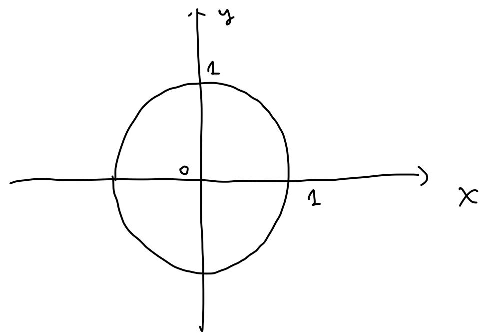
Trigonometric functions

Unit circle $x^2 + y^2 = 1$

circle, with center $(0,0)$, radius $= 1$



Question: How to locate a pt on the unit circle?

- Use coordinate
- Use signed distance

Signed distance: given $t > 0$, travel around the unit circle starting from $(1, 0)$, counterclockwise, to distance t

given $t < 0$, travel around the unit circle starting from $(1, 0)$, clockwise, to distance $-t$

Note: total length of the unit circle is 2π

e.g., π, 2π, $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{5\pi}{3}$, 4π

$-\pi$, $-\frac{\pi}{2}$, -3π, $-\frac{\pi}{3}$

terminal pt of signed distance

there should exist a correspondence between

coordinate & signed distance

π,	2π,	$\frac{\pi}{2}$,	$\frac{\pi}{3}$,	$\frac{2\pi}{3}$,	$\frac{5\pi}{3}$,	4π
$(-1,0)$	$(1,0)$	$(0,1)$	$(\frac{1}{2},\frac{\sqrt{3}}{2})$	$(-\frac{1}{2},\frac{\sqrt{3}}{2})$	$(\frac{1}{2},-\frac{\sqrt{3}}{2})$	$(1,0)$

$-\pi$,	$-\frac{\pi}{2}$,	-3π,	$-\frac{\pi}{3}$
$(-1,0)$	$(0,-1)$	$(-1,0)$	$(\frac{1}{2},-\frac{\sqrt{3}}{2})$

Define: trigonometric functions

given $t \in \mathbb{R}$

$\sin(t) = x$ - coordinate of the terminal pt of signed distance t

$\cos(t) = y$ - coordinate of the terminal pt of signed distance t

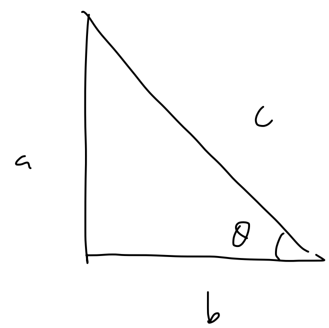
$\sec(t) = \frac{1}{\sin(t)} = \frac{1}{x\text{-coordinte}}$

$\csc(t) = \frac{1}{\cos(t)} = \frac{1}{y\text{-coo}}$

$\tan(t) = \frac{y}{x}$

$\cot(t) = \frac{x}{y}$

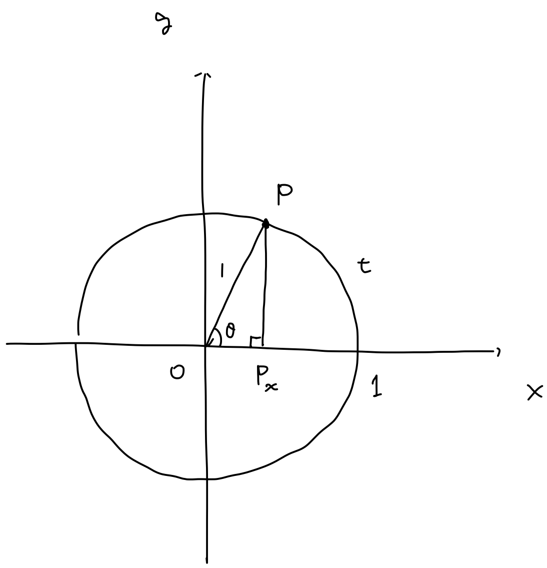
Relation to trigonometric function of angles



$\sin\theta = \frac{a}{c}$

$\cos\theta = \frac{b}{c}$

$\tan\theta = \frac{a}{b}$



given signed distance t, let P be the terminal pt. $P = (x, y) = (\cos t, \sin t)$

connect OP, draw a vertical line passing through P

set $\theta = \angle PO x$

then $\sin\theta = \frac{PP_x}{OP} = \frac{\sin t}{1} = \sin t$

$\cos\theta = \frac{OP_x}{OP} = \frac{\cos t}{1} = \cos t$

Secret: $\pi = 180^\circ$

$\Rightarrow \sin\frac{\pi}{2} = \sin 90^\circ = 1$

$\sin\frac{\pi}{3} = \sin 60^\circ = \frac{\sqrt{3}}{2}$

$\sin\frac{\pi}{4} = \sin 45^\circ = \frac{\sqrt{2}}{2}$

$\sin\frac{\pi}{6} = \sin 30^\circ = \frac{1}{2}$

Some special values:

	$\cos t$	$\sin t$	$\tan t$
0	1	0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	0	1	$+\infty$
$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{3}}$
π	-1	0	0

Fundamental Identities

The trigonometric functions are related to each other through equations called **trigonometric identities**. We give the most important ones in the following box.*

FUNDAMENTAL IDENTITIES

Reciprocal Identities

$$\csc t = \frac{1}{\sin t} \qquad \sec t = \frac{1}{\cos t} \qquad \cot t = \frac{1}{\tan t} \qquad \tan t = \frac{\sin t}{\cos t} \qquad \cot t = \frac{\cos t}{\sin t}$$

Pythagorean Identities

$$\sin^2 t + \cos^2 t = 1 \qquad \tan^2 t + 1 = \sec^2 t \qquad 1 + \cot^2 t = \csc^2 t$$

periodicity:

$\sin(t + 2\pi) = \sin(t)$. $\cos(t+2\pi) = \cos(t)$

$\tan(t + \pi) = \tan(t)$

How to find the value of $\sin(t)$ & $\cos(t)$.?

given any $t \in \mathbb{R}$, use translation $t + 2\pi k$ to find a value $t' \in [0, 2\pi]$. s.t. $t - t' \in 2\pi k$, $k \in \mathbb{Z}$

$\left.\begin{aligned} \sin(t) &= \sin(t') \\ \cos(t) &= \cos(t') \end{aligned}\right\} \Rightarrow$ locate t' on the unit circle e.g.

8. **(a)** $\cos \frac{19\pi}{6}$ **(b)** $\cos\left(-\frac{7\pi}{6}\right)$ **(c)** $\cos\left(-\frac{\pi}{6}\right)$

$$\frac{19\pi}{6} = 2\pi + \frac{7\pi}{6}$$

$$\cos\left(\frac{19\pi}{6}\right) = \cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

Graphs

TABLE 1

t	$\sin t$	$\cos t$
$0 \to \frac{\pi}{2}$	$0 \to 1$	$1 \to 0$
$\frac{\pi}{2} \to \pi$	$1 \to 0$	$0 \to -1$
$\pi \to \frac{3\pi}{2}$	$0 \to -1$	$-1 \to 0$
$\frac{3\pi}{2} \to 2\pi$	$-1 \to 0$	$0 \to 1$

So the sine and cosine functions repeat their values in any interval of length 2π. To sketch their graphs, we first graph one period. To sketch the graphs on the interval $0 \le t \le 2\pi$, we could try to make a table of values and use those points to draw the graph. Since no such table can be complete, let's look more closely at the definitions of these functions.

Recall that $\sin t$ is the y-coordinate of the terminal point $P(x, y)$ on the unit circle determined by the real number t. How does the y-coordinate of this point vary as t increases? It's easy to see that the y-coordinate of $P(x, y)$ increases to 1, then decreases to -1 repeatedly as the point $P(x, y)$ travels around the unit circle. (See Figure 1.) In fact, as t increases from 0 to $\pi/2$, $y = \sin t$ increases from 0 to 1. As t increases from $\pi/2$ to π, the value of $y = \sin t$ decreases from 1 to 0. Table 1 shows the variation of the sine and cosine functions for t between 0 and 2π.

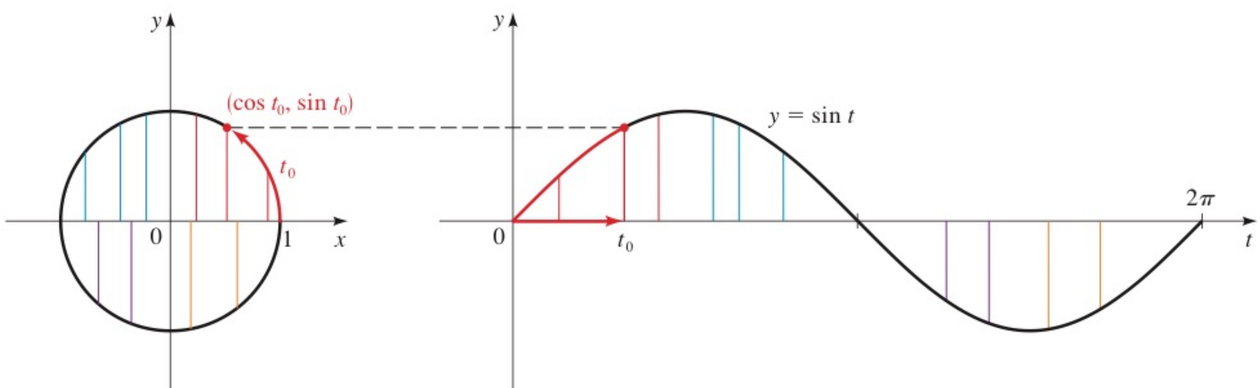


FIGURE 1

To draw the graphs more accurately, we find a few other values of $\sin t$ and $\cos t$ in Table 2. We could find still other values with the aid of a calculator.

TABLE 2

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0
$\cos t$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

SINE AND COSINE CURVES

The sine and cosine curves

$$y = a \sin kx \qquad \text{and} \qquad y = a \cos kx \qquad (k > 0)$$

have **amplitude** $|a|$ and **period** $2\pi/k$.

An appropriate interval on which to graph one complete period is $[0, 2\pi/k]$.

EXAMPLE 3 ■ Amplitude and Period

Find the amplitude and period of each function, and sketch its graph.

(a) $y = 4 \cos 3x$ **(b)** $y = -2 \sin \frac{1}{2}x$

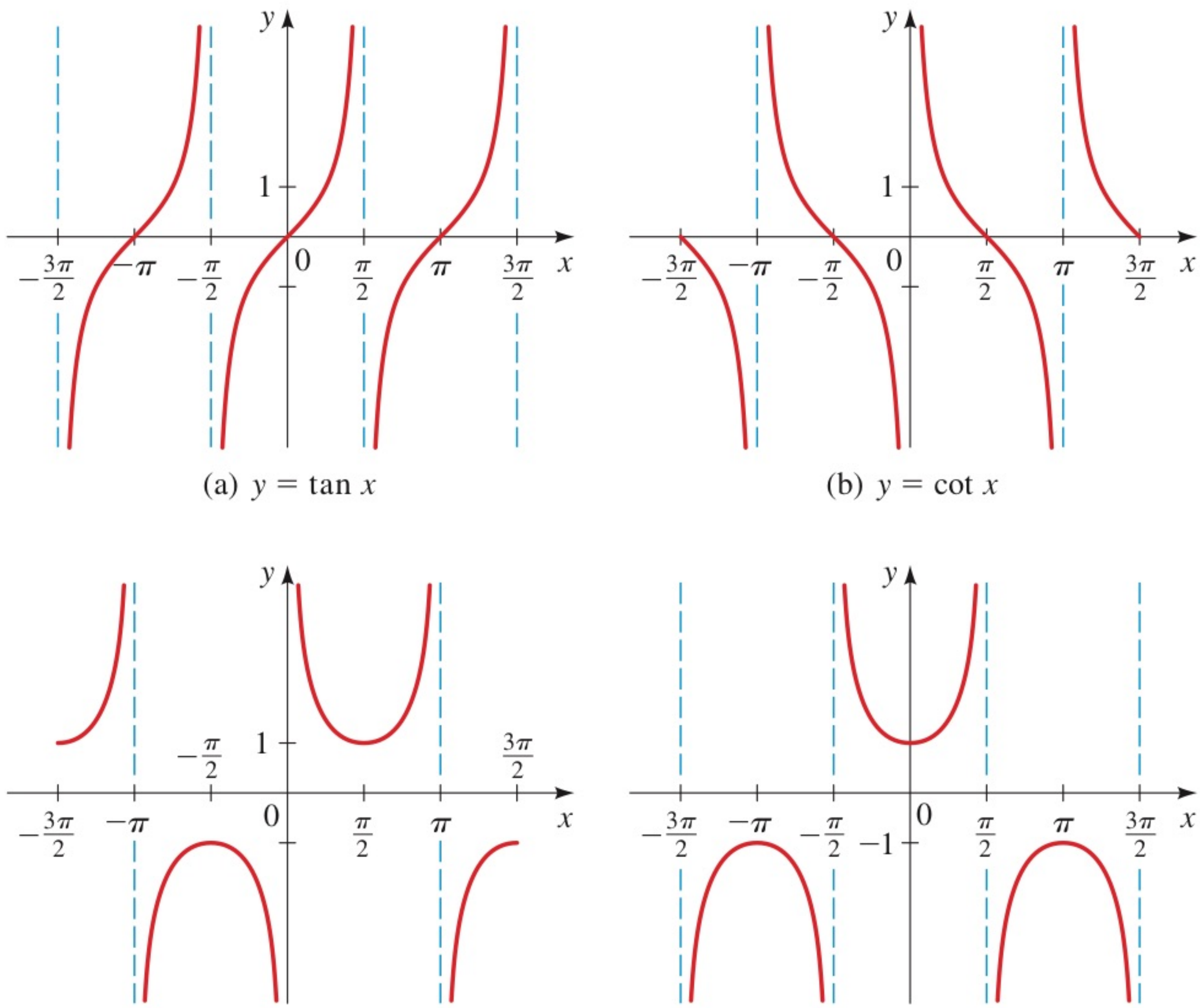
SHIFTED SINE AND COSINE CURVES

The sine and cosine curves

$$y = a \sin k(x - b) \qquad \text{and} \qquad y = a \cos k(x - b) \qquad (k > 0)$$

have **amplitude** $|a|$, **period** $2\pi/k$, and **horizontal shift** b.

An appropriate interval on which to graph one complete period is $[b, b + (2\pi/k)]$.



(a) $y = \tan x$ (b) $y = \cot x$

(c) $y = \csc x$ (d) $y = \sec x$

> **TANGENT AND COTANGENT CURVES**
>
> The functions
>
> $$y = a \tan kx \qquad \text{and} \qquad y = a \cot kx \qquad (k > 0)$$
>
> have period π/k.

Thus one complete period of the graphs of these functions occurs on any interval of length π/k. To sketch a complete period of these graphs, it's convenient to select an interval between vertical asymptotes:

To graph one period of $y = a \tan kx$, an appropriate interval is $\left(-\dfrac{\pi}{2k}, \dfrac{\pi}{2k}\right)$.

To graph one period of $y = a \cot kx$, an appropriate interval is $\left(0, \dfrac{\pi}{k}\right)$.

EXAMPLE 2 ■ Graphing Tangent Curves

Graph each function.

(a) $y = \tan 2x$ **(b)** $y = \tan 2\left(x - \dfrac{\pi}{4}\right)$