4/2.

Review: <u>Trigonometric functions</u>

- Signed distance on the unit circle

$t > 0$: counterclockwise travel
$t < 0$: clockwise travel
$\Big\}$ $\Rightarrow$ to the terminal pt $P(t)$

- $\sin(t)$ = y-coordinate of $P(t)$
$\cos(t)$ = x-coordinate of $P(t)$
$\Big\}$ $\Rightarrow$ defined over $\mathbb{R}$

$\tan(t) = \dfrac{\sin(t)}{\cos(t)}$, $\quad \sec(t) = \dfrac{1}{\cos(t)}$ $\Rightarrow$ domain: $\{x \in \mathbb{R} : x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$

$\cot(t) = \dfrac{\cos(t)}{\sin(t)}$, $\quad \csc(t) = \dfrac{1}{\sin(t)}$, $\Rightarrow$ domain: $\{x \in \mathbb{R} : x \neq k\pi, k \in \mathbb{Z}\}$

- equations:

$\sin^2(t) + \cos^2(t) = 1$, $\qquad 1 + \tan^2(t) = \dfrac{1}{\cos^2(t)} = \sec^2(t)$

$\sin(t+2\pi) = \sin(t)$, $\qquad \cos(t+2\pi) = \cos(t)$

$\sin(t+\pi) = -\sin(t)$, $\qquad \cos(t+\pi) = -\cos(t)$

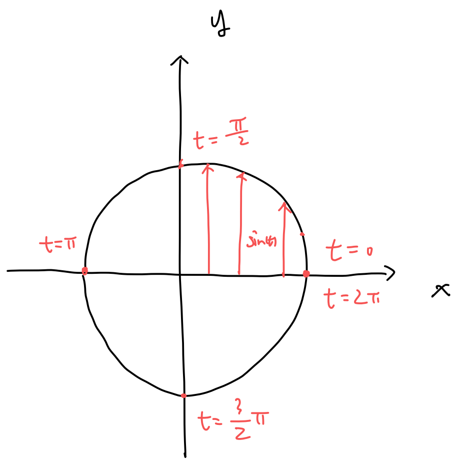
$\sin(t+\frac{\pi}{2}) = \cos(t)$, $\qquad \cos(t+\frac{\pi}{2}) = -\sin(t)$

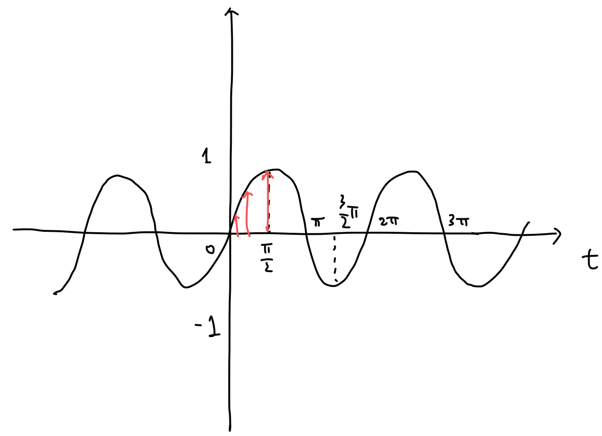
- $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

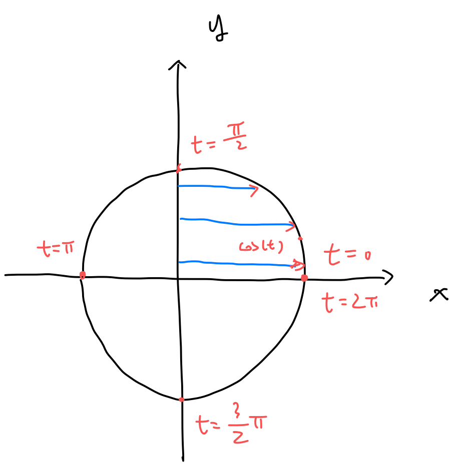
Graph of trigonometric functions

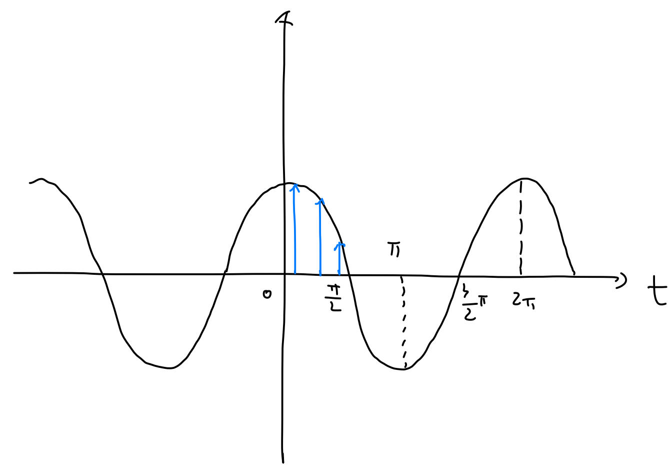
$\sin(t)$





$\cos(t)$





$$\sin\left(t + \frac{\pi}{2}\right) = \cos(t)$$

$$\cos\left(t + \frac{\pi}{2}\right) = -\sin(t)$$

SHIFTED SINE AND COSINE CURVES

The sine and cosine curves

$$y = a \sin k(x - b) \qquad \text{and} \qquad y = a \cos k(x - b) \qquad (k > 0)$$

have **amplitude** $|a|$, **period** $2\pi/k$, and **horizontal shift** b.

An appropriate interval on which to graph one complete period is $[b, b + (2\pi/k)]$.

e.g.

Find the amplitude, period, and horizontal shift of $y = 3 \sin 2\left(x - \frac{\pi}{4}\right)$, and graph one complete period.

SOLUTION We get the amplitude, period, and horizontal shift from the form of the function as follows:

$$\text{amplitude} = |a| = 3 \qquad \text{period} = \frac{2\pi}{k} = \frac{2\pi}{2} = \pi$$

$$y = 3 \sin 2\left(x - \frac{\pi}{4}\right)$$

$$\text{horizontal shift} = \frac{\pi}{4}\ (\text{to the right})$$

Since the horizontal shift is $\pi/4$ and the period is π, one complete period occurs on the interval

$$\left[\frac{\pi}{4}, \frac{\pi}{4} + \pi\right] = \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$$

EXAMPLE 5 ■ A Horizontally Shifted Cosine Curve

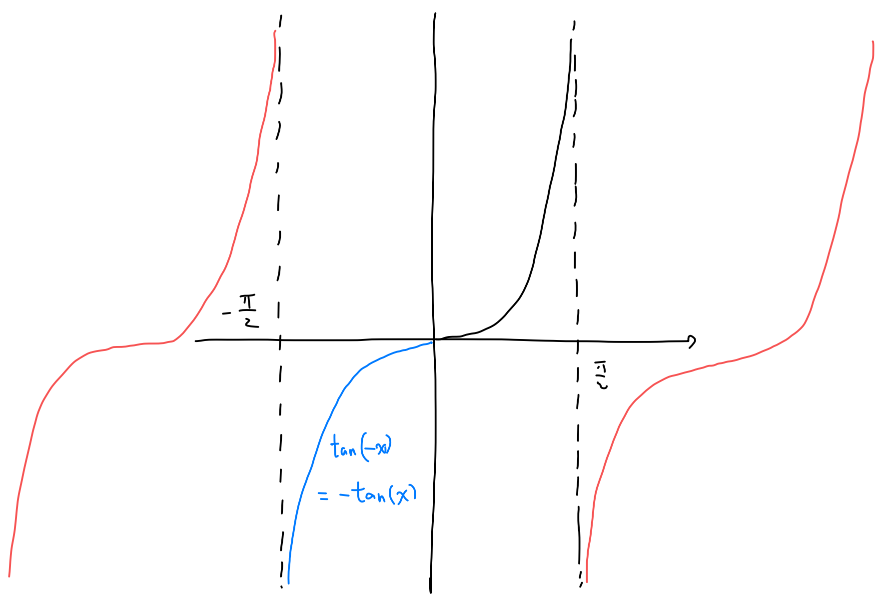
Find the amplitude, period, and horizontal shift of $y = \frac{3}{4} \cos\left(2x + \frac{2\pi}{3}\right)$, and graph one complete period.

More graphs:

$$y = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

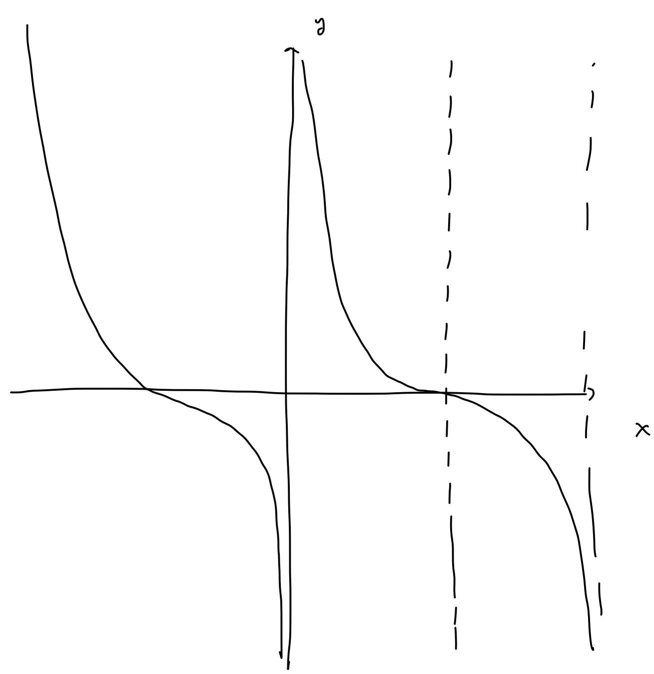
- $x \in [0, \frac{\pi}{2})$, $\tan(x) > 0$, increasing $\left(\begin{matrix} \sin \nearrow \\ \cos x \searrow \end{matrix} \right)$

  and $x \to \frac{\pi}{2}^-$, $\tan(x) \to +\infty$



$\tan(x+\pi) = \tan(x)$

$$\cot(x) = \frac{1}{\tan(x)}$$



To graph the cosecant and secant functions, we use the reciprocal identities

$$\csc x = \frac{1}{\sin x} \qquad \text{and} \qquad \sec x = \frac{1}{\cos x}$$

So to graph $y = \csc x$, we take the reciprocals of the y-coordinates of the points of the graph of $y = \sin x$. (See Figure 3.) Similarly, to graph $y = \sec x$, we take the reciprocals of the y-coordinates of the points of the graph of $y = \cos x$. (See Figure 4.)

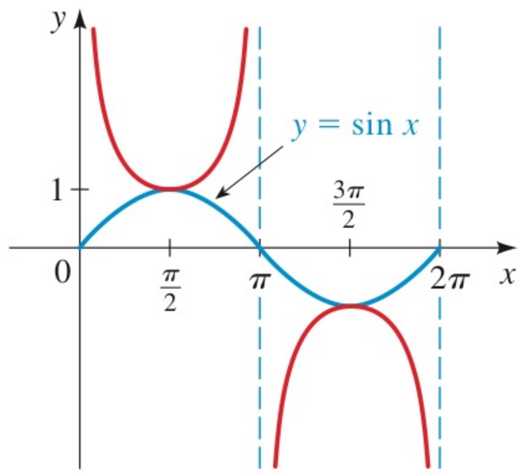


FIGURE 3 One period of $y = \csc x$

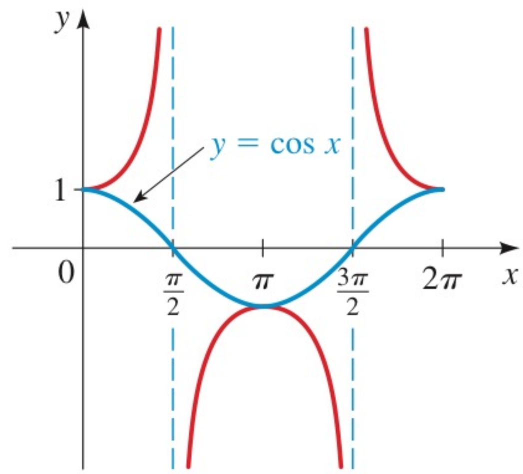


FIGURE 4 One period of $y = \sec x$

Let's consider more closely the graph of the function $y = \csc x$ on the interval $0 < x < \pi$. We need to examine the values of the function near 0 and π, since at these values $\sin x = 0$, and $\csc x$ is thus undefined. We see that

$$\csc x \to \infty \qquad \text{as} \qquad x \to 0^+$$

$$\csc x \to \infty \qquad \text{as} \qquad x \to \pi^-$$

TANGENT AND COTANGENT CURVES

The functions

$$y = a \tan kx \qquad \text{and} \qquad y = a \cot kx \qquad (k > 0)$$

have period π/k.

$a \tan k(x-b)$ $\qquad$ $a \cot k(x-b)$

COSECANT AND SECANT CURVES

The functions

$$y = a \csc kx \qquad \text{and} \qquad y = a \sec kx \qquad (k > 0)$$

have period $2\pi/k$.

$a \csc k(x-b)$ $\qquad$ $a \sec k(x-b)$

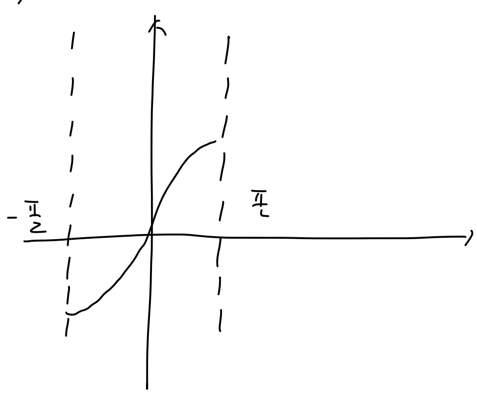
Inverse trigonometric functions

For a one-to-one function $f: A \to B$

$$f^{-1}: B \to A$$

trigonometric functions are not one-to-one, how to define inverse?

Example: $\sin(x)$



consider $f(x) = \sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2} \; : \; [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$

$\Rightarrow$ one-to-one, so that we can define

$$f^{-1}(x) = \sin^{-1}(x) = \arcsin(x) \cdot \quad [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$\left(\sin^{-1}(x) = t\right) :=$ t is the unique signed distance such that $\sin(t) = x$

Find each value.

(a) $\sin^{-1}\frac{1}{2}$ **(b)** $\sin^{-1}\left(-\frac{1}{2}\right)$

$$\sin^{-1}\frac{1}{2} = \frac{\pi}{6} \qquad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

$$\sin(\sin^{-1} x) = x \quad \text{for} \quad -1 \le x \le 1$$
$$\sin^{-1}(\sin x) = x \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

e.g

Find each value.

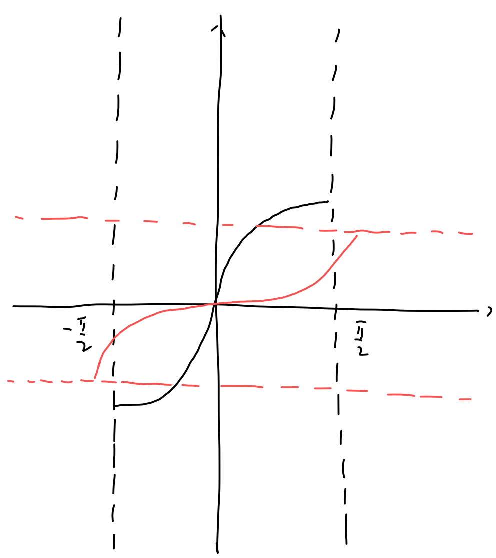
(a) $\sin^{-1}\left(\sin \frac{\pi}{3}\right)$

$= \frac{\pi}{3}$

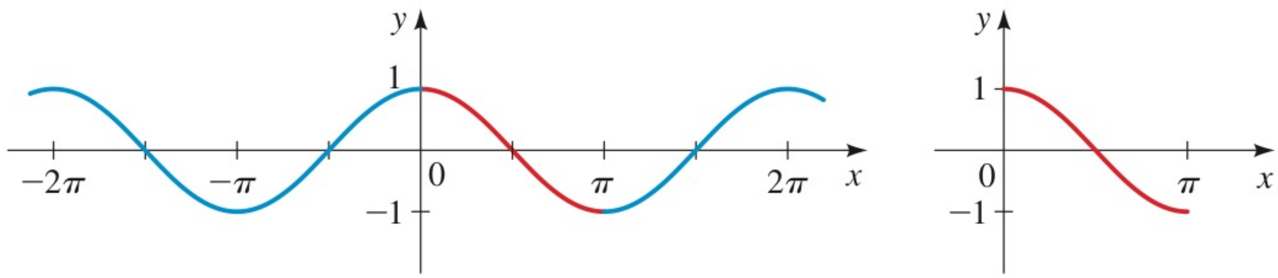
(b) $\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$

$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ not $\frac{2\pi}{3}$

Graph:



If the domain of the cosine function is restricted to the interval $[0, \pi]$, the resulting function is one-to-one and so has an inverse. We choose this interval because on it, cosine attains each of its values exactly once (see Figure 3).



$y = \cos x$ $\qquad$ $y = \cos x,\ 0 \le x \le \pi$

DEFINITION OF THE INVERSE COSINE FUNCTION

The **inverse cosine function** is the function $\cos^{-1}$ with domain $[-1, 1]$ and range $[0, \pi]$ defined by

$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x$$

The inverse cosine function is also called **arccosine**, denoted by **arccos**.

EXAMPLE 4 ■ Evaluating the Inverse Cosine Function

Find each value.

(a) $\cos^{-1} \dfrac{\sqrt{3}}{2}$ $= \dfrac{\pi}{6}$ $\qquad$ **(b)** $\cos^{-1} 0$ $= \dfrac{\pi}{2}$ $\qquad$ **(c)** $\cos^{-1}\left(-\dfrac{1}{2}\right)$ $= \dfrac{2\pi}{3}$

Thus $y = \cos^{-1} x$ *is the number in the interval* $[0, \pi]$ *whose cosine is* x. The following **cancellation properties** follow from the inverse function properties.

$$\boxed{\begin{array}{lll} \cos(\cos^{-1} x) = x & \text{for} & -1 \le x \le 1 \\ \cos^{-1}(\cos x) = x & \text{for} & 0 \le x \le \pi \end{array}}$$

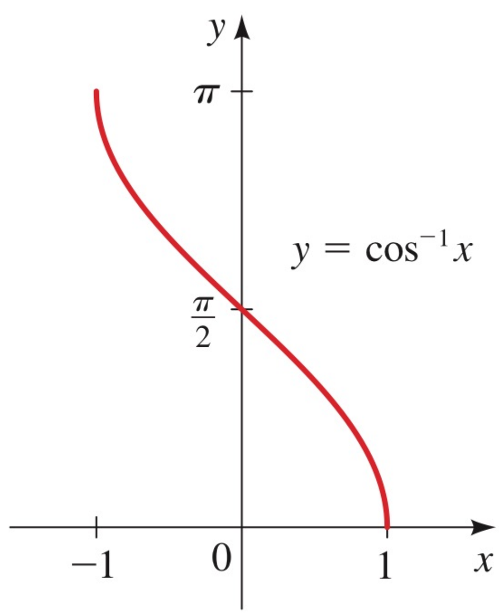
The graph of $y = \cos^{-1} x$ is shown in Figure 4; it is obtained by reflecting the graph of $y = \cos x$, $0 \le x \le \pi$, in the line $y = x$.

Find each value.

(a) $\cos^{-1}\left(\cos \dfrac{2\pi}{3}\right)$ **(b)** $\cos^{-1}\left(\cos \dfrac{5\pi}{3}\right)$

(a) $= \dfrac{2\pi}{3}$

(b) $= \cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3} \neq \dfrac{5\pi}{3}$



4.4

DEFINITION OF THE INVERSE TANGENT FUNCTION

The **inverse tangent function** is the function $\tan^{-1}$ with domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$ defined by

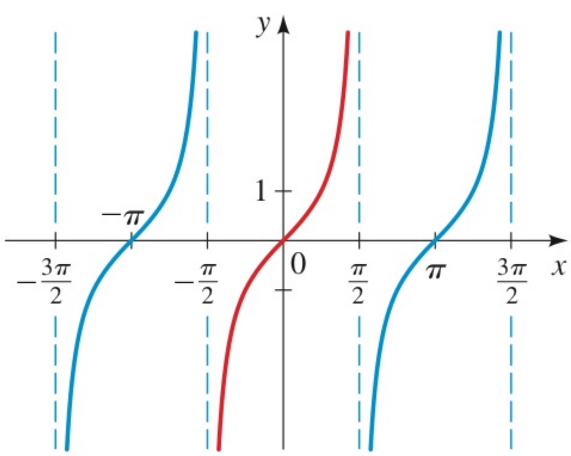
$$\tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x$$

The inverse tangent function is also called **arctangent**, denoted by **arctan**.

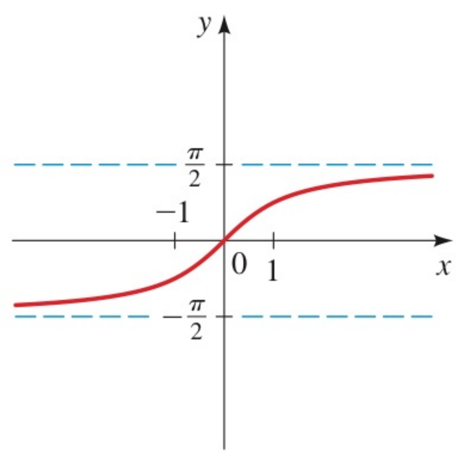
Thus $y = \tan^{-1} x$ *is the number in the interval* $(-\pi/2, \pi/2)$ *whose tangent is* x. The following **cancellation properties** follow from the inverse function properties.

$$\tan(\tan^{-1} x) = x \quad \text{for} \quad x \in \mathbb{R}$$
$$\tan^{-1}(\tan x) = x \quad \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Figure 5 shows the graph of $y = \tan x$ on the interval $(-\pi/2, \pi/2)$ and the graph of its inverse function, $y = \tan^{-1} x$.



$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$



$y = \tan^{-1} x$

EXAMPLE 6 ■ Evaluating the

Find each value.

(a) $\tan^{-1} 1$ **(b)** $\tan^{-1} \sqrt{3}$

||

$\frac{\pi}{4}$

||

$\frac{\pi}{3}$

Angle measure

Q: How to measure an angle?

360° → whole angle.

but this is not universal!

on another universal, people may use 720°, 330° to denote the "whole angle"

A good way:

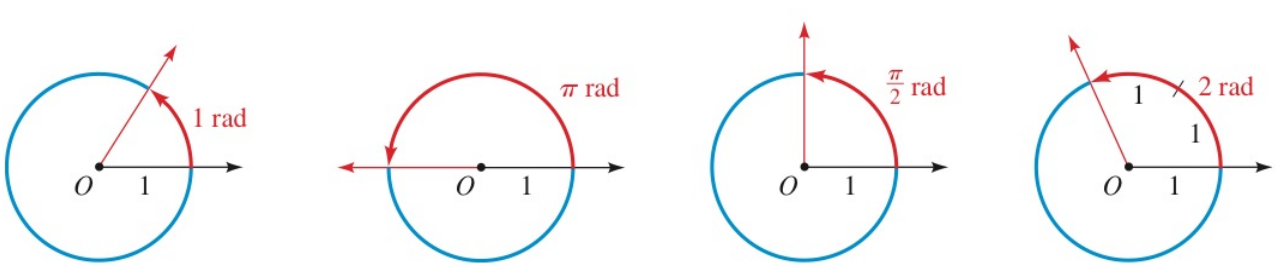
Angle Measure

The **measure** of an angle is the amount of rotation about the vertex required to move R_1 onto R_2. Intuitively, this is how much the angle "opens." One unit of measurement for angles is the **degree**. An angle of measure 1 degree is formed by rotating the initial side $\frac{1}{360}$ of a complete revolution. In calculus and other branches of mathematics a more natural method of measuring angles is used: *radian measure*. The amount an angle opens is measured along the arc of a circle of radius 1 with its center at the vertex of the angle.

DEFINITION OF RADIAN MEASURE

If a circle of radius 1 is drawn with the vertex of an angle at its center, then the measure of this angle in **radians** (abbreviated **rad**) is the length of the arc that subtends the angle (see Figure 2).

The circumference of the circle of radius 1 is 2π, so a complete revolution has measure 2π rad, a straight angle has measure π rad, and a right angle has measure $\pi/2$ rad. An angle that is subtended by an arc of length 2 along the unit circle has radian measure 2 (see Figure 3).



RELATIONSHIP BETWEEN DEGREES AND RADIANS

$$180° = \pi \text{ rad} \qquad 1 \text{ rad} = \left(\frac{180}{\pi}\right)^{\circ} \qquad 1° = \frac{\pi}{180} \text{ rad}$$

1. To convert degrees to radians, multiply by $\dfrac{\pi}{180}$.
2. To convert radians to degrees, multiply by $\dfrac{180}{\pi}$.

To get some idea of the size of a radian, notice that

$$1 \text{ rad} \approx 57.296° \qquad \text{and} \qquad 1° \approx 0.01745 \text{ rad}$$

An angle θ of measure 1 rad is shown in Figure 4.

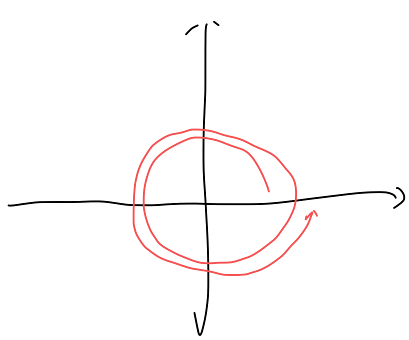
EXAMPLE 1 ■ Converting Between Radians and Degrees

(a) Express 60° in radians. **(b)** Express $\dfrac{\pi}{6}$ rad in degrees.

SOLUTION The relationship between degrees and radians gives

(a) $60° = 60\left(\dfrac{\pi}{180}\right) \text{ rad} = \dfrac{\pi}{3} \text{ rad}$ **(b)** $\dfrac{\pi}{6} \text{ rad} = \left(\dfrac{\pi}{6}\right)\left(\dfrac{180}{\pi}\right) = 30°$

<u>Note</u>: radian can exceeds 2π!



Length of a Circular Arc

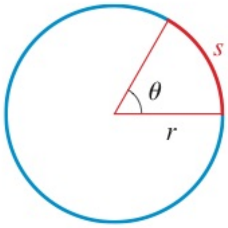


FIGURE 9 $s = \theta r$

An angle whose radian measure is θ is subtended by an arc that is the fraction $\theta/(2\pi)$ of the circumference of a circle. Thus in a circle of radius r the length s of an arc that subtends the angle θ (see Figure 9) is

$$s = \frac{\theta}{2\pi} \times \text{circumference of circle}$$

$$= \frac{\theta}{2\pi}(2\pi r) = \theta r$$

LENGTH OF A CIRCULAR ARC

In a circle of radius r the length s of an arc that subtends a central angle of θ radians is

$$s = r\theta$$

Solving for θ, we get the important formula

$$\boxed{\theta = \frac{s}{r}}$$

Area of a Circular Sector

The area of a circle of radius r is $A = \pi r^2$. A sector of this circle with central angle θ has an area that is the fraction $\theta/(2\pi)$ of the area of the entire circle (see Figure 11). So the area of this sector is

$$A = \frac{\theta}{2\pi} \times \text{area of circle}$$

$$= \frac{\theta}{2\pi}(\pi r^2) = \frac{1}{2}r^2\theta$$

AREA OF A CIRCULAR SECTOR

In a circle of radius r the area A of a sector with a central angle of θ radians is

$$A = \frac{1}{2}r^2\theta$$

Sloving triangles

Applications of Trigonometry of Right Triangles

A triangle has six parts: three angles and three sides. To **solve a triangle** means to determine all of its parts from the information known about the triangle, that is, to determine the lengths of the three sides and the measures of the three angles.

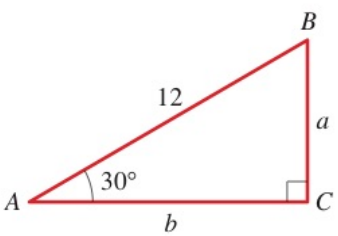


FIGURE 7

EXAMPLE 4 ■ Solving a Right Triangle

Solve triangle ABC, shown in Figure 7.

SOLUTION It's clear that $\angle B = 60°$. From Figure 7 we have

$$\sin 30° = \frac{a}{12} \qquad \text{Definition of sine}$$

$$a = 12 \sin 30° \qquad \text{Multiply by 12}$$

$$= 12\left(\tfrac{1}{2}\right) = 6 \qquad \text{Evaluate}$$

Also from Figure 7 we have

$$\cos 30° = \frac{b}{12} \qquad \text{Definition of cosine}$$

$$b = 12 \cos 30° \qquad \text{Multiply by 12}$$

$$= 12\left(\frac{\sqrt{3}}{2}\right) = 6\sqrt{3} \qquad \text{Evaluate}$$

Area of triangles

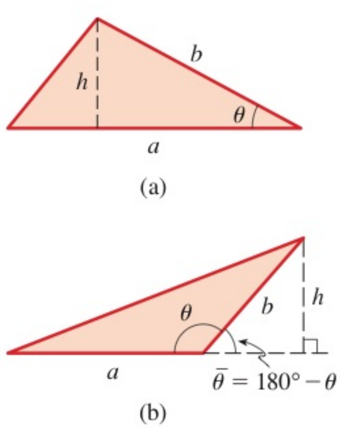


FIGURE 16

Areas of Triangles

We conclude this section with an application of the trigonometric functions that involves angles that are not necessarily acute. More extensive applications appear in Sections 6.5 and 6.6.

The area of a triangle is $\mathscr{A} = \frac{1}{2} \times \text{base} \times \text{height}$. If we know two sides and the included angle of a triangle, then we can find the height using the trigonometric functions, and from this we can find the area.

If θ is an acute angle, then the height of the triangle in Figure 16(a) is given by $h = b \sin \theta$. Thus the area is

$$\mathscr{A} = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2} ab \sin \theta$$

If the angle θ is not acute, then from Figure 16(b) we see that the height of the triangle is

$$h = b \sin(180° - \theta) = b \sin \theta$$

The Law of Sines

The **Law of Sines** says that in any triangle the lengths of the sides are proportional to the sines of the corresponding opposite angles. To state this law (or formula) more easily, we follow the convention of labeling the angles of a triangle as A, B, and C and the lengths of the corresponding opposite sides as a, b, and c, as in Figure 2.

THE LAW OF SINES

In triangle ABC we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

A satellite orbiting the earth passes directly overhead at observation stations in Phoenix and Los Angeles, 340 mi apart. At an instant when the satellite is between these two stations, its angle of elevation is simultaneously observed to be 60° at Phoenix and 75° at Los Angeles. How far is the satellite from Los Angeles?

SOLUTION We need to find the distance b in Figure 4. Since the sum of the angles in any triangle is 180°, we see that $\angle C = 180° - (75° + 60°) = 45°$ (see Figure 4), so we have

$$\frac{\sin B}{b} = \frac{\sin C}{c} \qquad \text{Law of Sines}$$

$$\frac{\sin 60°}{b} = \frac{\sin 45°}{340} \qquad \text{Substitute}$$

$$b = \frac{340 \sin 60°}{\sin 45°} \approx 416 \qquad \text{Solve for } b$$

The distance of the satellite from Los Angeles is approximately 416 mi.

EXAMPLE 2 ■ Solving a Triangle (SAA)

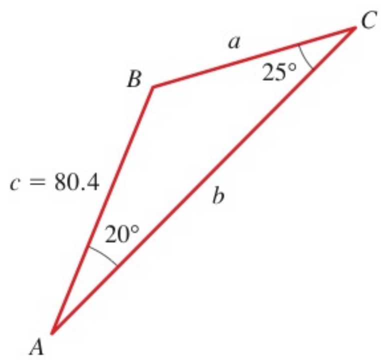


FIGURE 5

Solve the triangle in Figure 5.

SOLUTION First, $\angle B = 180° - (20° + 25°) = 135°$. Since side c is known, to find side a, we use the relation

$$\frac{\sin A}{a} = \frac{\sin C}{c} \qquad \text{Law of Sines}$$

$$a = \frac{c \sin A}{\sin C} = \frac{80.4 \sin 20°}{\sin 25°} \approx 65.1 \qquad \text{Solve for } a$$

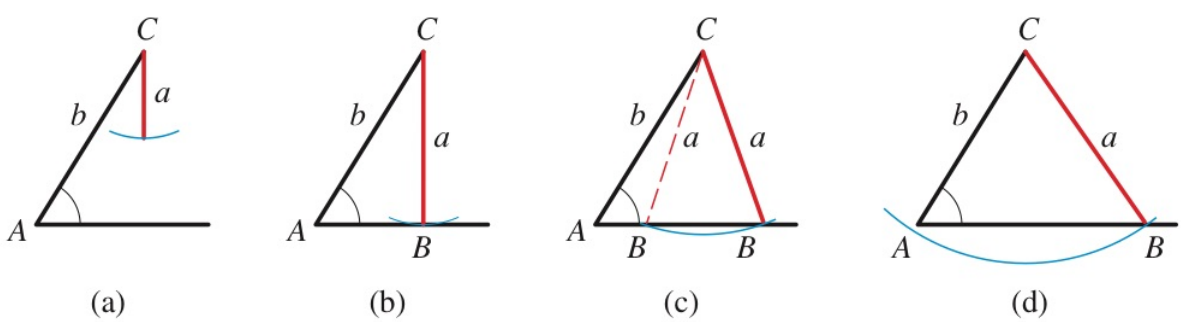
Similarly, to find b, we use

$$\frac{\sin B}{b} = \frac{\sin C}{c} \qquad \text{Law of Sines}$$

$$b = \frac{c \sin B}{\sin C} = \frac{80.4 \sin 135°}{\sin 25°} \approx 134.5 \qquad \text{Solve for } b$$

Ambiguous case:

In Examples 1 and 2 a unique triangle was determined by the information given. This is always true of Case 1 (ASA or SAA). But in Case 2 (SSA) there may be two triangles, one triangle, or no triangle with the given properties. For this reason, Case 2 is sometimes called the **ambiguous case**. To see why this is so, we show in Figure 6 the possibilities when angle A and sides a and b are given. In part (a) no solution is possible, since side a is too short to complete the triangle. In part (b) the solution is a right triangle. In part (c) two solutions are possible, and in part (d) there is a unique triangle with the given properties. We illustrate the possibilities of Case 2 in the following examples.



$$\left.\begin{array}{l}\text{ASA:}\\ \text{SAA:}\end{array}\right\} \Rightarrow \text{only 1 solution}$$

SSA: 0, 1, 2

EXAMPLE 3 ■ SSA, the One-Solution Case

Solve triangle ABC, where $\angle A = 45°$, $a = 7\sqrt{2}$, and $b = 7$.

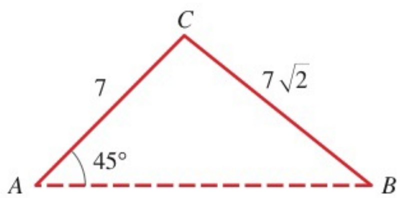


FIGURE 7

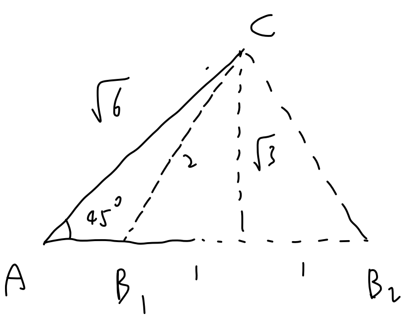
SOLUTION We first sketch the triangle with the information we have (see Figure 7). Our sketch is necessarily tentative, since we don't yet know the other angles. Nevertheless, we can now see the possibilities.

We first find $\angle B$.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \text{Law of Sines}$$

$$\sin B = \frac{b \sin A}{a} = \frac{7}{7\sqrt{2}} \sin 45° = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2} \qquad \text{Solve for } \sin B$$

SSA, two solutions case:



$b = \sqrt{6}$, $a = 2$

$A = 45° = \frac{\pi}{4}$

EXAMPLE 5 ■ SSA, the No-Solution Case

Solve triangle ABC, where $\angle A = 42°$, $a = 70$, and $b = 122$.

SOLUTION To organize the given information, we sketch the diagram in Figure 10. Let's try to find $\angle B$. We have

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad \text{Law of Sines}$$

$$\sin B = \frac{b \sin A}{a} = \frac{122 \sin 42°}{70} \approx 1.17 \qquad \text{Solve for } \sin B$$

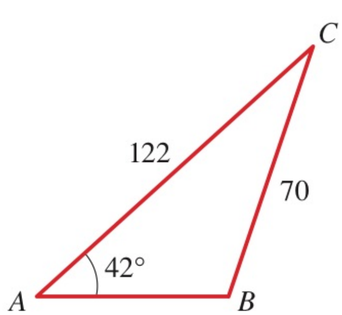


FIGURE 10

SSA: find the other sin

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \sin B \text{ may have } 0, 1, 2 \text{ choice(s)}$$

The Law of Cosines

The Law of Sines cannot be used directly to solve triangles if we know two sides and the angle between them or if we know all three sides (these are Cases 3 and 4 of the preceding section). In these two cases the **Law of Cosines** applies.

THE LAW OF COSINES

In any triangle ABC (see Figure 1) we have

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

EXAMPLE 2 ■ SSS, the Law of Cosines

The sides of a triangle are $a = 5$, $b = 8$, and $c = 12$ (see Figure 4). Find the angles of the triangle.

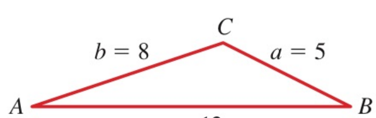


FIGURE 4

SOLUTION We first find $\angle A$. From the Law of Cosines, $a^2 = b^2 + c^2 - 2bc \cos A$. Solving for $\cos A$, we get

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{8^2 + 12^2 - 5^2}{2(8)(12)} = \frac{183}{192} = 0.953125$$

Using a calculator, we find that $\angle A = \cos^{-1}(0.953125) \approx 18°$. In the same way we get

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{5^2 + 12^2 - 8^2}{2(5)(12)} = 0.875$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{5^2 + 8^2 - 12^2}{2(5)(8)} = -0.6875$$

Using a calculator, we find that

$$\angle B = \cos^{-1}(0.875) \approx 29° \qquad \text{and} \qquad \angle C = \cos^{-1}(-0.6875) \approx 133°$$

Of course, once two angles have been calculated, the third can more easily be found from the fact that the sum of the angles of a triangle is 180°. However, it's a good idea to calculate all three angles using the Law of Cosines and add the three angles as a check on your computations.

EXAMPLE 3 ■ SAS, the Law of Cosines

Solve triangle ABC, where $\angle A = 46.5°$, $b = 10.5$, and $c = 18.0$.

SOLUTION We can find a using the Law of Cosines.

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$= (10.5)^2 + (18.0)^2 - 2(10.5)(18.0)(\cos 46.5°) \approx 174.05$$

Thus $a \approx \sqrt{174.05} \approx 13.2$. We also use the Law of Cosines to find $\angle B$ and $\angle C$, as in Example 2.

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{13.2^2 + 18.0^2 - 10.5^2}{2(13.2)(18.0)} \approx 0.816477$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{13.2^2 + 10.5^2 - 18.0^2}{2(13.2)(10.5)} \approx -0.142532$$

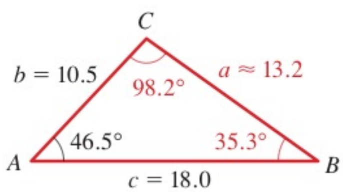


FIGURE 5

Using a calculator, we find that

$$\angle B = \cos^{-1}(0.816477) \approx 35.3° \qquad \text{and} \qquad \angle C = \cos^{-1}(-0.142532) \approx 98.2°$$

To summarize: $\angle B \approx 35.3°$, $\angle C \approx 98.2°$, and $a \approx 13.2$. (See Figure 5.)

Now Try Exercise 13 ■

Heron's formula :

The Area of a Triangle

An interesting application of the Law of Cosines involves a formula for finding the area of a triangle from the lengths of its three sides (see Figure 8).

HERON'S FORMULA

The area $\mathscr{A}$ of triangle ABC is given by

$$\mathscr{A} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a + b + c)$ is the **semiperimeter** of the triangle; that is, s is half the perimeter.

Proof We start with the formula $\mathscr{A} = \frac{1}{2}ab \sin C$ from Section 6.3. Thus

$$\begin{aligned}
\mathscr{A}^2 &= \tfrac{1}{4}a^2b^2 \sin^2 C \\
&= \tfrac{1}{4}a^2b^2(1 - \cos^2 C) && \text{Pythagorean identity} \\
&= \tfrac{1}{4}a^2b^2(1 - \cos C)(1 + \cos C) && \text{Factor}
\end{aligned}$$

Next, we write the expressions $1 - \cos C$ and $1 + \cos C$ in terms of a, b, and c. By the Law of Cosines we have

$$\begin{aligned}
\cos C &= \frac{a^2 + b^2 - c^2}{2ab} && \text{Law of Cosines} \\
1 + \cos C &= 1 + \frac{a^2 + b^2 - c^2}{2ab} && \text{Add 1} \\
&= \frac{2ab + a^2 + b^2 - c^2}{2ab} && \text{Common denominator} \\
&= \frac{(a+b)^2 - c^2}{2ab} && \text{Factor} \\
&= \frac{(a+b+c)(a+b-c)}{2ab} && \text{Difference of squares}
\end{aligned}$$

Similarly,

$$1 - \cos C = \frac{(c + a - b)(c - a + b)}{2ab}$$

Substituting these expressions in the formula we obtained for $\mathscr{A}^2$ gives

$$\begin{aligned}
\mathscr{A}^2 &= \tfrac{1}{4}a^2b^2 \frac{(a+b+c)(a+b-c)}{2ab}\, \frac{(c+a-b)(c-a+b)}{2ab} \\
&= \frac{(a+b+c)}{2}\, \frac{(a+b-c)}{2}\, \frac{(c+a-b)}{2}\, \frac{(c-a+b)}{2} \\
&= s(s-c)(s-b)(s-a)
\end{aligned}$$

Heron's Formula now follows from taking the square root of each side. ■