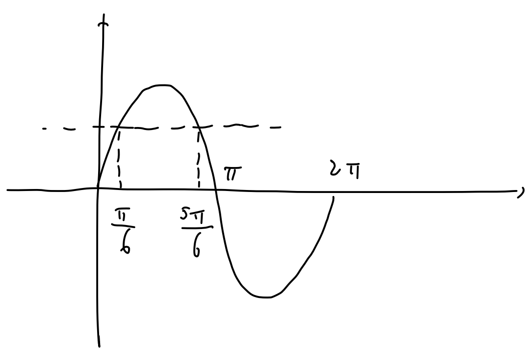
Basic trigonometric equation

Solving any trigonometric equation always reduces to solving a **basic trigonometric equation**—an equation of the form $T(\theta) = c$, where T is a trigonometric function and c is a constant. In the next three examples we solve such basic equations.

EXAMPLE 1 ■ Solving a Basic Trigonometric Equation

Solve the equation $\sin\theta = \dfrac{1}{2}$.

Step 1 : Find all the solutions in one complete period $[0, 2\pi)$



$\theta = \dfrac{\pi}{6}$ & $\dfrac{5\pi}{6}$

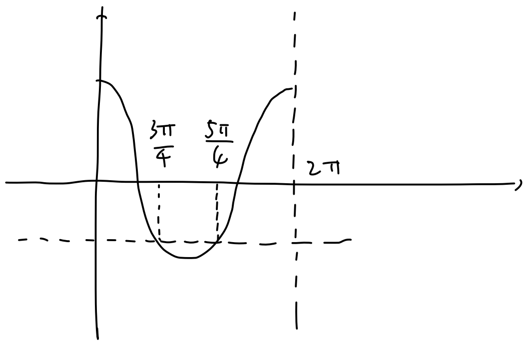
Step 2 : Find all the solutions (by adding multiples of the period)

$\theta = \dfrac{\pi}{6} + 2k\pi, \quad \dfrac{5\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}$

EXAMPLE 2 ■ Solving a Basic Trigonometric Equation

Solve the equation $\cos\theta = -\dfrac{\sqrt{2}}{2}$, and list eight specific solutions.

Step 1: Find all the solutions in one complete period $[0, 2\pi)$



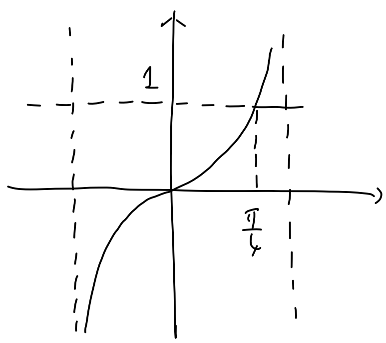
$$\theta = \frac{3\pi}{4} \quad \& \quad \frac{5\pi}{4}$$

Step 2: Find all the solutions (by adding multiples of the period)

$$\theta = \frac{3\pi}{4} + 2k\pi, \quad \frac{5\pi}{4} + 2k\pi, \quad k \in \mathbb{Z}$$

e.g. $\tan\theta = 1$

Step 1: Find all the solutions in one complete period $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



,

$\theta = \frac{\pi}{4}$

Step 2: Find all the solutions (by adding multiples of the period)

$\theta = \frac{\pi}{4} + k\pi, \quad k \in \mathbb{Z}$

Generally:

$$\sin(k(\theta - \theta_0)) = c$$

complete period: $[\theta_0, \theta_0 + \frac{2\pi}{|k|})$

period $= \frac{2\pi}{|k|}$

$$\cos(k(\theta - \theta_0)) = c$$

complete period: $[\theta_0, \theta_0 + \frac{2\pi}{|k|})$

period $= \frac{2\pi}{|k|}$

$$\tan(k(\theta - \theta_0)) = c$$

complete period: $(\theta_0 - \frac{\pi}{2|k|}, \theta_0 + \frac{\pi}{2|k|})$

period $= \frac{\pi}{|k|}$

Trigonometric of quadratic type:

$$f(T(\theta)) = 0, \quad f: \text{quadratic polynomial}$$

Step 1: solving the original quadratic equation: $f(x) = 0$

get x_1, x_2

Step 2: Solving $T(\theta) = x_1$ or x_2

EXAMPLE 6 ■ A Trigonometric Equation of Quadratic Type

Solve the equation $2\cos^2\theta - 7\cos\theta + 3 = 0$.

$$f(x) = 2x^2 - 7x + 3, \quad T(\theta) = \cos\theta$$

$$f(x) = (2x-1)(x-3) \Rightarrow x = \frac{1}{2} \text{ or } 3$$

$$\Rightarrow \cos\theta = \frac{1}{2} \text{ or } \underbrace{\cos\theta = 3}_{\text{no solution}}$$

$$\theta = \frac{\pi}{3} + 2k\pi \text{ or } \frac{5\pi}{3} + 2k\pi, \ k \in \mathbb{Z}$$

Solving trigonometric equations by using identities:

$\sin^2(x) + \cos^2(x) = 1$ & Sum+Product formulas

EXAMPLE 1 ■ Using a Trigonometric Identity

Solve the equation $1 + \sin\theta = 2\cos^2\theta$.

$1 + \sin\theta = 2(1 - \sin^2\theta)$

$\Rightarrow \quad 2\sin^2\theta + \sin\theta - 1 = 0$

$f(x) = 2x^2 + x - 1 = (2x-1)(x+1) \Rightarrow x = \frac{1}{2}$ or -1

$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} + 2k\pi, \ \frac{5\pi}{6} + 2k\pi, \ k \in \mathbb{Z}$

$\sin\theta = -1 \Rightarrow \theta = \frac{3\pi}{2} + 2k\pi, \ k \in \mathbb{Z}$

EXAMPLE 2 ■ Using a Trigonometric Identity

Solve the equation $\sin 2\theta - \cos\theta = 0$.

$2\sin\theta\cos\theta - \cos\theta = \cos\theta(2\sin\theta - 1) = 0$

$\cos\theta = 0, \quad \sin\theta = \frac{1}{2}$

$\downarrow \quad \downarrow$

$\frac{\pi}{2}, \ \frac{3\pi}{2}, \ \frac{\pi}{6}, \ \frac{5\pi}{6} + 2k\pi, \ k \in \mathbb{Z}$