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§1.1 Real Numbers

- Number systems

Natural numbers: 1, 2, 3, 4, 5, ... $\mathbb{N}$

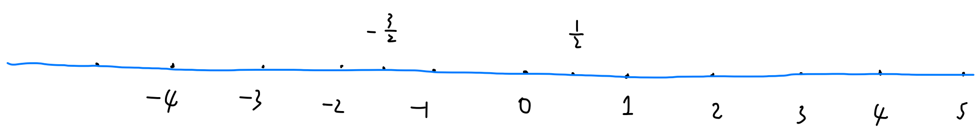
$\Downarrow$

Integers: ... -3, -2, -1, 0, 1, 2, 3, ... $\mathbb{Z}$

$\Downarrow$

Rational numbers: $\frac{1}{2}$, $\frac{7}{5}$, $-\frac{19}{43}$, ... $\frac{a}{b}$ $\mathbb{Q}$

We can use a "line" to represent them:



__Question__: Are there any other numbers on this line?

__Ans__: Yes: Irrational numbers:

$\sqrt{2}$, $\sqrt{5}$, π, e ...

these numbers can't be represented by $\frac{a}{b}$ where a, b are integers

Finally: Real numbers: all the "points" in the blue $\mathbb{R}$

= Union of rationals & irrationals

· Operations on real numbers

addition	$a+b$	$4+7=11$
subtraction	$a-b$	$11-7=4$
multiplication	$a\cdot b$	$4\times 7=28$
division	$a\div b$	$4\div 7=\frac{4}{7}=0.\overline{571428}$

Properties:

commutativity: $a+b=b+a$, $a\cdot b=b\cdot a$ $\quad$ $4+7=7+4=11$, $4\cdot 7=7\cdot 4=28$

associativity: $(a+b)+c=a+(b+c)$ $\quad$ $\underbrace{(4+7)+5}_{11+5}=16=\underbrace{4+(7+5)}_{4+12}$

$(a\cdot b)\cdot c=a\cdot(b\cdot c)$ $\quad$ $\underbrace{(4\cdot 7)\cdot 5}_{28\cdot 5}=140=\underbrace{4\cdot(7\cdot 5)}_{4\cdot 35}$

distributivity $a\cdot(b+c)=a\cdot b+a\cdot c$ $\quad$ $\underbrace{4\cdot(7+5)}_{4\times 12}=48=\underbrace{4\cdot 7+4\cdot 5}_{28+20}$

existence of reciprocal & negative $\quad$ $a\cdot\frac{1}{a}=1\ (a\neq 0)$ $\quad$ $4\cdot\frac{1}{4}=1$

$a+(-a)=0$ $\quad$ $4+(-4)=0$

· Addition trick for rationals: LCD method

$$\frac{1}{6}+\frac{5}{7}=\frac{1\cdot 7}{6\cdot 7}+\frac{5\cdot 6}{7\cdot 6}=\frac{7}{42}+\frac{30}{42}=\frac{37}{42}$$

$LCD(6,7)=42$

$$\frac{1}{6}+\frac{5}{7}\neq\frac{1+5}{6+7}=\frac{6}{13}$$

· Absolute value & Distances

On the real line, for any real number a

$$|a| = \text{distance from } a \text{ to } 0 \geq 0$$

<u>example</u>: $|1| = 1, \quad |-1| = 1$

$$|a| = \begin{cases} a, & \text{if } a \geq 0 ; \\ -a, & \text{if } a < 0 . \end{cases}$$

For any two numbers,

$$\text{distance}(a, b) = |a - b|$$

· set & intervals

number/ people

<u>set</u>: collection of objects (satisfying some condition)

ex. $S = \{2, 3, 4, 5, 6\} = \{x \in \mathbb{Z} \mid 1 < x < 7\}$

$2 \in S$ means "2 belongs to the set S"

$7 \notin S$

" $\cap$ " : $S_1 \cap S_2$

" $\cup$ " : $S_1 \cup S_2$

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

<u>e.x.</u> $(1, 3) \cap [2, 7]$

$(1, 3) \cup [2, 7]$

§ 1.2. Exponents & Radicals

· Integer Exponents

Def: a is a real number, for any positive integer n,

n-th power of a : $a^n = \underbrace{a \cdot a \cdot a \cdots \cdot a}_{n}$

ex: $6^3 = 6 \cdot 6 \cdot 6 = 216$

Some conventions:

$a^0 = 1, \qquad a^{-n} = \frac{1}{a^n}$

ex. $(-2)^{-3} = \frac{1}{(-2)^3} = \frac{1}{-2 \cdot -2 \cdot -2} = \frac{1}{-8} = -\frac{1}{8}$

Rules:

- $a^m \cdot a^n = a^{m+n}, \qquad \frac{a^m}{a^n} = a^{m-n}$ $\qquad \underset{4}{2^2} \cdot \underset{8}{2^3} = 32 = 2^{2+3}$

- $(a^m)^n = a^{mn} = (a^n)^m$

  ex: $(3^2)^3 = 3^6 = (3^3)^2$

  $\quad\ \ \| \qquad\quad \| \qquad\quad \|$

  $9^3 = 729 = 27^2$

- $(ab)^n = a^n \cdot b^n, \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

  ex: $(2 \cdot 3)^3 = 2^3 \cdot 3^3$

  $\quad\ \| \qquad\quad \|$

  $6^3 = 216 = 8 \cdot 27$

- $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$

  ex: $\left(\frac{2}{3}\right)^{-2} \qquad\qquad \left(\frac{3}{2}\right)^2$

  $\quad \| \qquad\qquad\qquad \|$

  $\frac{1}{\left(\frac{2}{3}\right)^2} = \frac{1}{\frac{4}{9}} = \frac{9}{4}$

Simplifying expressions:

e.g. $2a^3b^2 \cdot (3ab^4)^3 = 2a^3b^2 \cdot 3^3 \cdot a^3 \cdot (b^4)^3$

$= 2\cdot 3^3 \cdot a^3 \cdot b^2 \cdot a^3 \cdot b^{12}$

$= 2\cdot 3^3 \cdot a^3 \cdot a^3 \cdot b^2 \cdot b^{12} = 54\, a^6 \cdot b^{14}$

e.g. $\left(\frac{x}{y}\right)^3 \left(\frac{y^2 x}{z}\right)^4 = \frac{x^3}{y^3} \cdot \frac{y^8 x^4}{z^4} = \frac{x^3 y^8 x^4}{y^3 z^4} = \frac{x^3 x^4 \cdot y^{8-3}}{z^4} = \frac{x^7 \cdot y^5}{z^4}$

e.g. $\frac{6st^{-4}}{2s^{-2}t^2} = \frac{6s}{2s^{-2}\cdot t^2 \cdot t^4} = \frac{3s\cdot s^2}{t^6} = \frac{3s^3}{t^6}$

$= 3\cdot st^{-4} \cdot (s^{-2}t^2)^{-1} = 3st^{-4}\cdot s^2 t^{-2} = 3s^3 t^{-6}$

Radicals:

Motivation: to give meaning to expressions such as $2^{\frac{5}{4}}$

Def: Let n be a positive integer, a is a real number the principal n-th root of a is:

$$\underbrace{\sqrt[n]{a}}_{\text{notation}} = a^{\frac{1}{n}} = \text{the unique number } b \text{ such that } b^n = a$$

with the convention that when n is even, $b > 0$

e.g. $n = 4$, $a = 81$.

$\sqrt[4]{81}$ = a positive number such that $b^4 = 81 \Rightarrow b = 3 / -3 \Rightarrow \sqrt[4]{81} = 3$

$n = 2$, $a = 16$

$\sqrt[2]{16} = \ldots$ $b^2 = 16 \Rightarrow b = 4 / -4 \Rightarrow \sqrt[2]{16} = 4$

$n = 3$, $a = -8$

$\sqrt[3]{-8} = \ldots$ $b^3 = -8 \Rightarrow b = -2$

Rules:

- $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ $\qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

e.g. $\sqrt[3]{-8 \cdot 27} = \sqrt[3]{-8} \cdot \sqrt[3]{27}$ $\qquad \sqrt[4]{\frac{16}{81}} = \frac{2}{3} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}}$

$\sqrt[3]{-216} = -6 = -2 \cdot 3$,

- $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a} = \sqrt[n]{\sqrt[m]{a}}$

e.g. $\sqrt[2]{\sqrt[3]{729}} \qquad \sqrt[6]{729} \qquad \sqrt[3]{\sqrt[2]{729}}$

$\sqrt[2]{9} = 3 = \sqrt[3]{27}$

- $$\sqrt[n]{a^n} = \begin{cases} a & \text{if } n \text{ is odd} \\ |a|, & \text{if } n \text{ is even} \end{cases}$$

e.g.: $\sqrt[3]{(-5)^3} = \sqrt[3]{-125} = -5$

$\sqrt[4]{(-3)^4} = \sqrt[4]{81} = 3 = |-3|$

Now we can make sense of expressions like $2^{5/4}$

Def: Rational exponents:

m, n are integers such that $n > 0$

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

e.g: $4^{\frac{5}{2}} = \left(\sqrt[2]{4}\right)^5 = 2^5 = 32$

$2^{\frac{5}{4}} = \left(\sqrt[4]{2}\right)^5 = \sqrt[4]{32}$

$125^{-\frac{1}{3}} = \left(\sqrt[3]{125}\right)^{-1} = 5^{-1} = \frac{1}{5}$

Simplifation

$$\sqrt{x\sqrt{x}} = \sqrt{x \cdot x^{\frac{1}{2}}} = \sqrt{x^{\frac{3}{2}}} = x^{\frac{3}{4}}$$

Rationaling the denominators

we will encounter expressions of the followng form

$\frac{1}{\sqrt[n]{a}}$, it's convention that we remove all the rational radicals on the denominator:

$$\frac{1}{\sqrt[n]{a}} = \frac{1}{a^{\frac{1}{n}}} = \frac{a^{\frac{n-1}{n}}}{a}$$

ex. $n=2$. $\frac{1}{\sqrt{a}} = \frac{\sqrt{a}}{a}$

: $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

$$\frac{1}{\sqrt[3]{5}} = \frac{1}{5^{\frac{1}{3}}} = \frac{\sqrt[3]{25}}{5}$$

§1.3. Algebraic expressions

- variables : a letter that can represent any number from a given set

- algebraic expressions :

variables & real numbers + addition, substraction, multiplication, division, powers, radicals

e.g. $2x^2 - 3x + 4$, $\sqrt{x} + 10$, $\dfrac{y-2z}{y^2+2x}$

- monomial : ax^k → non negative number

real number, variable

e.g. $2x^2$, $-3x$, 4

- Polynomials : a sum of monomials :

Def : $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

- $a_n, a_{n-1}, \ldots, a_0$ real number
- $n \geq 0$
- $(a_n \neq 0)$ deg is n
- $a_k x^k$: terms

Polynomial	Type	Terms	Degree
$2x^2 - 3x + 4$	trinomial	$2x^2, -3x, 4$	2
$x^8 + 5x$	binomial	$x^8, 5x$	8
$8 - x + x^2 - \frac{1}{2}x^3$	four terms	$-\frac{1}{2}x^3, x^2, -x, 3$	3
$5x + 1$	binomial	$5x, 1$	1
$9x^5$	monomial	$9x^5$	5
6	monomial	6	0

· Addition & Substraction:

idea: Combine like terms: same variable & same degree

$3x^7 + 5x^7 = 8x^7$

EXAMPLE 1 ■ Adding and Subtracting Polynomials

(a) Find the sum $(x^3 - 6x^2 + 2x + 4) + (x^3 + 5x^2 - 7x)$.

(b) Find the difference $(x^3 - 6x^2 + 2x + 4) - (x^3 + 5x^2 - 7x)$.

· Multiplication:

$$(2x + 1)(3x - 5) = \underset{\mathbf{F}}{6x^2} - \underset{\mathbf{O}}{10x} + \underset{\mathbf{I}}{3x} - \underset{\mathbf{L}}{5}$$

$$= 6x^2 - 7x - 5$$

EXAMPLE 3 ■ Multiplying Polynomials

Find the product: $(2x + 3)(x^2 - 5x + 4)$

· Special product formula

■ Special Product Formulas

Certain types of products occur so frequently that you should memorize them. You can verify the following formulas by performing the multiplications.

SPECIAL PRODUCT FORMULAS

If A and B are any real numbers or algebraic expressions, then

1. $(A + B)(A - B) = A^2 - B^2$	Sum and difference of same terms
2. $(A + B)^2 = A^2 + 2AB + B^2$	Square of a sum
3. $(A - B)^2 = A^2 - 2AB + B^2$	Square of a difference
4. $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$	Cube of a sum
5. $(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$	Cube of a difference

$(x-2)(x+2) = x^2 - 4$

$(x+1)^2 = x^2 + 2x + 1$

$(x-1)^2 = x^2 - 2x - 1$

$(x+1)^3$

$(x-1)^3$

· Factoring polynomials

reverse the process of product! usually very hard!!

FACTORING →

$$x^2 - 4 = (x - 2)(x + 2)$$

← EXPANDING

We say that $x - 2$ and $x + 2$ are **factors** of $x^2 - 4$.

The easiest type of factoring occurs when the terms have a common factor.

EXAMPLE 6 ■ Factoring Out Common Factors

Factor each expression.

(a) $3x^2 - 6x$ **(b)** $8x^4y^2 + 6x^3y^3 - 2xy^4$ **(c)** $(2x + 4)(x - 3) - 5(x - 3)$

· Factoring quadratic polynomials $\Longleftrightarrow$ find roots of quadratic equations

Guiding example:

$$x^2 - a = (x - \sqrt{a})(x + \sqrt{a}) \qquad a \geq 0$$

In general, quadratic polynomial takes the following form

$$x^2 + bx + c$$

how to factorize it?

key idea: reduce to the Guiding case

$$\begin{aligned} x^2 + bx + c &= \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c \\ &= \left(x + \frac{b}{2}\right)^2 - \frac{b^2 - 4c}{4} \\ &= \left(x + \frac{b}{2} + \sqrt{\frac{b^2-4c}{4}}\right)\left(x + \frac{b}{2} - \sqrt{\frac{b^2-4c}{4}}\right) \\ &= \left(x + \frac{b + \sqrt{b^2-4c}}{2}\right)\left(x + \frac{b - \sqrt{b^2-4c}}{2}\right) \end{aligned}$$

§1.4. Rational expressions

Fractional expressions: quotients of algebraic expressions

e.g. $\dfrac{2x}{x-1}$ $\quad \dfrac{y-2}{y^2+4}$ $\quad \dfrac{x^3-x}{x^2-5x+6}$ $\quad \dfrac{x}{\sqrt{x^2+1}}$

Rational expressions: quotients of polynomials

fractional, but not rational

Domain: sometimes the fractional expression may not be defined for some values of the variables

Domain: the set where the expression is defined

e.g. $\dfrac{2x}{x-1}$, x can't be 1, because you can't divide 0!

Domain: $\{x \in \mathbb{R} : x \neq 1\}$ or $\mathbb{R} \setminus \{1\}$

$= (-\infty, 1) \cup (1, +\infty)$

$\dfrac{x-1}{\sqrt{x}}$, first: x can't be 0 by the same reason

second. $x \geq 0$ so that $\sqrt{x}$ is defined

hence domain: $\{x \in \mathbb{R} : x > 0\} = (0, +\infty)$

e.g. $\dfrac{x}{x^2-5x+6}$: $x^2 - 5x + 6$

$= \left(x + \dfrac{-5+\sqrt{25-24}}{2}\right)\left(x + \dfrac{-5-\sqrt{25-24}}{2}\right)$

$= (x-2)(x-3)$

$\mathbb{R} \setminus \{2, 3\}$

Simplifying expressions by factorization

To **simplify rational expressions**, we factor both numerator and denominator and use the following property of fractions:

$$\frac{AC}{BC} = \frac{A}{B}$$

This allows us to **cancel** common factors from the numerator and denominator.

EXAMPLE 2 ■ Simplifying Rational Expressions by Cancellation

Simplify: $\dfrac{x^2 - 1}{x^2 + x - 2}$

SOLUTION

$$\begin{aligned} \frac{x^2 - 1}{x^2 + x - 2} &= \frac{(x - 1)(x + 1)}{(x - 1)(x + 2)} && \text{Factor} \\ &= \frac{x + 1}{x + 2} && \text{Cancel common factors} \end{aligned}$$

Multiplying & Dividing

To **multiply rational expressions**, we use the following property of fractions:

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$$

This says that to multiply two fractions, we multiply their numerators and multiply their denominators.

EXAMPLE 3 ■ Multiplying Rational Expressions

Perform the indicated multiplication and simplify: $\dfrac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \dfrac{3x + 12}{x - 1}$

SOLUTION We first factor.

$$\begin{aligned} \frac{x^2 + 2x - 3}{x^2 + 8x + 16} \cdot \frac{3x + 12}{x - 1} &= \frac{(x - 1)(x + 3)}{(x + 4)^2} \cdot \frac{3(x + 4)}{x - 1} && \text{Factor} \\ &= \frac{3(x - 1)(x + 3)(x + 4)}{(x - 1)(x + 4)^2} && \text{Property of fractions} \\ &= \frac{3(x + 3)}{x + 4} && \text{Cancel common factors} \end{aligned}$$

Diving

To **divide rational expressions**, we use the following property of fractions:

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}$$

This says that to divide a fraction by another fraction, we invert the divisor and multiply.

EXAMPLE 4 ■ Dividing Rational Expressions

Perform the indicated division and simplify: $\dfrac{x-4}{x^2-4} \div \dfrac{x^2-3x-4}{x^2+5x+6}$

SOLUTION

$$\frac{x-4}{x^2-4} \div \frac{x^2-3x-4}{x^2+5x+6} = \frac{x-4}{x^2-4} \cdot \frac{x^2+5x+6}{x^2-3x-4} \qquad \text{Invert and multiply}$$

$$= \frac{(x-4)(x+2)(x+3)}{(x-2)(x+2)(x-4)(x+1)} \qquad \text{Factor}$$

$$= \frac{x+3}{(x-2)(x+1)} \qquad \text{Cancel common factors}$$

Adding & Substracting:

To **add or subtract rational expressions**, we first find a common denominator and then use the following property of fractions:

$$\frac{A}{C} + \frac{B}{C} = \frac{A + B}{C}$$

Although any common denominator will work, it is best to use the **least common denominator** (LCD) as explained in Section 1.1. The LCD is found by factoring each denominator and taking the product of the distinct factors, using the highest power that appears in any of the factors.

EXAMPLE 5 ■ Adding and Subtracting Rational Expressions

Perform the indicated operations and simplify.

(a) $\dfrac{3}{x - 1} + \dfrac{x}{x + 2}$ **(b)** $\dfrac{1}{x^2 - 1} - \dfrac{2}{(x + 1)^2}$

§1.5 Equations

Equation: mathematical expression 1 = mathematical expression 2

e.g.

$$3 + 5 = 8$$
$$x = 0$$
$$5x + 2 = 17$$
$$4x + 3 = 19$$
$$2x^2 + 8x + 1 = x^2 + 13x - 5$$

$x^2 - 5x + 6$

Solving equation: for equations with unknown variables, find all the possible values of the variable such that the "statement" is true

$x = 0 \rightsquigarrow x = 0$, x can't be anything else, like $x = 1$

$1 = 0 \Rightarrow$ not true

$5x + 2 = 17 \Longleftrightarrow x = 3$, for all other values, not true

$4x + 3 = 19 \Longleftrightarrow x = 4$

$2x^2 + 8x + 1 = x^2 + 13x - 5 \Longleftrightarrow x^2 - 5x + 6 = 0 \Longleftrightarrow (x-2)(x-3) = 0$

$\Longleftrightarrow x = 2$ or 3

PROPERTIES OF EQUALITY

Property	Description
1. $A = B \Leftrightarrow A + C = B + C$	Adding the same quantity to both sides of an equation gives an equivalent equation.
2. $A = B \Leftrightarrow CA = CB \quad (C \neq 0)$	Multiplying both sides of an equation by the same nonzero quantity gives an equivalent equation.

Linear equation

Solving Linear Equations

The simplest type of equation is a *linear equation*, or first-degree equation, which is an equation in which each term is either a constant or a nonzero multiple of the variable.

LINEAR EQUATIONS

A **linear equation** in one variable is an equation equivalent to one of the form

$$ax + b = 0$$

where a and b are real numbers and x is the variable.

Here are some examples that illustrate the difference between linear and nonlinear equations.

Linear equations	**Nonlinear equations**	
$4x - 5 = 3$	$x^2 + 2x = 8$	Not linear; contains the square of the variable
$2x = \frac{1}{2}x - 7$	$\sqrt{x} - 6x = 0$	Not linear; contains the square root of the variable
$x - 6 = \dfrac{x}{3}$	$\dfrac{3}{x} - 2x = 1$	Not linear; contains the reciprocal of the variable

Quadratic equations

Solving Quadratic Equations

Linear equations are first-degree equations like $2x + 1 = 5$ or $4 - 3x = 2$. Quadratic equations are second-degree equations like $x^2 + 2x - 3 = 0$ or $2x^2 + 3 = 5x$.

QUADRATIC EQUATIONS

A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$.

Some quadratic equations can be solved by factoring and using the following basic property of real numbers.

ZERO-PRODUCT PROPERTY

$$AB = 0 \quad \text{if and only if} \quad A = 0 \quad \text{or} \quad B = 0$$

This means that if we can factor the left-hand side of a quadratic (or other) equation, then we can solve it by setting each factor equal to 0 in turn. This method works only when the right-hand side of the equation is 0.

We can use the technique of completing the square to derive a formula for the roots of the general quadratic equation $ax^2 + bx + c = 0$.

THE QUADRATIC FORMULA

The roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$