Affinoid Space

Tate algebra & Affinoid algebra

Def: n-th Tate algebra for NA local field K is defined as

$$T_n = \left\{ f = \sum_{\mu \in N^n} a_\mu \zeta^\mu \;\middle|\; \lim_{|\mu| \to +\infty} a_\mu = 0 \right\}$$

Rmk: This is called restricted power series. here

$$\sum_{\mu \in N^n} a_\mu \zeta^\mu = \sum_{\mu_1, \dots, \mu_r \geq 0} a_{\mu_1 \cdots \mu_r} \zeta_1^{\mu_1} \cdots \zeta_r^{\mu_r} \in K[[\zeta_1, \dots, \zeta_n]]$$

We fix an algebraic closure $\bar{K}$ of K, the the NA valuation admits a unique extension to $\bar{K}$ and we denote $B^n(\bar{K})$ as the unit ball to be:

$$B^n(\bar{K}) = \{ (x_1, \dots, x_n) \mid |x_i| \leq 1 \}$$

then the Tate algebra $K\langle \zeta_1, \dots, \zeta_n \rangle$ has another interpretation:

$$f \in K[[\zeta_1, \dots, \zeta_n]] \text{ converges for } \forall\, x \in B^n(\bar{K}) \Leftrightarrow f \in T_n$$

i.e. it's the "function ring" of the unit ball $B^n(\bar{K})$

Def: a K-algebra A is called an affinoid K-algebra if there is an epimorphism of K-algebras

$$\alpha : T_n \longrightarrow A$$

for some $n \geq 1$

Rmk: In the category of K-affinoid algebras, fiber product exists

K-affinoid algebras and its associated spaces will play the similar role of "local affine piece" for the general rigid analytic spaces. let's first state some purely algebraic properties of K-affinoid algebras:

Prop: Suppose A is an affinoid K-algebra, then

- A is noetherian, Jacobson
- A admits a noetherian normalization, i.e. $\exists\, d \geq 0$, s.t. there is a finite monomorphism:

$$T_d \longrightarrow A$$

$\Rightarrow$ Nullstellsatz: if $\mathfrak{m}$ is a maximal ideal of A, then $A/\mathfrak{m}$ is a finite extension of K because there would exits $T_d \hookrightarrow A/\mathfrak{m}$, hence since $A/\mathfrak{m}$ is a field $\Rightarrow d = 0 \Rightarrow A/\mathfrak{m}$ is finite K-mod, therefore a finite extension of K

Residue norm & Supreme norm

There are two kinds of "norm" on a K-affinoid algebra. Let's state the definition & relations:

- residue norm:

  $\forall\ \alpha: T_n \longrightarrow A$ surjective, since T_n is a K-Banach space (w.r.t the norm defined by coefficients)

  $$|f|_\alpha = \inf_{\substack{g \in \ker\alpha \\ \| \\ \mathfrak{a}}} |f-g|$$

  Rk: Any K-homo of any K-affinoid algebras A & B is continuous w.r.t any residue norm
  hence any two residue norms are equivalent

- Supreme norm:

  consider the set $Max(A)$, if we are given $\alpha: T_n \to A$, then, $Max(A) = V(\overset{\ker\alpha}{\overset{\|}{\mathfrak{a}}}) \underset{closed}{\subset} B^n(\bar{K})$

  hence $f(x)$, $x \in Max(A)$ makes sense as an element in $\bar{K}$, i.e. just evaluation at $x \in B^n(\bar{K})$

  or, it can be interpreted as the image of f under $A \longrightarrow A/m\ (\hookrightarrow \bar{K})$, since A/m is finite over K

  the valuation on K extends uniquely to $\bar{K}$, hence $|f(x)|$. therefore we define:

  $$|f|_{sup} = \sup_{x \in Max(A)} |f(x)|$$

  Maximal principle: $\exists\, x \in Max(A)$, s.t. $|f|_{sup} = |f(x)|$

  Obviously, Supreme norm is only a "semi-norm" because f nilpotent $\Leftrightarrow |f|_{sup} = 0$

Affinoid spaces

Def: For an K-affinoid algebra A, we define the associated affinoid space to be

$$Sp\, A = (\underset{\substack{\uparrow \\ \text{"topological space"}}}{\mathrm{Max}\, A},\ A) \quad \nwarrow \text{functions on it}$$

Rmk: We only focus on "Maximal ideal" because general prime ideals don't behave well under localization (P62, Example 22) and as we have already seen, $\mathrm{Max}(A)$ can be identified with $V(\mathfrak{a}) \subseteq B^n(\bar{K})$, A can be viewed as functions on it

Topology on SpA

The main goal for running away from K-varieties to K-rigid analytic space it to give a way to compute cohomology. Remember in the case of $\mathbb{C}$-varieties, cohomology of constant sheaf is not interesting at all, but when we do cohomology in $\mathbb{C}$-manifolds, cohomology of constant sheaf behaves very good, why? the main reason is that Zariski topology is too coarse and only good enough for quasi-coherent sheaves, here we introduce rigid analytic space with the same hope with a good topology on it, cohomology can be computed "correctly"! Hence defining the "right" topology is very important

Natural question: $H_{\acute{e}t}(X, A) = \prod H(X(\bar{\mathbb{Q}}_\ell), \mathbb{Z}_\ell)$?

Initial idea: the right topology on SpA should be consistent with the topology on K

Naive try: 1. $K = \bar{K}$, then $\mathrm{Max}(A) \overset{\text{closed}}{\hookrightarrow} B^n(K) = \{(x_1, \dots, x_n) \mid |x_i| \le 1\}$ ($\sim$ Zariski)

since there is a natural topology on K^n, hence a natural topology on $B^n(k)$

then this topology restricts to $\mathrm{Max}(A)$

2. $K \neq \bar{K}$, we have $\mathrm{Max}(A) = V(\mathfrak{a}) \subseteq \mathrm{Max}(T_n)$, our naive idea is to identify $\mathrm{Max}(T_n)$ with $B^n(\bar{K})$, which is given by $m_x \mapsto (\bar{x}_1, \dots, \bar{x}_n)$, $\bar{x}_i \in \bar{K}$, but actually this is not rigorous, because we only have $\bar{x}_i \in T_n/m_x$ is finite extension of K, we can't identify it directly with a point in $B^n(\bar{K})$, essentially, we can only get a Galois orbit!

i.e.

$$\mathrm{Max}(T_n) \simeq B^n(\bar{K})/G_K \qquad T_n/m_x \xrightarrow{G.1} T_n/m_{gx} \quad \text{automorphisms of fields}$$
$$\bar{x}_i \mapsto \bar{x}_i$$

here $G_K = \mathrm{Gal}(\bar{K}/K)$, then there is a natural quotient topology on $\mathrm{Max}(T_n)$ induced from $B^n(\bar{K}) \subseteq \bar{K}^n$, now $\mathrm{Max}(A) \overset{\text{closed}}{\subseteq} \mathrm{Max}(T_n)$, there is a natural topology on $\mathrm{Max}(A)$

Both of these definitions seem to be satisfactory, but remember: We fix a surjective $\alpha: T_n \to A$!
We should prove that this topology is independent of our choice of α!
This follows from the equivalence of arbitrary residue norm on A

Now we will give this topology on $Sp(A)$ by another way, by this method, we can give an open basis by a simple kind of open subsets, which will be useful when calculating cohomology.

Affinoid subdomains

For an K-affinoid space $X = Sp A$, define

$$X(f, \varepsilon) = \{ x \in Max(A) \mid |f(x)| < \varepsilon \}, \quad f \in A, \ \varepsilon > 0$$

Def: (canonical topology)

The topology generated by $X(f, \varepsilon)$ for all $f \in A$, $\varepsilon > 0$ is called the canonical topology on X

Does this canonical topology agree with the topology we defined earlier? The answer is yes!

- $X(f, \varepsilon)$ is open in "naive" topology: this simple because $|\cdot|$ is continuous on $B^n(\bar{K})$
- $X(f, \varepsilon)$ form an open base: $\forall x \in Max(A) \longrightarrow B^n(\bar{K})/G_K$, then $\forall z \in B^n(\bar{K})$, $z \in B(x, \varepsilon)$ can be interpreted as the condition $f_i(z_i) = \prod_{\sigma \in G_K} (z_i - x_i^\sigma)$, $|f_i(z_i)| < \varepsilon^n$, $1 \le i \le n$

only finitely many,

then $\bigcap_{i=1}^{n} X(f_i, \varepsilon^n) \subseteq B(x, \varepsilon)$ (Krasner's lemma)

Clearly, this version of definition of topology is much more "canonical" than the naive one.
Actually, we can even only focus on a more special kind of open subsets

$$X(f) = X(f, 1)$$

Claim: $\forall X(f, \varepsilon)$ can be written as union of $X(g)$

pf:

$$X(f, \varepsilon) = \bigcup_{\substack{\varepsilon' \le \varepsilon \\ \varepsilon' \in |\bar{K}^*|}} X(f, \varepsilon')$$

then for $\forall \varepsilon' \in |\bar{K}^*|$, $\exists c \in K^*$, s.t., $\varepsilon'^s = |c|$, $s \in \mathbb{N}$, then

$$|f(x)| < \varepsilon' \Leftrightarrow |f^s(x)| < |c| \Leftrightarrow |c^{-1} f^s(x)| < 1 \Leftrightarrow x \in X(c^{-1} f^s)$$

Then we know: every open subset is union of $X(f_1, \dots, f_r)$, $f_i \in A$

Now I'm going to state a key lemma in proving openness for some subset:

Lemma: For K-affinoid space $X = Sp A$, and $f \in A$, suppose $\varepsilon = |f(x)| > 0$, then $\exists g \in A$, s.t.

- $g(x) = 0$
- $X(g) \subseteq \{ y \in Sp A \mid |f(y)| = \varepsilon \}$

this lemma tells us the following subsets are open:

$\{ x \in Sp(A) \mid f(x) \neq 0 \}$

$\{ x \in Sp(A) \mid |f(x)| \le \varepsilon, \ \varepsilon > 0 \}$

$\{ x \in Sp(A) \mid |f(x)| = \varepsilon, \ \varepsilon > 0 \}$

$\{ x \in Sp(A) \mid |f(x)| \ge \varepsilon |$

W-domain: $X(f_1, \dots, f_r)$

L-domain: $X(f_1, \dots, f_r, g_1^{-1}, \dots, g_s^{-1}) = \{ x \in X \mid |f_r(x)| \le 1, \ |g_j(x)| \ge 1 \}$

R-domain: $X\left(\frac{f_1}{f_0}, \dots, \frac{f_r}{f_0}\right) = \{ x \in X \mid |f_i(x)| \le |f_0(x)|, \ 1 \le i \le r \}$

where we need the condition: $f_0, \dots, f_r$ has no common zero

$\Leftrightarrow (f_0, \dots, f_r) = A$

W is L, L is R

Let's now give the definition of affinoid subdomains, they play the role of "open affine subscheme"

<u>Def</u>: X is a K-affinoid space. $U \subseteq X$ is called an affinoid subdomain if

- $\exists\ \iota: X' \to X$ as a morphism of affinoid spaces, s.t. $\iota(X') \subseteq U$
- universal property: $\forall$ affinoid K-space morphism $\varphi: Y \to X$ s.t. $\varphi(Y) \subseteq U$ admits a unique factorization:

$$\varphi:\ Y \xrightarrow{\tilde{\varphi}} X' \xrightarrow{\iota} X$$

this definition looks very confusing, we explain it now,

- In the setting of this definition, we can immediately get $\iota: X' \xrightarrow{\sim} U$ is a bijection, hence we can transfer the K-affinoid space structure of X' to U, making U a affinoid K-space
- When endowed with this affinoid K-space structure, we can define the canonical topology on U, this topology agrees with the induced topology from X
- U is an open subset of X

Hence we see that, affinoid subdomain is essentially: open sub-affinoid K-space of $\mathrm{Sp}A = X$ satisfying universal property

Especially, W, L, R-domain are all affinoid subdomains

$X' = X(f_1, \dots, f_r)$, then $A_{X'} = A\langle \zeta_1, \dots, \zeta_r \rangle / (\zeta_i - f_i)$

$X' = X(f_1, \dots, f_r, g_1^{-1}, \dots, g_s^{-1})$, then $A_{X'} = A\langle \zeta_1, \dots, \zeta_r, \eta_1, \dots, \eta_s \rangle / (\zeta_i - f_i, 1 - g_j \eta_j)$

$X' = X(\frac{f_1}{f_0}, \dots, \frac{f_r}{f_i})$, then $A_{X'} = A\langle \zeta_1, \dots, \zeta_r \rangle / (f_0 \zeta_i - f_i)$

Affinoid subdomains form a distinguished class of open subset of an affinoid K-space $X = \mathrm{Sp}A$.

- Composition (Transitivity): if $V \to X$ is an affinoid subdomain, $U \to V$ is an affinoid subdomain then $U \to V \to X$ is an affinoid subdomain of X
- Fiber product: Suppose $X' \to X$ is an affinoid subdomain, then for $\forall\, Y$ affinoid K-space, $Y' := Y \times_X X'$ is an affinoid subdomain of Y, i.e. the following cartesian diagram transfers affinoid subdomain

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

moreover, if X' is W, L, R-domain, then Y' is also W, L, R-domain respectively

Now we state a lemma cerning the local nature of morphisms of affinoid K-spaces. this theorem will give us the result that every affinoid is open

<u>Lemma</u>: (local nature of affinoid morphism) $\varphi: Y \to X$ corresponds to $\varphi^*: A \to B$, $x \in X$, corresponds $\mathfrak{m} \subset A$

1. Assume $\varphi^*: A/\mathfrak{m} \to B/\mathfrak{m}B$ is surj, then $\exists$ special affinoid subdomain $X' \to X$ containing x, s.t.

$$\begin{tikzcd} Y' \arrow[r, "\varphi'", "\text{closed imm}"'] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "\varphi"] & X \end{tikzcd}$$

2. Assume φ^* induces $A/\mathfrak{m}^n \xrightarrow{\sim} B/\mathfrak{m}^n B$ for $\forall n \in \mathbb{N}$, then $\exists$ special affinoid subdomain $X' \to X$ containing x, s.t.

$$\begin{tikzcd} Y' \arrow[r, "\varphi'", "\sim"'] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "\varphi"] & X \end{tikzcd}$$

Locally closed immersion & Gerritzen-Grauert theorem

In this section, let's view X more geometrically, we will make it into a locally ringed spaces. We have already defined affinoid subdomains, this is an open subset, which is also a affinoid space, s.t the affinoid canonical topology = restriction topology, which also satisfies a universal property. We also proved that actually there are lots of such domains, say W, L, R-domain, (and G-G theorem will tell us that every affinoid subdomain is a finite union of R-domains) Since W-domain already form an open base, we can define sheaf, germs on X.

- presheaf of affinoid functions: affinoid subdomain $U \longmapsto \mathcal{O}_X(U)$ = the affinoid algebra of U

  germ: $$\mathcal{O}_{X,x} = \varinjlim_{x \in U} \mathcal{O}_X(U)$$

  By the fiber product properties of affinoid subdomain, for $\forall \varphi : Y \to X$, we get an induced homo:
  $$\varphi^*_{\varphi(y)} : \mathcal{O}_{X,\varphi(y)} \longrightarrow \mathcal{O}_{Y,y}$$

Prop: Germs of $x \in X$

- $\mathcal{O}_{X,x}$ is a Noetherian local ring, with maximal ideal $\mathfrak{m}\mathcal{O}_{X,x}$ $\rightsquigarrow$ essentially, we only need to show $\mathcal{O}_{X,x}$ is $\mathfrak{m}$-adically separated
- φ^*_x is a local homomorphism
- $\mathcal{O}_{X,x}$ not necessarily equals to $A_\mathfrak{m}$, but we have a sequence
  $$A \longrightarrow A_\mathfrak{m} \hookrightarrow \mathcal{O}_{X,x}$$

  and we get isomorphisms for $\forall n \in \mathbb{N}$
  $$A/\mathfrak{m}^n \xrightarrow{\sim} A_\mathfrak{m}/\mathfrak{m}^n \xrightarrow{\sim} \mathcal{O}_{X,x}/\mathfrak{m}^n\mathcal{O}_{X,x}$$

  so there are isomorphisms:
  $$\hat{A} \xrightarrow{\sim} \hat{A}_\mathfrak{m} \xrightarrow{\sim} \hat{\mathcal{O}}_{X,x}$$
  $\Rightarrow$ local 映射若诱导局部环之间同构
  那么 naive topology 下是同构的!

this prop has lots of implications, (like in the world of Scheme), we list them below

- $f = 0 \Leftrightarrow f_x = 0, \forall x \in \operatorname{Sp} A$

  pf: $$A \hookrightarrow \prod_\mathfrak{m} A_\mathfrak{m} \hookrightarrow \prod_x \mathcal{O}_{X,x}$$

- If X is covered by $\{X_i\}_{i \in I}$, then we get injection:
  $$\mathcal{O}_X(X) \hookrightarrow \prod_i \mathcal{O}_X(X_i)$$

- For affinoid subdomain $X' \hookrightarrow X$, we get $A \underset{\text{?}}{\to} A'$, then A' is flat over A

  not necessarily inj
  consider $A \to A_f$

For a morphism of affinoids, say $\varphi: Y \to X$, we have associated $\varphi^*: A \to B$, then φ^* factors into

$$A \twoheadrightarrow A/_{\ker\varphi^*} \hookrightarrow B \quad \rightsquigarrow \quad Y \xrightarrow{\tilde{\varphi}} X' \xhookrightarrow{i} X$$

here i is a closed immersion, so the general structure of φ only involves the structure of $\tilde{\varphi}$

In general, there is no "simple" description of $\tilde{\varphi}$ (i.e. a morphism with injective affinoid algebra homomorphism)

We will focus on a special kind of morphism:

<u>Def</u>: $\varphi: Y \to X$ is called a locally closed immersion (open immersion) if

- φ is injective on the underlying topological space
- $\varphi^*_x: \mathcal{O}_{X,\varphi(y)} \to \mathcal{O}_{Y,y}$ is surjective (bijective) for $\forall y \in Y$

<u>Rmk</u>: By the lemma on the local nature of affinoid space morphisms, we can conclude:

$\forall x \in X$, $\exists$ a special affinoid subdomain U_x, s.t. $U_x \times_X Y \to U_x$ is a closed immersion.

then X can be covered by $\{U_i\}_{i\in I}$, s.t. $Y \times_X U_i = V_i \to U_i$ is closed immersion for $\forall i \in I$

Same is true for open immersion, by $\mathcal{O}_{X,\varphi(y)} \xrightarrow{\sim} \mathcal{O}_{Y,y}$, we get $\exists U_x$, s.t. $U_x \times_X Y \xrightarrow{\sim} U_x$, i.e.

X can be covered by $\{U_i\}_{i\in I}$, s.t. $Y \times_X U_i = V_i \xrightarrow{\sim} U_i$, here topologically, $Y = \bigcup_{i\in I} Y \times_X U_i$, for $\forall i \in I$, $Y \times_X U_i \overset{\exists !}{\simeq} U_i$

and actually these isomorphisms can be glued: $Y \times_X U_{ij} = Y \times_X (U_i \cap U_j) = Y \times_X U_i \cap Y \times_X U_j$, this holds for only finitely many U_i

G-G theorem will tell us that U_i can be chosen to be sufficiently large, and

locally closed immersions form a "distinguished" class of morphisms

Prop:
- Composition of locally closed immersion (closed, open immersion) is still a locally closed immersion
- Locally closed immersions (closed, open immersions) are stable under base change
- Finite locally closed immersion is closed immersion

Among all the locally closed immersions, a particular type is important

<u>Def</u>: (Runge Immersion) A morphism $\varphi: Y \to X$ is Runge immersion if φ can be factored into:

$$\underset{\text{closed.im}}{Y \xrightarrow{\quad} W} \xrightarrow[\text{W-domain}]{\quad} X$$

Let's give a criterion for Runge immersion, it gives an answer to the general structure of $\tilde{\varphi}$

Prop: For a homo $\sigma: A \to A'$ of affinoid K-algebras, TFAE

- $\varphi: \mathrm{Sp}A' \to \mathrm{Sp}A$ is a Runge immersion
- <u>$\sigma(A)$ is dense in A'</u>
- $\sigma(A)$ contains a system of affinoid generators of A' over A

this prop easily implies that Runge immersion is stable under composition & base change

Finally, let's come to the G-G theorem

Thm: $\varphi: X' \to X$ is a locally closed immersion, then $\exists$ covering $\{U_i\}_{i=1}^n$ of X by finitely many R-domains, s.t. $\varphi^{-1}(U_i) \to U_i$ is Runge immersion for $\forall i$

Rmk: In particular, if φ is an open immersion, then each $\varphi^{-1}(U_i) \to U_i$ is W-domain then for $\forall$ affinoid subdomain $X' \subseteq X$, $\exists$ finitely many R-domain U_i, s.t

$$X' = \bigcup_{i=1}^n X' \cap U_i$$

$$X' \cap U_i \to U_i \text{ is W-domain}$$

hence X' is a finite union of R-domains of X

Rmk: This theorem tells us, affinoid subdomain looks just like finitely many union of principal open subsets

Rigid Spaces

Roughly speaking, affinoid spaces is an analogue of affine schemes. rigid spaces is an analogue of schemes, they can be obtained/defined as as the definition of schemes in terms of affine schemes, i.e. a locally ringed space which "locally" looks likes an affinoid space, but here we don't use "topology". we use Grothendieck topology. they are good enough to define Structure sheaf, cohomology and so on.

Grothendieck Topology

Def: $\mathcal{C}$ is a category, for $\forall X \in Ob\,\mathcal{C}$. we specify a set of family of morphisms, $Cov(X)$, s.t.

- if $Y \xrightarrow{\sim} X$ is an isomorphism, then $(Y \to X) \in Cov(X)$
- if $(X_i \to X)_{i\in I} \in Cov(X)$, $(X_{ij} \to X_i)_{j\in J_i} \in Cov(X_i)$, then
$$(X_{ij} \longrightarrow X)_{i\in I, j\in J_i} \in Cov(X)$$
- if $(X_i \to X)_{i\in I} \in Cov(X)$, $Y \to X$ is any morphism, then
  - (i) $X_i \times_X Y$ exists in $\mathcal{C}$
  - (ii) $(X_i \times_X Y \to Y)_{i\in I} \in Cov(Y)$

usually, we will have a space Ω, the $Ob\,\mathcal{C}$ is just "some good" subset of Ω, they are to be understood as "open subset"

$(X, Cov(X))_{X\in Ob(\mathcal{C})}$ is called a Grothendieck topology on the category $\mathcal{C}$

Grothendieck topology is all we need to define sheaf, cohomology. because all of these notions only involve "covering", not topology itself. we are going to give a Grothendieck topology on an affinoid space

Def: Weak G-topology on affinoid space $X = Sp\,A$

For affinoid space $X = Sp\,A$, define category $\mathcal{C}$ to be:
objects: affinoid subdomains of X
morphisms: inclusion

Now for $\forall$ U affinoid subdomain $\subset X$, we define $Cov(U)$ to be
$$Cov(U) = \{ (U_i \to U)_{i\in I} \mid |I| < +\infty,\ U_i \text{ affinoid subdomain of } X, \text{ s.t. } \bigcup_{i\in I} U_i = U \}$$ finite covering

Rmk: Obviously, there are lots of "open" subset in X, remember our intuition is, affinoid space is kind of analytification of affine schemes over complete field, then by canonical metric on K^n, there is a canonical topology as we have already shown before. But we **only** focus on those "good" ones, i.e. affinoid subdomains, G-G theorem tells us that they are finitely many union of finitely many R-subdomains. and R-subdomains are very easy to understand, and very easy to do cohomology. This idea also shows up in $\mathbb{R}/\mathbb{C}$, when we do cohomology, we only use those good-shaped open subsets.

Rmk: Why this gives a G-topology? We only need to check the last condition, which essentially the transitivity of affinoid spaces & fiber product property

Def: Strong G-topology on affinoid space $X = \mathrm{Sp}\,A$

In this definition, we enlarge both the category $\mathcal{C}$, and the covering data for every single object

Objects: U open in X, and $\exists$ a covering of U by affinoid subdomain $\{U_i\}_{i\in I}$, not necessarily finite, s.t. $\forall$ morphism of affinoid spaces $\varphi: Z \to X$, s.t. $\varphi(Z) \subset U$, $\varphi^{-1}(U_i)$ covers Z, (they are also affinoid subdomains), $\exists$ a finite refinement of $(\varphi^{-1}(U_i) \to Z)$ by affinoid subdomains

they are called admissible open a.o

morphisms: inclusion

Now for $\forall V \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Cov}(V)$ is the following:

$(V_i \to V)_{i\in I} \in \mathrm{Cov}(V)$ if:

$\forall$ morphism of affinoid spaces $\varphi: Z \to X$, s.t. $\varphi(Z) \subset V$, $\varphi^{-1}(V_i)$ covers Z, $\exists$ a finite refinement of $(\varphi^{-1}(V_i) \to Z)_{i\in I}$ by affinoid subdomains.

admissible covering a.c

Let's check this really defines a G-topology

- trivial, just rephrase the covering condition
- suppose now $(V_i \to V)_{i\in I} \in \mathrm{Cov}(V)$, $(V_{ij} \to V_i)_{j\in J_i} \in \mathrm{Cov}(V_i)$, then for $\forall\, \varphi: Z \to X$, s.t. $\varphi(Z) \subset V$ we know $(\varphi^{-1}(V_i) \to Z)_{i\in I}$ admits a finite refinement by affinoid subdomains, i.e. $\exists\, Z_i \hookrightarrow \varphi^{-1}(V_i)$, and Z_i is affinoid subdomains of Z, here $1 \le i \le n$, now we have $Z_i \longrightarrow \varphi^{-1}(V_i) \xrightarrow{\varphi} V_i$, then $(\varphi^{-1}(V_{ij}) \to Z_i)_{j\in J_i}$ covers Z_i then $\exists$ finite refinement by affinoid subdomains, therefore we get a finite refinement by affinoid subdomains of $(V_{ij} \to V)$
- Suppose $(V_i \to V)_{i\in I} \in \mathrm{Cov}(V)$, and $W \to V$ is an inclusion.
  - (i) $V_i \times_V W$ exists: $\exists\, (W_j \to W)_{j\in J}$ satisfy finiteness condition, then we have $(\underset{W_j\cap V_i}{W_j \times_V V_i} \to \underset{W\cap V_i}{W \times_V V_i})_{j\in J}$, let's check it satisfies finiteness condition: $\forall\, Z \xrightarrow{\varphi} X$, s.t. $\varphi(Z) \subset W \times_V V_i \,(\hookrightarrow W)$, let's consider $(\varphi^{-1}(W_j \cap V_i) \longrightarrow Z)_{j\in J}$, we also have $(\varphi^{-1}(W_j) \to Z)_{j\in J}$, the latter one admits a finite refinement by affinoid subdomains, $Z_j \hookrightarrow \varphi^{-1}(W_j) \hookrightarrow Z$, obviously $Z = \varphi^{-1}(V_i) \Rightarrow Z_j \subset \varphi^{-1}(V_i)$ have $Z_j \hookrightarrow \varphi^{-1}(W_j \cap V_i)$, i.e. $V_i \times_V W$ exists in the G-topology
  - (ii) $(V_i \times_V W \longrightarrow W)_{i\in I} \in \mathrm{Cov}(W)$: for $\forall\, \varphi: Z \to X$, s.t. $\varphi(Z) \subset W\,(\hookrightarrow V)$, obviously $(\varphi^{-1}(V_i \cap W) \to Z)_{i\in I}$ covers Z, and moreover $(\varphi^{-1}(V_i) \to Z)$, but $\varphi^{-1}(V_i \cap W) = \varphi^{-1}(V_i) \cap \varphi^{-1}(W) = Z \cap \varphi^{-1}(V_i) = \varphi^{-1}(V_i)$, we know the latter one admits finite refinement by affinoid subdomains, hence $(V_i \times_V W \to W) \in \mathrm{Cov}(W)$

Rmk: What's the relation between weak G-topology & strong G-topology?

1. Weak G-topology only contains a very small family of open sets, recall that affinoid subdomain is simply a finitely many union of R-domains, but we also point out that this already gives us a basis of the canonical topology. However, Strong G-topology contains much more "open sets", for example
   - finite union of affinoid subdomains (obviously it is not necessarily affinoid subdomain)
   - Zariski open. i.e. Suppose $X = \mathrm{Sp}\,A$, $f \in A$, then $D(f) = \{x \in X \mid f(x) \neq 0\}$ is admissible open

pf: $D(f) = \bigcup_{n=0}^{\infty} X(f^{-1}, \varepsilon^n)$, $X(f^{-1}, \varepsilon^n) = \{x \in X \mid |f(x)| \ge \varepsilon^n\}$ is an affinoid subdomain

now $\forall\, \varphi: Z \to X$, s.t. $\varphi(Z) \subset D(f)$, we get $\varphi^*: A \to B$, $(Z = \mathrm{Sp}\,B)$, $g = \varphi^*(f)$. since $\varphi(Z) \subset D(f)$ we know $g(z) \neq 0$, $\forall z \in Z$, $\varphi^{-1}(X(f^{-1}, \varepsilon^n)) = \{z \in Z \mid |g(z)| \ge \varepsilon^n\}$, by maximum principle, $\exists n_0 \ge 0$, s.t. $|g(z)| \ge \varepsilon^{n_0}$, $\forall z \in Z$, i.e. Z is covered by $\bigcup_{n \ge n_0} \varphi^{-1}(X(f^{-1}, \varepsilon^n)) = \varphi^{-1}(X(f^{-1}, \varepsilon^{n_0}))$

this claim tells us that Strong G-topology on affinoid spaces is "finer" than Zariski topology

actually we also have to prove every affine covering is admissible, this simply because affine space is quasi-compact!

2. every (pre-)sheaf on weak G-topology extends uniquely to a (pre-) sheaf on strong G-topology. Remember that every a.o. is a union of affinoid subdomains, so we can apply the usual criterion for sheaf

$$\mathcal{F}(U) \overset{?}{=} \text{equalizer of } \Big(\prod_i \mathcal{F}(U_i) \to \prod_{i,i'} \mathcal{F}(U_i \cap U_{i'}) \Big) =: H^0((U_i), \mathcal{F})$$

but obviously we can have many covering $(U_i' \to U)_{i \in I}$ satisfying the condition for an a.o., it is not obvious that different choice of covering gives the same answer, hence we define

$$\mathcal{F}(U) = \varinjlim_{\mathfrak{U}=(U_i)} H^0((U_i), \mathcal{F})$$

the partial order is given by refinement relation

3. Strong G-topology satisfies "completeness condition", and actually, Strong G-topology is determined by the following conditions:

G0: $\emptyset$, X are admissible open

G1: $(U_i \to U)$ is a.c., then suppose $V \subset X$, and $V \cap U_i$ is a.o., $\forall i \Rightarrow V \cap U$ is a.o.

G2: Suppose $(U_i)_{i \in I}$ covers a.o. U, where U_i are a.o., and $\exists$ a refinement of $(U_i)_{i \in I}$ by a.c. of U, then $(U_i)_{i \in I}$ itself is an a.c.

Remember that we would like to construct a global rigid space which is locally an affinoid, this tells us that we should give a G-topology on a "global space" whose locally restricted G-topology is an affinoid. Now that if we know the local G-topology, how to recover the "global" one? The most naive idea is that if a subset is "locally a.o.", then it is globally a.o, and if a covering is "locally a.o.", then it is "globally a.o" This claim tells us this most naive idea is right when the space is affinoid

Now we can give the definition of rigid K-space

Def: A rigid K-space is a locally ringed G-space, $(X, \mathcal{O}_X)$, s.t.

- The G-topology satisfies G0, G1, G2
- $\exists$ an admissible covering $(X_i)_{i \in I}$ of X, s.t. $(X_i, \mathcal{O}_X|_{X_i})$ is an affinoid K-space, $\forall i \in I$

Rmk: G1 tells us that $V \subset X$ is a.o. $\Leftrightarrow$ $V \cap X_i$ is a.o. in X, $\forall i \in I$

but we know nothing about the G-topology of the global space X, we only know the G-topology on X_i, our naive expectation is that $V \cap X_i$ is a.o. in X_i, so that we could have a sense about the local shape of V, so what is the relation between G-topology on X_i & G-topology on X?

Local-to-Globale lemma: X is a set, (X_i) is a covering of X, $\mathfrak{T}_i$ is a G-topology on X_i, $\forall i \in I$ s.t. G0, G1, G2 are satisfied. For all $i, j \in I$, suppose

- $X_i \cap X_j$ is $\mathfrak{T}_i$-open
- $\mathfrak{T}_i$ & $\mathfrak{T}_j$ restricts to the same G-topology on $X_i \cap X_j$

then $\exists$ a unique G-topology $\mathfrak{T}$ on X, s.t.

① X_i is $\mathfrak{T}$-open, $\mathfrak{T}|_{X_i} = \mathfrak{T}_i$, ② $\mathfrak{T}$ satisfies G0, G1, G2, ③ $(X_i \to X) \in Cov(X)$ in $\mathfrak{T}$

By this lemma and second condition for rigid K-spaces, we know, a.o. in X_i are all of the shape: $X_i \cap V$, where V is a.o. in X, same claim also holds for a.c.

Rmk : G2 tells us the shape of o.c. := U a.o. in X, then $(U_j \to U)_{j\in J}$ is o.c. iff
(continued) $(U_j \times_X X_i \longrightarrow U \times_X X_i)_{j\in J}$ is o.c. of $U \times_X X_i$ for $\forall i$

pf: if $(U_j \times_X X_i \longrightarrow U \times_X X_i)_{j\in J}$ is o.c. of X_i for $\forall i$, then $(U_j \times_X X_i \to U)_{i\in I, j\in J}$ is o.c. of U
obviously this is a refinement of $(U_j \to U)_{j\in J} \Rightarrow (U_j \to U)_{j\in J}$ is o.c.
Now if $(U_j \to U)_{j\in J}$ is o.c. then by def $(U_j \times_X X_i \longrightarrow U \times_X X_i)_{j\in J}$ is o.c. of $U \times_X X_i$

Structure sheaf $\mathcal{O}_X$

In the definition of rigid K-space, we use a sheaf $\mathcal{O}_X$. let's give the definition and some properties of it
Under the weak G-topology on affinoid space $X = \mathrm{Sp}A$. we define:

$$\mathcal{O}_X(U) = \text{the K-algebra of } U$$

since U is an affinoid subdomain, it has the shape $U = \mathrm{Sp}B$, $\mathcal{O}_X(U)$ is simply B

Tate's Acyclicity Theorem: $\mathcal{O}_X$ forms a sheaf for the weak G-topology on $X = \mathrm{Sp}A$

Rmk: By the property of strong G-topology, there is a unique extension of $\mathcal{O}_X$ to a sheaf for strong G-topology on $X = \mathrm{Sp}A$
this is the local sheaf $\mathcal{O}_X|_{X_i}$ of X_i in the definition of rigid K-spaces.

pf:

I would like to talk more about the G-topology on the rigid K-space X. In AG, we know affine schemes are quasi-compact & quasi-separated. So in some cases, section of function ring is easy to compute. Look closely into the definition of strong G-topology of affinoid K-spaces, we have similar results.

<u>Prop</u>: · Affinoid K-space is quasi-compact w.r.t the weak & strong G-topology.

· Rigid K-space X is quasi-separated, i.e. if U & V are two quasi-compact a.o. subset of X, then $U \cap V$ is quasi-compact a.o. subset.

pf · · Suppose we have an a.c $(U_i \to U)_{i\in I}$, where $U = \operatorname{Sp} A$, then consider $U \xrightarrow{id} U$, we get there is a finite refinement by affinoid subdomains $V_j \subset U$, s.t. $V_j \hookrightarrow U_{\sigma(j)} \hookrightarrow U$, hence the covering is essentially finite

· By the definition of a.o subset, we only need to prove this when X is affinoid K-space, because if U_1 & U_2 is quasi-compact w.r.t the G-topology, then $U_1 \cup U_2$ is also qc w.r.t the G-topology, because if $(W_i \to U_1 \cup U_2)_{i\in I}$, then $(W_i \cap U_1 \to U_1)$, $(W_i \cap U_2 \to U_2)$ are a.c of U_1 & U_2 respectively, hence they are both essentially finite. By the same reason, we can reduce to the case that U & V are R-domains, then $U\cap V$ is also an R-domain, therefore $U \cap V$ must be qc.

For a quasi-compact rigid K-spaces, we can compute its global sections by the following lemma

Lemma: Suppose X is a quasi-compact rigid K-space, then take any admissible covering by affinoids $(X_i \to X)_{i=1}^{n}$

$$0 \longrightarrow \mathcal{O}_X(X) \to \prod_{i=1}^{n} \mathcal{O}_X(X_i) \to \prod_{i=1}^{n} \mathcal{O}_X(X_i \cap X_j)$$

pf · We know $\mathcal{O}_X(X) = \varinjlim_{\mathfrak{U}} \Gamma(\mathfrak{U}, \mathcal{O}_X)$, we have a natural map $\Gamma(\mathfrak{U}, \mathcal{O}_X) \to \mathcal{O}_X(X)$

Now suppose $\mathfrak{U}'$ is a refinement of $\mathfrak{U}$, i.e. $\mathfrak{U}' = (U_i' \to X)_{i\in I'}$, and $U_i' \hookrightarrow X_{\sigma(i)} \hookrightarrow X$

then for $\forall k$, $(U_i' \cap X_k \to X_k)_{i\in I'}$ is an a.c. of X_k, hence $\exists$ a finite refinement by affinoid subdomains, i.e. $\exists\ \mathfrak{U}'' \to \mathfrak{U}' \to \mathfrak{U}$, where $\mathfrak{U}''$ is a finite covering, so it suffices to show that $\Gamma(\mathfrak{U}, \mathcal{O}_X) \xrightarrow{\sim} \Gamma(\mathfrak{U}'', \mathcal{O}_X)$

this covering is actually better than a simply covering, if we denote $\mathfrak{U}'' = (U_i'' \to X)_{i\in I''}$, $|I''| < +\infty$, define subset $I''(k) = \{ i \in I'' \mid \sigma(i) = k\}$, then $\mathfrak{U}''_k = (U_i'' \to X_k)_{i\in I''(k)}$ is an a.c. of X_k, then

$$0 \to \Gamma(\mathfrak{U}'', \mathcal{O}_X) \to \prod_{i\in I''(k)} \mathcal{O}_X(U_i'') \to \prod \mathcal{O}_X(U_i'' \cap U_j'')$$

We have

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \Gamma(\mathfrak{U}, \mathcal{O}_X) \arrow[r] \arrow[d, dashed] & \prod_{k=1}^{n} \mathcal{O}_X(X_k) \arrow[r] \arrow[d] & \prod_{k,l} \mathcal{O}_X(X_k \cap X_l) \arrow[d] \\
0 \arrow[r] & \Gamma(\mathfrak{U}'', \mathcal{O}_X) \arrow[r] & \prod_{k=1}^{n} \prod_{i\in I''(k)} \mathcal{O}_X(U_i'') \arrow[r] \arrow[d] & \prod_{k=1}^{n} \prod_{i,j} \mathcal{O}_X(U_i'' \cap U_j'') \arrow[dl, phantom, "//"] \\
 & & \prod_{k=1}^{n} \prod_{i,j} \mathcal{O}_X(U_i'' \cap U_j'') &
\end{tikzcd}$$

take $\forall s \in \Gamma(\mathfrak{U}'', \mathcal{O}_X)$.

$$\begin{tikzcd}
s \arrow[r] & (s_i) \arrow[r] \arrow[d] & 0 \arrow[dl, phantom, "//"] \\
 & 0 &
\end{tikzcd}$$

which means (s_i) gives rise to $\prod_{k=1}^{n} \mathcal{O}_X(X_k)$

hence $\Gamma(\mathfrak{U}, \mathcal{O}_X) \simeq \Gamma(\mathfrak{U}'', \mathcal{O}_X)$

only need to check: $(U_i'' \cap U_j'', i \in I''(k), j \in I''(l) \to X_k \cap X_l)$ is an a.c.

because $(U_i'' \cap X_l \to X_k \cap X_l)_{i\in I''(k)}$, and $(U_i'' \cap U_j'' \to U_i'' \cap X_l)_{j\in I''(l)}$ both a.c.

Prop: X is a quasi-compact rigid K-space, and $A = \mathcal{O}_X(X)$, suppose $(g_0, \dots, g_r) = A$, then

$$\mathcal{O}_X\left(X\langle \frac{g_1}{g_0}, \dots, \frac{g_r}{g_0}\rangle\right) = A\langle \frac{g_1}{g_0}, \dots, \frac{g_r}{g_0}\rangle$$

question: does A give rise to an affinoid K-alg?

pf: Since X is quasi-compact, $\exists (X_i)_{i=1}^{n}$, s.t. X_i are affinoid K-spaces, define $A_i = \mathcal{O}_X(X_i)$, then by the lemma,

$$0 \to A \to \prod_{i=1}^{n} A_i \longrightarrow \prod_{i,j} \mathcal{O}_X(X_i \cap X_j)$$

Obviously, $X\langle \frac{g_1}{g_0}, \dots, \frac{g_r}{g_0}\rangle$ is quasi-compact, therefore we get an exact sequence:

$$0 \to \mathcal{O}_X(X\langle \frac{g_\cdot}{g_0}\rangle) \to \prod_{i=1}^{n} \mathcal{O}_X(X_i\langle \frac{g_\cdot}{g_0}\rangle) \to \prod_{i,j} \mathcal{O}_X(X_i \cap X_j\langle \frac{g_\cdot}{g_0}\rangle)$$

here $\mathcal{O}_X(X_i\langle \frac{g_\cdot}{g_0}\rangle) = A_i\langle \frac{g_\cdot}{g_0}\rangle = A_i \hat{\otimes}_A A\langle T_1, \dots, T_r\rangle/(g_0T_i - g_i)$, since $A\langle T_1, \dots, T_r\rangle/(g_0 T_i - g_i)$ is flat over A, then compare these two exact sequences it suffices to show that

$$\mathcal{O}_X(X_i \cap X_j) \hat{\otimes}_A A\langle T_1, \dots, T_r\rangle/(g_0T_i - g_i) \simeq \mathcal{O}_X(X_i \cap X_j)\langle \frac{g_i}{g_0}\rangle$$

but we have shown $X_i \cap X_j$ is q.c. it can be covered by finitely many affinoid subdomains, X_{ijk}, then both these two exact sequence can be extended to $\mathcal{O}_X(X_{ijk})$ & $\mathcal{O}_X(X_{ijk}\langle \frac{g_\cdot}{g_0}\rangle)$

Similar to AG, we can define properties of morphisms between rigid K-spaces, such as separatedness, properness

GAGA

Keep in mind that rigid K-space is the analogue of manifold in the non-Archimedean field, for varieties $/\mathbb{C}$ the famous GAGA principle gives a correspondence to $\mathbb{C}$-manifold, and some sheaf category are equivalent. Now in this section we give GAGA principle for varieties X/K, where K is non-Archimedean local field, we will define the so-called GAGA functor, which takes X to a rigid space X^{rig}, we will first construct for $X = \mathbb{A}^n_K$, then for $X = \operatorname{Sp} A$ affinoids, then for general rigid K-spaces

Main goal: $\{\text{locally of finite type } K\text{-schemes}\} \xrightarrow{GAGA} \{\text{rigid analytic } K\text{-spaces}\}$

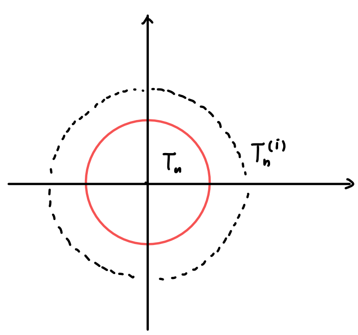
$$Z \longrightarrow Z^{rig}$$

Of course, this functor should satisfy some universal properties

<u>Def</u>: (Rigid analytification) For a locally of finite type K-scheme Z, the rigid analytification of Z is a pair (Z^{rig}, ι), where Z^{rig} is a rigid K-space, and $\iota: (Z^{rig}, \mathcal{O}_{Z^{rig}}) \to (Z, \mathcal{O}_Z)$ is a morphism of locally G-ringed K-spaces, which satisfies the following universal property

- For a K-rigid space $(Y, \mathcal{O}_Y)$, and a morphism of locally G-ringed K-spaces $f: Y \to Z$, $\exists$ a unique morphism $(Y, \mathcal{O}_Y) \to (Z^{rig}, \mathcal{O}_{Z^{rig}})$ factoring through f by ι

- $\mathbb{A}^n_K$: Contrary to our naive idea, the rigid analytic space associated to $\mathbb{A}^n_K$ is not $(\operatorname{Sp} T_n, T_n)$, because the latter one is just a "unit ball around 0", and by intuition, $\mathbb{A}^{n,rig}_K$ should be the whole affine space "K^n" (or $\overline{K}^n/\mathrm{Gal}_K$), the picture is:



to include all points, we consider $c \in K$, $|c| > 1$, and

$$T_n^{(i)} = T_n(|c|^i) = K\langle c^{-i}\zeta_1, \dots, c^{-i}\zeta_n\rangle$$

then $\operatorname{Sp} T_n^{(i)}$ can be understood as $B^n(|c|^i)$, i.e. ball of radius $|c|^i$

we have natural map:

$$T_n = T_n^{(0)} \leftarrow T_n^{(1)} \leftarrow T_n^{(2)} \leftarrow T_n^{(3)} \leftarrow \cdots$$
$$\zeta_i \longleftarrow \zeta_i$$

hence we get a sequence of morphisms of affinoid K-spaces:

$$\operatorname{Sp} T_n \longrightarrow \operatorname{Sp} T_n^{(1)} \longrightarrow \operatorname{Sp} T_n^{(2)} \longrightarrow \operatorname{Sp} T_n^{(3)} \longrightarrow \cdots$$

then by gluing these $\operatorname{Sp} T_n^{(i)}$, we get $\mathbb{A}^{n,rig}_K = \bigcup_{n=0}^{\infty} \operatorname{Sp} T_n^{(i)}$

- $\operatorname{Spec} A$: Similarly, consider $A^{(i)} = T_n^{(i)}/\mathfrak{a}$. hence $\mathfrak{a}$ is the image of $\mathfrak{a}$ under $K[\zeta] \longrightarrow T_n^{(i)}$, $\forall i$

$K[\zeta]/\mathfrak{a} \simeq A$

then we have natural map: $\underset{\substack{V(\mathfrak{a}) \text{ in} \\ \text{the 1-ball}}}{A^{(0)}} \leftarrow \underset{\substack{V(\mathfrak{a}) \text{ in} \\ \text{the } |c|\text{-ball}}}{A^{(1)}} \leftarrow \underset{\substack{V(\mathfrak{a}) \text{ in} \\ |c|^2\text{-ball}}}{A^{(2)}} \leftarrow \cdots \rightsquigarrow \operatorname{Sp} A^{(0)} \to \operatorname{Sp} A^{(1)} \to \operatorname{Sp} A^{(2)} \longrightarrow \cdots$

$$(\operatorname{Spec} A)^{rig} = \bigcup_{i=0}^{\infty} \operatorname{Sp} A^{(i)}$$

Rmk: Obviously, the above construction involves the choice of $|c|>1$, and the presentation of A. Once we can show that they satisfy the universal property of rigid analytification, then these choices have no effects on the final space. the universal property relies on the following lemma:

Lemma: Z is an affine scheme over K of finite type. Y is a rigid K-space, then we have

$$\underset{\substack{\text{morphisms as} \\ \text{locally ringed G-spaces.}}}{\mathrm{Mor}(Y, Z)} \simeq \mathrm{Hom}_K(\mathcal{O}_Z(Z), \mathcal{O}_Y(Y))$$

Once we have established this lemma, consider $\forall\, Y \xrightarrow{\varphi} \mathrm{Spec} A = \mathrm{Spec}\, K[\zeta]/\mathfrak{a}$, φ corresponds to $\sigma: K[\zeta]/\mathfrak{a} \longrightarrow \mathcal{O}_Y(Y)$
consider $|\sigma(\bar{\zeta}_i)|_{sup} = \sup_{y\in Y} |\sigma(\bar{\zeta}_i)(y)|$, choose $N>0$, s.t. $|\sigma(\bar{\zeta}_i)|_{sup} \le |c|^N$, then σ admits a factorization:

$$\begin{array}{ccccc} K[\zeta]/\mathfrak{a} & \longrightarrow & T_n^{(N)}/\mathfrak{a} & \longrightarrow & \mathcal{O}_Y(Y) \\ & & \bar{X}_i & \longmapsto & c^{-N}\sigma(\bar{\zeta}_i) \\ \bar{\zeta}_i & \longmapsto & \bar{\zeta}_i & \longmapsto & \sigma(\bar{\zeta}_i) \end{array}$$

hence we get $Y \longrightarrow \mathrm{Sp}\, T_n^{(N)}/\mathfrak{a} \hookrightarrow (\mathrm{Spec} A)^{rig} \xrightarrow{\iota} \mathrm{Spec} A$

Now we state the main theorem in this section

Thm: Every locally of finite type K-scheme Z admits a rigid analytification $(Z^{rig}, Z^{rig} \xrightarrow{\iota} Z)$. Futhermore, as a set Z^{rig} = closed points of Z

pf: Suppose $Z = \bigcup_i U_i$, U_i open affine, finite type$/K$, morally speaking, $Z^{rig} = "\bigcup_i" U_i^{rig}$
this union is actually a "gluing" data: for $\forall i$, $\exists\, U_i^{rig} \xrightarrow{\iota_i} U_i$, and for $U_i \cap U_j$, easy to show that $\iota_i^{-1}(U_i\cap U_j)$ & $\iota_j^{-1}(U_i \cap U_j)$ are two rigid analytifications of $U_i \cap U_j$, hence $\exists!\ \varphi_{ij}: \iota_i^{-1}(U_i\cap U_j) \subset U_i^{rig} \xrightarrow{\sim} \iota_j^{-1}(U_i\cap U_j) \subset U_j^{rig}$, then $(U_i^{rig}, \varphi_{ij})$ can be glued to be Z^{rig}

Rmk: For $\forall$ morphism of schemes $f: Z_1 \longrightarrow Z_2$, we get $Z_1^{rig} \xrightarrow{\iota_1} Z_1 \xrightarrow{f} Z_2$, then by universal property, $\exists f^{rig}$

$$\begin{tikzcd} Z_1^{rig} \arrow[r, "f^{rig}"] \arrow[d, "\iota_1"'] & Z_2^{rig} \arrow[d, "\iota_2"] \\ Z_1 \arrow[r, "f"] & Z_2 \end{tikzcd}$$

Now we have partially arrived our goal! We constructed rigid analytification for each locally finite type K-scheme, this rigid analytification agrees with our intuition: it consists of closed points. Especially, we focus our eyes on K-varieties, The classical GAGA functor says that for schemes $X/\mathbb{C}$, $(X^{an}, \mathcal{O}_{X^{an}})$ & $(X, \mathcal{O}_X)$ are "same" when computing cohomology of coherent sheaves, moreover, the latter one has much more "non-algebraic" cohomologies, In our situation, we will later show that X^{rig}/K also good for coherent sheaves.

Example: Tate curve

In this section, I will give an explanation for the following isomorphism

$$\mathbb{G}_m^{rig}/q^{\mathbb{Z}} \simeq E_q^{rig}$$

here E_q is an elliptic curve$/K$, $q \in K^\times$, $|q|<1$, and E_q is defined by the following equation

$$y^2 + xy = x^3 + a_4(q)x + a_6(q)$$

- $\mathbb{G}_m^{rig}$, $\mathbb{G}_m^{rig}/q^{\mathbb{Z}}$

We know that $\mathbb{G}_m = \mathrm{Spec}\, K[T,S]/(1-ST)$, then for $|q^{-1}|>1$, consider

$$\mathbb{G}_m^{(i)} = \mathrm{Sp}\left(K\langle q^i T, q^i S\rangle/(1-ST)\right)$$ as a set, simply $x \in \bar{K}^\times/_{G_K}$, $|x| < |q|^{-i}$

by the construction of rigid analytification, $\mathbb{G}_m^{rig} = \bigcup_{i=0}^{\infty} \mathbb{G}_m^{(i)}$

what is the meaning of $\mathbb{G}_m^{rig}/q^{\mathbb{Z}}$? As a set, it is $\bar{K}^\times/q^{\mathbb{Z}}$, i.e. $x \sim y$ if $y = xq^n$, $n \in \mathbb{Z}$

Obviously, we know that $\forall x \in \bar{K}^\times$ can be equivalent to a unique $x' \in \bar{K}^\times$, s.t. $|q| < |x| \leq 1$. therefore we consider

$$\mathrm{Sp}\, K\langle T,S\rangle/(1-ST)\,\langle |q|T^{-1}\rangle \simeq \mathrm{Sp}\, K\langle T,S\rangle\,\langle |q|S\rangle$$

$$\simeq \mathrm{Sp}\, K\langle T,S,U\rangle/(1-ST,\ U-|q|S)$$

- E_q & E_q^{rig}

We know $E_q = \mathrm{Proj}\, K[X,Y,Z]/(Y^2Z + XYZ - X^3 - a_4(q)XZ^2 - a_6(q)Z^3)$

$$= \mathrm{Spec}\, K[x,y]/(y^2+xy-x^3-a_4(q)x-a_6(q)) \cup \mathrm{Spec}\, K[x,z]/((1+x)z - x^3 - a_4(q)xz^2 - a_6(q)z^3)$$

$$E_q^{(i)} = \mathrm{Sp}\, K\langle |q|^i x, |q|^i y\rangle/(y^2+xy-x^3-a_4(q)x-a_6(q)) \cup \mathrm{Sp}\, K\langle |q|^i x, |q|^i z\rangle/((1+x)z - x^3 - a_4(q)xz^2 - a_6(q)z^3)$$

as a set, $E_q^{(i)} = E_q(\bar{K})/_{G_K}$ is a union of $\begin{cases} \text{radius } |q|^{-i}\text{-ball around } [0:0:1] \\ \text{radius } |q|^{-i}\text{-ball around } [0:1:0] \end{cases}$

- The isomorphism $\mathbb{G}_m^{rig}/q^{\mathbb{Z}} \longrightarrow E_q^{rig}$

Define $s_k(q) = \sum_{n=1}^{\infty} \frac{n^k q^n}{1-q^n}$, then $\mathbb{G}_m^{rig}/q^{\mathbb{Z}} \longrightarrow E_q^{rig}$ can be given by:

$$\varphi:\ \bar{K}^\times/q^{\mathbb{Z}} \longrightarrow E_q(\bar{K})$$

$$u \longmapsto (X(u,q),\ Y(u,q))$$

Trick that I have never noticed before: for a variety $X/_k$, $\mathrm{Max}\, X \neq \underset{\sim}{X(\bar{k})} = \mathrm{Mor}_k(\mathrm{Spec}\,\bar{k}, X)$

$\mathrm{Max}\, X = X(\bar{k})/_{G_k}$.

here $X(u,q) = \sum_{n\in\mathbb{Z}} \frac{q^n u}{(1-q^n u)^2} - 2s_1(q)$, $Y(u,q) = \sum_{n\in\mathbb{Z}} \frac{(q^n u)^2}{(1-q^n u)^3} + s_1(q)$

Both sides have Galois action, and this morphism is $\mathrm{Gal}(\bar{K}/K)$-equivariant, hence reduced to the temporarily set-theoretically map from $\mathbb{G}_m^{an}/q^{\mathbb{Z}}$ to E_q^{rig}, this map is bijective since φ is bijective

Rigid Cohomology

In this part, I try to summarize some further topics on rigid analytic spaces, such as overconvergent isocrystals, the main goal is to understand how to associate a overconvergent isocrystal to a crystalline crystal.

Let's fix notations, V is a mixed characteristic $(0,p)$ DVR, k = residue field, π uniformizer, $K = \mathrm{Frac}(V)$. We usually consider the following two types of geometric objects

- X: k-scheme locally of finite type,
- $\mathcal{P}$: π-adic formal scheme / $\mathrm{Spf}\, V$, i.e. $\varprojlim_n \mathcal{O}_P / \pi^n \mathcal{O}_P \simeq \mathcal{O}_P$

  s.t. northerian

  locally of topologically finite type. i.e. locally is a quotient of $V\langle t_1, \dots, t_n\rangle$ ← Tate algebra

Suppose Y is a noetherian scheme over V, locally of finite type, then we can associate two rigid analytic spaces to it:

- Y_K^{rig}: Y_K is a K-scheme, locally of finite type, construction by the previous GAGA
- $(\hat{Y})_K$: $\hat{Y}$ is the π-adic completion of Y, it's a π-adic formal scheme $/V$

  then $(\hat{Y})_K$ is locally a rigid analytic space.

<u>Prop</u>: there is a canonical morphism, which is an open immersion

$$\iota: \ (\hat{Y})_K \longrightarrow Y_K^{rig}$$

ι realizes $(\hat{Y})_K$ as an admissible open of Y_K^{rig} and ι is an isomorphism is Y is proper over V.

<u>Example</u>: $Y = \mathbb{A}_V^n$, then

$$(\hat{Y})_K = \hat{\mathbb{A}}_V^n = \{(t_1, \dots, t_n) \in \bar{K} \mid |t_i| \le 1\} / \mathrm{Gal}_K$$

$$Y_K^{rig} = \{(t_1, \dots, t_n) \in \bar{K}\} / \mathrm{Gal}_K$$

Frames & Tubes

Suppose X is a k-variety, P π-adic formal scheme$/V$, a formal embedding

$$\iota : X \hookrightarrow P$$

is a locally closed immersion, the tube of X in P is defined to be

$$]X[_P = sp^{-1}(\iota(X))$$

i.e. we have the following diagram:

$$\begin{tikzcd}]X[_P \arrow[r, "\iota"] \arrow[d] & P_K \arrow[d, "sp"] \\ X \arrow[r, "\iota"'] & P \end{tikzcd}$$

$sp: P_K \longrightarrow P$

Thus, using Proposition 7, we get a canonical specialization map

$$\mathrm{sp}: X_{\mathrm{rig}} \longrightarrow X_k$$

that is characterized as follows. Consider a point $x \in X_{\mathrm{rig}}$. To determine its image $\mathrm{sp}(x) \in X_k$, choose an affine open subscheme $U = \mathrm{Spf}\, A$ in X such that x belongs to $U_{\mathrm{rig}} = \mathrm{Sp}\, A \otimes_R K$, and let $\mathfrak{m} \subset A \otimes_R K$ be the corresponding maximal ideal. Then consider the projection $\tau_K : A \otimes_R K \longrightarrow (A \otimes_R K)/\mathfrak{m} = K'$ where K' is finite over K by 3.1/4. Let $\tau : A \longrightarrow B$ for $B = \tau_K(A)$ be the restriction of τ_K. As we have seen, B is a local integral domain of dimension 1 lying between R and the valuation ring of K'. In fact, τ gives rise to the rig-point of X corresponding to x, and the surjections $A \otimes_R k \longrightarrow B \otimes_R k \longrightarrow B \otimes_R k / \mathrm{rad}(B \otimes_R k)$ determine the closed point of the special fiber $U_k = \mathrm{Spec}\, A \otimes_R k \subset X_k$ that equals the image of x under the specialization map sp. Note that the construction of sp is similar to the one considered in [BGR], 7.1.5, although we never use "canonical reductions"

basically, if f is a function on P cuts out X, then

$$|f(x)| < 1, \ \forall x \in]X[_P$$

Example:

Proposition 2.2.11 *Let P be a (locally closed) formal subscheme of $\widehat{\mathbf{P}}^N_{\mathcal{V}}$ and $X := \mathbf{A}^N_k \cap P_k$. Then, we have*

$$]X[_P = \mathbf{B}^N(0, 1^+) \cap P_K.$$

Corollary 2.2.12 *Let A be a $\mathcal{V}$-algebra of finite presentation, $X := \mathrm{Spec}\, A$ and $V := X_K^{\mathrm{rig}}$. Let*

$$\mathcal{V}[T_1, \ldots, T_N] \to A$$

be a presentation of A, $X \subset \mathbf{A}^N_{\mathcal{V}}$ the corresponding inclusion, Y the algebraic closure of X in $\mathbf{P}^N_{\mathcal{V}}$ and $P := \widehat{Y}$. Then, we have

$$]Y_k[_{\widehat{Y}} = \widehat{Y}_K = Y_K^{\mathrm{rig}} \supset X_K^{\mathrm{rig}}$$

and

$$]X_k[_{\widehat{Y}} = \widehat{X}_K = \mathbf{B}^N(0, 1^+) \cap X_K^{\mathrm{rig}} \subset Y_K^{\mathrm{rig}}.$$

Def: A frame is a diagram

$$\underbrace{X \hookrightarrow Y}_{\substack{\text{open immersion} \\ \text{of } k\text{-schemes}}} \underbrace{\hookrightarrow P}_{\substack{\text{closed immersion of a } k\text{-scheme } Y \\ \text{into a } \pi\text{-adic formal scheme } P}}$$

Example: X is a quasi-projective variety$/k$, take $\mathbb{P}^N_k$ for some ambient space of X

then take $Y =$ closure of X in $\mathbb{P}^N_k$, $P = \widehat{\mathbb{P}^N_V}$

A morphism between frames is a diagram

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "g"] & Y' \arrow[r, hook] \arrow[d, "f"] & P' \arrow[d, "u"] \\ X \arrow[r, hook] & Y \arrow[r, hook] & P \end{tikzcd}$$

Note that u_k induces $P_k' \to P_k$, and the corresponding

$$\left.\begin{array}{l}]X'[_{P'} \longrightarrow]X[_{P,} \\ ]Y'[_{P'} \longrightarrow]X[_{P} \end{array}\right\} \Rightarrow$$ this two maps has nothing to do with f & g so usually we will only mention u

We say a morphism of two frames flat (resp. smooth, étale) if u is flat (resp. smooth. étale) in a neighborhood of X' (i.e. $\exists\, U' \subset P'$ open, and $u|_{U'} : U' \to P$ is flat (resp. smooth, étale))

We say a morphism of two frames quasi-compact (affine), if u is qc (affine)

Prop: Flat (smooth, étale) morphism u induces flat (smooth, étale) morphism of

$$]X[_P \longrightarrow]X'[_{P'}$$

Tubes of small radius

The previous definition $]X[_P$ doesn't depend on the scheme structure of X, but what we are going to do next does depend on the scheme structure of X

We first consider the following lemma

Lemma 2.3.1 *Let $P = \operatorname{Spf} A$ be a formal affine $\mathcal{V}$-scheme,*

$$X := V(f_1, \ldots, f_r) \cap D(g_1, \ldots g_s) \cap P_k \subset P$$

and X' a subvariety of X defined in P as

$$X' := V(f_1', \ldots, f_{r'}') \cap D(g_1', \ldots g_{s'}') \cap P_k.$$

Then, there exists $\eta_0 < 1$ such that for all $\eta_0 \leq \eta < 1$, the subset

$$[X']_{P\eta} := \{x \in P_K, |f_1'(x)|, \ldots, |f_{r'}'(x)| \leq \eta \\ \text{and} \quad \exists j \in \{1, \ldots, s'\}, |g_j'(x)| = 1\}$$

is contained in

$$[X]_{P\eta} := \{x \in P_K, |f_1(x)|, \ldots, |f_r(x)| \leq \eta \\ \text{and} \quad \exists j \in \{1, \ldots, s\}, |g_j(x)| = 1\}.$$

When the valuation is discrete, we may choose $\eta_0 = |\pi|$.

especially, it tells us that, $\exists \eta_0 < 1$, s.t. for $\forall \eta_0 < \eta \le 1$, $[X]_{P,\eta}$ is a well-defined open subset of P_K

Proposition 2.3.2 *Let $P = \mathrm{Spf} A$ be a formal affine $\mathcal{V}$-scheme and*

$$X := V(f_1, \cdots, f_r) \cap D(g_1, \cdots g_s) \cap P_k \subset P$$

a finite presentation of X as a formal subscheme of P. Define as before

$$[X]_{P\eta} := \{x \in P_K, |f_1(x)|, \ldots, |f_r(x)| \le \eta \text{ and } \exists j \in \{1, \ldots, s\}, |g_j(x)| = 1\}$$

and also

$$]X[_{P\eta} := \{x \in P_K, |f_1(x)|, \ldots, |f_r(x)| < \eta \text{ and } \exists j \in \{1, \ldots, s\}, |g_j(x)| = 1\}.$$

obviously, for general π-adic formal scheme $P/\mathcal{V}$ we can glue the affine piece to get a definition

Then we have:

(i) *If $P' \subset P$ is an open subset, then for all $\eta < 1$,*

$$[X \cap P'_k]_{P'\eta} = [X]_{P\eta} \cap P'_K \quad \text{and} \quad]X \cap P'_k[_{P'\eta} =]X[_{P\eta} \cap P'_K.$$

(ii) *When the valuation is discrete, $[X]_{P\eta}$ and $]X[_{P\eta}$ only depend on X and not on the choice of the presentation as soon as $\eta \ge |\pi|$ in the first case and $\eta > |\pi|$ in the second case.*

(iii) *In general, if*

$$X = V(f'_1, \cdots, f'_{r'}) \cap D(g'_1, \cdots g'_{s'}) \cap P_k$$

is another presentation of X as a formal subscheme of P, and $[X]'_{P\eta}$ and $]X['_{P\eta}$ denote the corresponding rigid analytic varieties, there exists $\eta_0 < 1$ such that for all $\eta_0 \le \eta < 1$, we have

$$[X]'_{P\eta} = [X]_{P\eta} \quad \text{and} \quad]X['_{P\eta} =]X['_{P\eta}.$$

Strict neighborhood

Suppose we are given a frame $X \hookrightarrow Y \hookrightarrow P$, then consider $]Z[_P =]Y-X[_P$ is open in P_K
V is a strict nbd of X if $]X[_P \subset V \subset]Y[_P$ is an admissible open in P_K and
$$(V,]Z[_P)$$ forms a admissible covering of $]Y[_P$

The notion of strict nbd doesn't depends to Y, actually.

Proposition 3.1.10 *Let $(X \subset Y \subset P)$ be a frame and $Y' \subset Y$ a closed subvariety containing X. Then, a subset V of $]Y'[_P$ is a strict neighborhood of $]X[_P$ in $]Y[_P$ if and only if it is a strict neighborhood of $]X[_P$ in $]Y'[_P$.*

Examples for strict nbd are the following:

Proposition 3.2.4 *Let $(X \subset Y \subset P)$ be a quasi-compact frame and Z a closed complement for X in Y. If $\lambda < 1$, then*

$$V^\lambda :=]Y[_P \setminus]Z[_{P\lambda}$$

is an admissible open subset of $]Y[_P$ and we have an admissible covering

$$]Y[_P = V^\lambda \cup [Z]_{P\lambda}.$$

Moreover, if $\eta < 1$, then

$$V^\lambda_\eta := [Y]_\eta \cap V^\lambda$$

is a quasi-compact admissible open subset of $]Y[_P$ and we have an admissible covering

$$[Y]_{P\eta} = V^\lambda_\eta \cup ([Y]_{P\eta} \cap [Z]_{P\lambda}).$$

Finally, we also have an admissible covering $V^\lambda = \cup V^\lambda_\eta$.

Calculus

Infinitesimal site of rigid spaces

Def: Suppose $V \to T$ is a morphism of rigid spaces, then $(V/T)_{inf}$ consists of the following

$$\begin{array}{ccc} V_0' & \hookrightarrow & V' \\ \downarrow & & \downarrow \\ V & \longrightarrow & T \end{array}$$

where $V_0' \to V'$ is a nil closed immersion

Def: A (finitely presented) infinitesimal crystal on V/T is a family of (coherent) sheaves $\mathcal{E}_{V'}$ on every object $(V_0' \to V')$ of $(V/T)_{inf}$, such that

for every $(V_0'' \to V'') \xrightarrow{u} (V_0' \to V')$ over V, i.e.
$$\begin{array}{ccc} V_0'' & \longrightarrow & V'' \\ \downarrow & & \downarrow u \\ V_0' & \longrightarrow & V' \end{array}$$

we have an isomorphism

$$\varphi_u : u^* \mathcal{E}_{V'} \xrightarrow{\simeq} \mathcal{E}_{V''}$$

$\{\varphi_u\}$ is compatible in the sense that

$$\varphi_{v \circ u} = \varphi_v \circ v^*(\varphi_u)$$

Stratification:

We now turn to stratifications. Let

$$\delta : V \hookrightarrow V \times_T V$$

be the diagonal embedding. We will write $V^{(n)}$ for the n-th infinitesimal neighborhood of V in $V \times_T V$ (defined by $\mathcal{I}^{n+1}$ if V is defined by $\mathcal{I}$) and

$$p_1^{(n)}, p_2^{(n)} : V^{(n)} \to V$$

for the projections (which are homeomorphisms). We will always consider $V \times_T V$ as well as $V^{(n)}$ as rigid analytic varieties over V using the first projection.

A *stratification* on an $\mathcal{O}_V$-module $\mathcal{E}$ is a compatible sequence of linear isomorphisms called the *Taylor isomorphisms*

$$\{\epsilon^{(n)} : p_2^{(n)*}\mathcal{E} \simeq p_1^{(n)*}\mathcal{E}\}_{n \in \mathbf{N}}$$

on $V^{(n)}$ with $\epsilon^{(0)} = Id_{\mathcal{E}}$ that satisfy a cocycle condition on triple products. A morphism of stratified modules is a morphism of $\mathcal{O}_V$-modules that is compatible with the data. Stratified $\mathcal{O}_V$-modules form an abelian category with exact and faithful forgetful functor. Moreover, this construction is functorial in V/T and also with respect to isometric extensions of K. The tensor product of two stratified modules has a canonical stratification and the same is true for the internal Hom if the first module is coherent.

We will see that stratification is closely related to connections

Connections

With the same notations as above, the sheaf of differential operators on V/T can be defined as

$$\mathcal{D}_{V/T} := \varinjlim \mathcal{D}_{n,V/T}.$$

with

$$\mathcal{D}_{n,V/T} = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{O}_{V^{(n)}}, \mathcal{O}_V).$$

Also, by definition, there is an exact sequence

$$0 \to \Omega^1_{V/T} \to \mathcal{O}_{V^{(1)}} \to \mathcal{O}_V \to 0$$

and

$$d := p_2^{(1)*} - p_1^{(1)*} : \mathcal{O}_V \to \Omega^1_{V/T} \subset \mathcal{O}_{V^{(1)}}$$

makes $\Omega^1_{V/T}$ universal for T-derivations into coherent modules. A *connection* on an $\mathcal{O}_V$-module $\mathcal{E}$ is an $\mathcal{O}_T$-linear map

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_V} \Omega^1_{V/T}$$

satisfying the *Leibniz rule*

$$\nabla(fs) = f\nabla(s) + s \otimes df.$$

A *horizontal* map is an $\mathcal{O}_V$-linear map compatible with the connections. Mod-

Now we want to extend the above theory to the case that we are cared about: $]X[_p \hookrightarrow]Y[_p \hookrightarrow P_K$
First we consider the following lemma

Recall that the generic fiber functor $P \mapsto P_K$ behaves as well as we may expect with respect to products and immersions, and in particular with respect to infinitesimal neighborhoods. More precisely, if S is a formal $\mathcal{V}$-scheme, P a formal S-scheme, and $P^{(n)} \subset P \times_S P$ denotes the n-th infinitesimal neighborhood of P over S, we have $(P_K)^{(n)} = P_K^{(n)}$.

Proposition 4.3.1 *Let $X \hookrightarrow P$ be an S-immersion of an algebraic S_k-variety into a formal S-scheme. If $X \hookrightarrow P \hookrightarrow P^{(n)}$ is the composite immersion, we have an isomorphism*

$$]X[_{P^{(n)}} \simeq]X[_P^{(n)}$$

and, when X is quasi-compact, for $\eta < 1$, an isomorphism

$$[X]_{P^{(n)}\eta} \simeq [X]_{P\eta}^{(n)} \quad (\text{resp.}]X[_{P^{(n)}\eta} \simeq]X[_{P\eta}^{(n)}).$$

Then we can define the concept of stratification on strict nbd as follows

Definition 4.3.4 *Let $(X \subset Y \subset P)$ be an S-frame, V a strict neighborhood of $]X[_P$ in $]Y[_P$ and $\mathcal{E}$ an $\mathcal{O}_V$-module. A stratification on $\mathcal{E}$ is said to be* overconvergent *if there exists a strict neighborhood*

$$V' \subset (V \times_{S_K} V) \cap]Y[_{P \times_S P}$$

of $]X[_{P \times_S P}$ in $]Y[_{P \times_S P}$ and an isomorphism

$$\epsilon : p_2^* \mathcal{E}_{|V'} \simeq p_1^* \mathcal{E}_{|V'}$$

such that the Taylor isomorphism of $\mathcal{E}$ is induced on $V' \cap]Y[_P^{(n)}$ by ϵ for each n. We will also say that the connection of $\mathcal{E}$ is overconvergent. *In the case $Y = X$, we simply say* convergent.

Note that in the convergent case, there is only one strict neighborhood, namely $]X[_P$ itself. And we see that a stratified $\mathcal{O}_{]X[_P}$-module $\mathcal{E}$ is convergent if the Taylor isomorphisms of $\mathcal{E}$ comes from an isomorphism $\epsilon : p_2^* \mathcal{E} \simeq p_1^* \mathcal{E}$ on $]X[_{P^2}$.

Overconvergent Sheaves

A general definition for overconvergent sheaf is the following

Definition 5.1.1 *Let V be a rigid analytic variety over K and T an admissible open subset of V. A sheaf of sets $\mathcal{E}$ on V is* overconvergent along T *if $\mathcal{E}_{|T} = 0$ (and not $\emptyset$).*

but we will only focus on a specific kind of sheaves: $j_V^\dagger \mathcal{O}_V$ -modules

Dagger operators

Definition: **Proposition 5.1.12** *Let $(X \subset Y \subset P)$ be a frame, V an admissible open subset of $]Y[_P$ and $\mathcal{E}$ a sheaf of sets on V. Then*

$j_{VV'} : V' \to V$

$\mathcal{E}$

(i) *We have*

$$j_X^\dagger \mathcal{E} = \varinjlim j_{VV'*} j_{VV'}^{-1} \mathcal{E}$$

where V' runs through all the strict neighborhoods of $]X[_P$ in $]Y[_P$ and

$$j_{VV'} : V \cap V' \hookrightarrow V$$

if $V'' \subset V' \Rightarrow$

$V \cap V'' \overset{j_1}{\subset} V \cap V' \overset{j'}{\subset} V \qquad j'' = j' \circ j_1$

denotes the inclusion map.

$j_X^\dagger$: Sheaf of sets on V $\rightsquigarrow$ Sheaf of sets on V

sections of $j_X^\dagger$ can be described explicitly

If W is a quasi-compact admissible open subset of V, then

$$\Gamma(W, j_X^\dagger \mathcal{E}) = \varinjlim_{V' \subset V} \Gamma(W \cap V', \mathcal{E})$$

where V' runs through the strict neighborhoods of $]X[_P$ in $]Y[_P$.

it also tells us the following:

- restriction of $j_X^\dagger \mathcal{E}$ to $]X[_P$ is $\mathcal{E}|_{]X[_P}$
- restriction of $j_X^\dagger \mathcal{E}$ to $V \cap]Z[_P$ is 0

because for $\lambda < 1$, $U_\lambda =]Y[_P -]Z[_{P,\lambda}$ is a strict nbd of $]X[_P$ in $]Y[_P$

then $\underbrace{j_X^\dagger \mathcal{E}|_{V \cap]Z[_{P,\lambda}}}_{\text{any open affinoid, section} = 0} = 0$ because $U_\lambda \cap]Z[_{P,\lambda} = \emptyset$ and by the section formula above

since $]Z[_{P,\lambda}$ covers $]Z[_P \Rightarrow j_X^\dagger \mathcal{E}|_{V \cap]Z[_P} = 0$

The above observation makes the following category equivalence clear

Proposition 5.3.1 *Let $(X \subset Y \subset P)$ be a frame, V an admissible open subset of $]Y[_P$ and $\mathcal{A}$ a sheaf of rings on V. Then the categories of overconvergent $\mathcal{A}$-modules and $j_X^\dagger \mathcal{A}$-modules are equivalent.*

More precisely, the forgetful functor from $j_X^\dagger \mathcal{A}$-modules to $\mathcal{A}$-modules is fully faithful with exact left adjoint $j_X^\dagger$ and its image is the subcategory of overconvergent $\mathcal{A}$-modules.

$]Z[_P \cap V$

We also has the following category equivalence

Theorem 5.4.4 *Let $(X \subset Y \subset P)$ be a quasi-compact frame and V an admissible open subset of $]Y[_P$. Then, the functors $j_X^\dagger$ induce an equivalence of categories*

$$\varinjlim_{V'} \mathrm{Coh}(\mathcal{O}_{V \cap V'}) \simeq \mathrm{Coh}(j_X^\dagger \mathcal{O}_{V^\bullet})$$

when V' runs through the strict neighborhoods of $]X[_P$ in $]Y[_P$.

(here, the coherence means:

for any A: sheaf of rings. E: sheaf of A-modules

E is of finite type A-mod: if locally $\exists\, A^n \twoheadrightarrow E$

E is of coherence A-mod: if for $\forall\, \varphi: A^n \twoheadrightarrow E$, $\ker\varphi$ is also finite type)

the down-to-earth language of this category equivalence is the following

Corollary 5.4.5 *We have the following*

(i) *If E is a coherent $j_X^\dagger \mathcal{O}_V$-module, there exists a strict neighborhood V' of $]X[_P$ in $]Y[_P$ and a coherent $\mathcal{O}_{V \cap V'}$-module $\mathcal{E}$ such that $E = j_X^\dagger \mathcal{E}$.*

(ii) *If $\mathcal{E}$ and $\mathcal{F}$ are coherent $\mathcal{O}_V$-modules and*

$$\varphi : j_X^\dagger \mathcal{E} \to j_X^\dagger \mathcal{F}$$

is any morphism, there exists a strict neighborhood V' of $]X[_P$ in $]Y[_P$ and a morphism

$$\psi : \mathcal{E}_{|V \cap V'} \to \mathcal{F}_{|V \cap V'}$$

such that $\varphi = j_X^\dagger \psi$.

(iii) *If $\mathcal{E}$ and $\mathcal{F}$ are coherent $\mathcal{O}_V$-modules and*

$$\psi, \psi' : \mathcal{E} \to \mathcal{F}$$

satisfy $j_X^\dagger \psi' = j_X^\dagger \psi$, then there exists a strict neighborhood V' of $]X[_P$ in $]Y[_P$ such that $\psi'_{|V \cap V'} = \psi_{|V \cap V'}$.

Note also that full faithfulness means that

$$\mathrm{Hom}(j_X^\dagger \mathcal{E}, j_X^\dagger \mathcal{F}) = \varinjlim_{V'} \mathrm{Hom}(\mathcal{E}_{|V \cap V'}, \mathcal{F}_{|V \cap V'}).$$

Stratification & Connection on overconvergent modules

Stratification:

Definition 6.1.1 *Let $(X \subset Y \subset P)$ be an S-frame and V an admissible open subset of $]Y[_P$. A* stratification on a $j_X^\dagger \mathcal{O}_V$-module *is a stratification as $\mathcal{O}_V$-module. And a morphism of stratified $j_X^\dagger \mathcal{O}_V$-modules is simply a morphism of stratified $\mathcal{O}_V$-modules.*

connection

Definition 6.1.8 *Let $(X \subset Y \subset P)$ be an S-frame and V an admissible open subset of $]Y[_P$. A* (integrable) connection on a $j_X^\dagger \mathcal{O}_V$-module *is a (integrable) connection as $\mathcal{O}_V$-module. And a morphism is simply a horizontal morphism in the usual sense.*

In other words, the category of $j_X^\dagger \mathcal{O}_V$-modules with a (integrable) connection over S_K is the full subcategory of $\mathcal{O}_V$-modules with a (integrable) connection over S_K that are overconvergent. Again, morphisms are automatically $j_X^\dagger \mathcal{O}_V$-linear.

How to view $j_X^\dagger \mathcal{O}_V$-mod as $\mathcal{O}_V$-mod?

$\forall \mathcal{E}$ is a $j_X^\dagger \mathcal{O}_V$-mod, then $\exists$ strict nbd V', s.t. $(V',]Z[\cap V)$ is an admissible covering of V

$\forall$ admissible open U of V, and $s \in \mathcal{E}(U)$, we know $s|_{]Z[\cap V \cap U} = 0$

hence for $\forall a \in j_X^\dagger \mathcal{O}_V(U)$, $a|_{U\cap V'} \in \mathcal{O}_V(U \cap V')$, we consider

$$a \cdot s : \quad \text{on } U \cap V' \quad a|_{U\cap V'} \cdot s|_{U \cap V'}$$
$$\text{on }]Z[\cap V \cap V' \quad 0$$

they glue to a section over U, hence we define a $\mathcal{O}_V$-mod structure on $\mathcal{E}$

We have the following category equivalence

Proposition 6.1.10 *Let $(X \subset Y \subset P)$ be a smooth S-frame and V an admissible open subset of $]Y[_P$. Then, the categories of stratified $j_X^\dagger \mathcal{O}_V$-modules, the category of left $j_X^\dagger \mathcal{D}_{V/S_K}$-modules and the category of $j_X^\dagger \mathcal{O}_V$-modules with an integrable connection are all equivalent.*

Overconvergent isocrystal

Now we come to the concept of overconvergent isocrystal

Definition 7.1.1 *A* (finitely presented) overconvergent isocrystal *on an S-frame $(X \subset Y \subset P)$ is*

⟺ a coherent $\mathcal{O}_V$-mod for some strict nbd

(i) *a family of (coherent) $j_{X'}^{\dagger}\mathcal{O}_{]Y'[_{P'}}$-modules $E_{P'}$ for each morphism of S-frames*

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d] & Y' \arrow[r, hook] \arrow[d] & P' \arrow[d, "u"] \\ X \arrow[r, hook] & Y \arrow[r, hook] & P \end{tikzcd}$$

(ii) *a family of isomorphisms*

$$\varphi_u : u^{\dagger}E_{P'} \simeq E_{P''} \qquad u^{\dagger} := j_{X'}^{\dagger} u_K^{*}$$

for each commutative diagram

$$\begin{tikzcd} & X' \arrow[rr, hook] \arrow[dd] & & Y' \arrow[rr, hook] \arrow[dd] & & P' \arrow[dddd] \\ X'' \arrow[ur, "f"] \arrow[rr, hook] \arrow[dr] & & Y'' \arrow[ur, "g"] \arrow[rr, hook] \arrow[dr] & & P'' \arrow[ur, "u"] \arrow[dddr] & \\ & X \arrow[rr, hook] & & Y \arrow[ddrr] & & \\ & & & & & \\ & & & & & S \end{tikzcd}$$

subject to the cocycle condition

$$\varphi_{v \circ u} = \varphi_v \circ u'^{*}\varphi_u.$$

We call $E_{P'}$ the realization *of E on $(X' \subset Y' \subset P'/S')$.*

A morphism of overconvergent isocrystals *is a family of compatible morphisms of $j_{X'}^{\dagger}\mathcal{O}_{]Y'[_{P'}}$-modules.*

When $Y = X$, we simply say convergent isocrystal.

We point out that the transition morphisms are not required to be compatible with the structural morphism to P, but only to S. This is very important. In other words, $E_{P'}$ depends only on P' and not on a particular morphism $P' \to P$. ⇝ this gives us stratification, connections...

We will denote by $\mathrm{Isoc}^{\dagger}(X \subset Y \subset P/S)$ the category of finitely presented overconvergent isocrystals on $(X \subset Y \subset P/S)$ and by $\mathrm{Isoc}(X \subset P/S)$ the category of finitely presented convergent isocrystals on $(X \subset P/S)$.

Proposition 7.1.2 *Overconvergent isocrystals on an S-frame $(X \subset Y \subset P)$ form an abelian category with faithful exact functor $E \mapsto E_P$. In particular,* $\mathrm{Isoc}^{\dagger}(X \subset Y \subset P/S)$ *is an abelian category.*

Obviously, this definition depends on a particular choice of a frame $X \hookrightarrow Y \hookrightarrow P$ over S

In the following, we will define the pullback functor on isocrystals and show that the category $\mathrm{Isoc}^{\dagger}(X \subset Y \subset P/S)$ is independent of P

Functoriality: we will define the pull-back of an isocrystal

Proposition 7.1.3 *Let*

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "f"] & Y' \arrow[r, hook] \arrow[d, "g"] & P' \arrow[d, "u"] \\ X \arrow[r, hook] & Y \arrow[r, hook] & P \end{tikzcd}$$

be a morphism from an S'-frame to an S-frame over some morphism $v : S' \to S$. Then, any overconvergent isocrystal E on $(X \subset Y \subset P/S)$ defines by restriction an overconvergent isocrystal

$$u^*E := E_{|(X' \subset Y' \subset P')}$$

on $(X' \subset Y' \subset P'/S')$. And this is functorial in E.

In particular, there is a pullback functor

$$u^* : \mathrm{Isoc}^\dagger(X \subset Y \subset P/S) \to \mathrm{Isoc}^\dagger(X' \subset Y' \subset P'/S').$$

actually, this pullback is independent of a specific choice of u, see the following:

Proposition 7.1.6 *Let*

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "f"] & Y' \arrow[r, hook] \arrow[d, "g"] & P' \arrow[d, shift right, "u_1"'] \arrow[d, shift left, "u_2"] \\ X \arrow[r, hook] & Y \arrow[r, hook] & P \end{tikzcd}$$

be two morphisms of frames over the same $S' \to S$ and E an overconvergent isocrystal on $(X \subset Y \subset P/S)$. Then, there exists a canonical isomorphism of overconvergent isocrystals

$$u_2^*E \simeq u_1^*E.$$

Short explain: this follows from the condition (ii) of the definition of isocrystals, because we know by (ii)

$$\begin{array}{ccccc} u_2^* E & \xleftarrow[\sim]{\varphi_{u_2}} & E_{P'} & \xrightarrow[\sim]{\varphi_{u_1}} & u_1^* E \\ \| & & & & \| \\ u_2^\dagger E & & & & u_1^\dagger E \end{array}$$

this observation implies the following

Corollary 7.1.7 *Assume that there exists two morphisms of S-frames*

$$\begin{tikzcd} & & P' \arrow[dd, shift right, "u"'] \\ X \arrow[r, hook] & Y \arrow[ur, hook] \arrow[dr, hook] & \\ & & P \arrow[uu, shift right, "v"'] \end{tikzcd}$$

Then u^* and v^* induce an equivalence of categories between overconvergent isocrystals on $(X \subset Y \subset P/S)$ and $(X \subset Y \subset P'/S)$.

therefore we see that $\mathrm{Isoc}^\dagger(X \hookrightarrow Y \hookrightarrow P/S)$ is independent of the choice of P, once there is a morphism connecting them

· Dependence on Y

Now we investigate the dependence of $\mathrm{Isoc}^{\dagger}(X \hookrightarrow Y \hookrightarrow P/S)$ on Y

Theorem 7.1.8 *Let S be a formal $\mathcal{V}$-scheme and*

$$\begin{tikzcd} & Y' \arrow[r, hook] \arrow[d, "g"] & P' \arrow[d, "u"] \\ X \arrow[ur, hook] \arrow[r, hook] & Y \arrow[r, hook] & P \end{tikzcd}$$

a proper smooth morphism of frames over S. Then the pullback functor is an equivalence of categories:

$$u^* : \mathrm{Isoc}^{\dagger}(X \subset Y \subset P/S) \simeq \mathrm{Isoc}^{\dagger}(X \subset Y' \subset P'/S).$$

Virtual frames

The "independence of P" motivates us the consider only the pair $(X \hookrightarrow Y)$, this is just so-called virtual frames

Roughly speaking, a virtual frame is a frame where the third term is missing. Alternately, it might be seen as a smooth frame whose third term is not specified.

Definition 7.3.1 *A* virtual frame $(X \subset Y)$ *is an open immersion $X \hookrightarrow Y$ of algebraic k-varieties. A* morphism of virtual frames *from $(X' \subset Y')$ to $(X \subset Y)$ is a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "f"] & Y' \arrow[d, "g"] \\ X \arrow[r, hook] & Y. \end{tikzcd}$$

Unless necessary, we will only mention g. The morphism is said to be cartesian *if the square is cartesian.*

we also have the following notion: frames over a virtual frames

Definition 7.3.5 *A* morphism from an S-frame $(X' \subset Y' \subset P')$ to a virtual S-frame $(X \subset Y)$ *is a commutative diagram*

$$\begin{tikzcd}
X' \arrow[r, hook] \arrow[d, "f"] & Y' \arrow[r, hook] \arrow[d, "g"] & P' \arrow[dd] \\
X \arrow[r, hook] & Y \arrow[dr] & \\
 & & S
\end{tikzcd}$$

We will also say that $(X' \subset Y' \subset P')$ is a frame over $(X \subset Y/S)$. *A* morphism of S-frames over the virtual frame $(X \subset Y/S)$ *is a commutative diagram*

$$\begin{tikzcd}
 & X' \arrow[rr, hook] \arrow[dd] & & Y' \arrow[rr, hook] \arrow[dd] & & P' \arrow[ddd] \\
X'' \arrow[ur, "f"] \arrow[rr, hook] \arrow[dr] & & Y'' \arrow[ur, "g"] \arrow[rr, hook] \arrow[dr] & & P'' \arrow[ur, "u"] \arrow[ddr] & \\
 & X \arrow[rr, hook] & & Y \arrow[drr] & & \\
 & & & & & S
\end{tikzcd}$$

Now we are able to define the notion of isocrystals over a virtual frames:

Definition 7.3.7 *A* (finitely presented) overconvergent isocrystal *on a virtual S-frame* $(X \subset Y)$ *is a family of (coherent)* $j^\dagger_{X'}\mathcal{O}_{]Y'[_{P'}}$*-modules* $E_{P'}$ *for each frame* $(X' \subset Y' \subset P')$ *over* $(X \subset Y/S)$ *and, for each morphism of S-frames*

$$\begin{tikzcd}
X'' \arrow[r, hook] \arrow[d, "f"] & Y'' \arrow[r, hook] \arrow[d, "g"] & P'' \arrow[d, "u"] \\
X' \arrow[r, hook] & Y' \arrow[r, hook] & P'
\end{tikzcd}$$

everything is copied in the previous definition of isocrystals over S-frames

over $(X \subset Y/S)$*, an isomorphism*

$$\varphi_u : u^\dagger E_{P'} \simeq E_{P''}.$$

Moreover, these isomorphisms are subject to the cocycle condition

$$\varphi_{v \circ u} = \varphi_v \circ u'^{*}\varphi_u.$$

We call $E_{P'}$ *the* realization *of* E *on* $(X' \subset Y' \subset P'/S')$.

A morphism of overconvergent isocrystals *is a family of compatible morphisms of* $j^\dagger_{X'}\mathcal{O}_{]Y'[_{P'}}$*-modules.*

When $Y = X$*, we simply say* convergent isocrystal.

We will denote by $\mathrm{Isoc}^\dagger(X \subset Y/S)$ the category of finitely presented overconvergent isocrystals on $(X \subset Y/S)$ and by $\mathrm{Isoc}(X/S)$ the category of finitely presented convergent isocrystals on (X/S).

The true "independence of P" should be stated as follows

Proposition 7.3.11 *Let $(X \subset Y \subset P)$ be a smooth S-frame. Then, the restriction functor*

$$\mathrm{Isoc}^{\dagger}(X \subset Y/S) \to \mathrm{Isoc}^{\dagger}(X \subset Y \subset P/S)$$

is an equivalence of categories.

pf: For $\forall\, E \in \mathrm{Isoc}^{\dagger}(X \subset Y \subset P/S)$, and $\forall$ S-frame $(X' \to Y' \to P'/S)$ over $(X \hookrightarrow Y/S)$ we consider the "intermediate frame"

$$\begin{tikzcd}
X' \arrow[r] \arrow[dd, "f"'] & Y' \arrow[r] \arrow[dd, "g"] & P' \\
 & & P \times_S P' \arrow[u, "p_2"'] \arrow[d, "p_1"] \\
X \arrow[r] & Y \arrow[r] & P
\end{tikzcd}$$

then $p_1^{\dagger} E \in \mathrm{Isoc}^{\dagger}(X' \to Y' \to P \times_S P'/S)$ by smoothness of $P \times_S P' \to P'$ proof of Thm 7.1.8 tells us $\exists\, E' \in \mathrm{Isoc}^{\dagger}(X' \to Y' \to P'/S)$, s.t. $p_1^{\dagger} E \simeq p_2^{\dagger} E'$

then we define $E \mapsto E'$ is quasi-inverse to the restriction functor

Stratification

Now we introduce the notion of stratification on S-frames (hence virtual frames)

Definition 7.2.1 *Let $(X \subset Y \subset P)$ be an S-frame and E a $j_X^{\dagger}\mathcal{O}_{]Y[_P}$-module. An* overconvergent stratification *on E is an isomorphism of $j_X^{\dagger}\mathcal{O}_{]Y[_{P\times_S P}}$-modules, also called the* (Taylor) isomorphism *of E,*

$$\epsilon : p_2^* E \simeq p_1^* E$$

such that

$$p_{13}^*(\epsilon) = p_{12}^*(\epsilon) \circ p_{23}^*(\epsilon)$$

on $]Y[_{P\times_S P\times_S P}$.

When $Y = X$, we simply say convergent.

A morphism of $j_X^{\dagger}\mathcal{O}_{]Y[_P}$-modules with overconvergent stratification is a morphism of $j_X^{\dagger}\mathcal{O}_{]Y[_P}$-modules compatible with the Taylor isomorphisms.

bear in mind that "isocrystal" ⟺ "stratification" we have the following category equivalence

Proposition 7.2.2 *The category of (finitely presented) overconvergent isocrystals on an S-frame $(X \subset Y \subset P)$ is naturally equivalent to the category of (coherent) $j_X^\dagger \mathcal{O}_{]Y[_P}$-modules with an overconvergent stratification.*

P & $P \times_S P$ & $P_S^{(n)}$

Proposition 7.2.5 *If E is an overconvergent isocrystal on an S-frame $(X \subset Y \subset P)$, then E_P has a natural stratification and this is functorial in E. In particular, E_P has an integrable connection and there is a forgetful functor*

$$\mathrm{Isoc}^\dagger(X \subset Y \subset P/S) \to \mathrm{MIC}(X \subset Y \subset P/S).$$

E_P : sheaf of modules on $]Y[_P$,

$\nabla : E_P \to E_P \otimes \Omega^1_{]Y[_P/S}$

we mention the concept of "overconvergent connection"

Definition 7.2.10 *Let $(X \subset Y \subset P)$ be a smooth S-frame. An integrable connection on a coherent $j_X^\dagger \mathcal{O}_{]Y[_P}$-module E is* overconvergent *if there exists a strict neighborhood V of $]X[_P$ in $]Y[_P$ and a coherent $\mathcal{O}_V$-module $\mathcal{E}$ with an overconvergent integrable connection (see Definition 4.3.4) such that $E = j_X^\dagger \mathcal{E}$ (and ∇ too comes from the connection of $\mathcal{E}$).*

Now we continue investigating the dependence on Y of $\mathrm{Isoc}^\dagger(X \to Y/S)$, the main result is the following category equivalence

Theorem 7.4.18 *Let*

$$\begin{tikzcd} & Y' \arrow[dd, "h"] \\ X \arrow[ur, hook] \arrow[dr, hook] & \\ & Y \end{tikzcd}$$

be a proper morphism of virtual S-frames. Then

(i) *The functor $E \mapsto h^*E$ is an equivalence of categories*

$$\mathrm{Isoc}^\dagger(X \subset Y/S) \simeq \mathrm{Isoc}^\dagger(X \subset Y'/S).$$

proof explanation:

we can reduce this problem locally and assume h is projective, then a geometric result implies that we can complete the diagram to be

$$\begin{tikzcd} & Y'' \arrow[r, hook] \arrow[dd] \arrow[dr, hook] & P' \arrow[dd, "u"] \\ X \arrow[ur, hook] \arrow[dr, hook] & Y' \arrow[d, "h"] & \\ & Y \arrow[r] & P \end{tikzcd}$$

such that u is proper étale

Lemma 6.5.1 *Let $(X \subset Y \subset P)$ be an S-frame, $X \hookrightarrow Y'$ an open immersion and $h : Y' \to Y$ be a projective morphism which induces the identity on X. Then, locally on $(X \subset Y \subset P)$, there exists a closed subvariety Y'' of Y' containing X such that the morphism induced by h extends to a proper étale morphism of frames*

$$\begin{tikzcd} & Y'' \arrow[r, hook] \arrow[dd, "h"] & P' \arrow[dd, "u"] \\ X \arrow[ur, hook] \arrow[dr, hook] & & \\ & Y \arrow[r, hook] & P. \end{tikzcd}$$

Now we come to the case that $X/_k$ is an algebraic variety, we introduce the notion of S-frames over X.

Definition 8.1.1 *Let X be an algebraic variety over S_k. An S-*frame over X *is an S-frame $(X' \subset Y' \subset P')$ endowed with a morphism of algebraic S_k-varieties $X' \to X$. A* morphism of S-frames over X *is a morphism of S-frames compatible with the morphisms to X:*

$$\begin{tikzcd}
 & X' \arrow[r, hook] \arrow[d] & & Y' \arrow[r, hook] & & P' \arrow[ddd] \\
X'' \arrow[ur, "f"] \arrow[r, hook] \arrow[dr] & X & Y'' \arrow[ur, "g"] \arrow[r, hook] & & P'' \arrow[ur, "u"] \arrow[ddr] & \\
 & & & & & \\
 & & & & & S \arrow[from=2-2, to=4-6]
\end{tikzcd}$$

With these definitions, it is clear that S-frames over X form a category.

Proposition 8.1.2 *If X is an algebraic variety over S_k, then S-frames over X form a category with non empty finite inverse limits.*

with this notion, we can continue talk about overconvergent isocrystals

Definition 8.1.3 *A* (finitely presented) overconvergent isocrystal *on an algebraic S_k-variety X is a family of (coherent) $j^\dagger_{X'}\mathcal{O}_{]Y'[_{P'}}$-modules $E_{P'}$ for each S-frame $(X' \subset Y' \subset P')$ over X and, for each morphism of S-frames*

$$\begin{tikzcd}
X'' \arrow[r, hook] \arrow[d, "f"] & Y'' \arrow[r, hook] \arrow[d, "g"] & P'' \arrow[d, "u"] \\
X' \arrow[r, hook] & Y' \arrow[r, hook] & P'
\end{tikzcd}$$

over X, an isomorphism

$$\varphi_u : u^\dagger E_{P'} \simeq E_{P''}.$$

Moreover, these isomorphisms are subject to the cocycle condition

$$\varphi_{v\circ u} = \varphi_v \circ u'^{*}\varphi_u.$$

We call $E_{P'}$ the realization *of E on $(X' \subset Y' \subset P'/S')$.*

A morphism of overconvergent isocrystals *is a family of compatible morphisms of $j^\dagger_{X'}\mathcal{O}_{]Y'[_{P'}}$-modules.*

We will denote by $\mathrm{Isoc}^\dagger(X/S)$ the category of finitely presented overconvergent isocrystals on X/S

an explicit description in the case of affine X is given by the following

Proposition 8.1.13 *Let $X = \mathrm{Spec} A$ be a smooth affine $\mathcal{V}$-scheme and Y denotes the closure of X in some projective space for a given presentation of A. Then, there is an equivalence of categories*

$$\begin{aligned} \mathrm{Isoc}^\dagger(X_k) &\xrightarrow{\simeq} \mathrm{MIC}^\dagger(A_K^\dagger) \\ E &\longmapsto M := \Gamma(\widehat{Y}_K^{\mathrm{rig}}, E_{\widehat{Y}}) \end{aligned}$$

between finitely presented overconvergent isocrystals on X and coherent $A_K^\dagger$-modules with an overconvergent integrable connection.

we compare this with virtual frames

Proposition 8.1.8 *If $(X \subset Y)$ is a proper virtual S-frame, restriction induces an equivalence of categories*

$$\mathrm{Isoc}^\dagger(X/S) \simeq \mathrm{Isoc}^\dagger(X \subset Y/S).$$

Corollary 8.1.9 *Let X be an algebraic variety over S_k.*

(i) *If $X \hookrightarrow Y$ is an open embedding into a proper variety over S_k, then the category* $\mathrm{Isoc}^\dagger(X \subset Y/S)$ *only depends on X/S, up to natural equivalence.*
(ii) *If moreover $Y \hookrightarrow P$ is a closed embedding into a formal S-scheme which is smooth in a neighborhood of X, the category* $\mathrm{Isoc}^\dagger(X \subset Y \subset P/S)$ *only depends on X/S, up to natural equivalence.*

Ogu's interpretation