PEL Moduli Problem

In this note, I try to explain the representability of PEL moduli problems

- Polarization

First we introduce the Picard functor, given projective Abelian scheme A/S, we consider the following

necessary condition for the existence of the Picard scheme

$$\mathrm{Pic}_{A/S} : Sch/S \longrightarrow Sets$$

for Abelian schemes we have a section $\varepsilon : S \to A$

$$T/S \longmapsto \frac{\mathrm{Pic}(A_T)}{p^*\mathrm{Pic}(T)} \simeq \text{isomorphism classes of } \mathcal{L}, \text{ where } \mathcal{L} \in \mathrm{Pic}(A_T) \text{ and } \varepsilon_T^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_T$$

$$\mathcal{L} \longmapsto p^*\varepsilon_T^*\mathcal{L}^{-1} \otimes \mathcal{L}$$

GROTHENDIECK proved that: $\mathrm{Pic}_{A/S}$ is represented by a smooth group scheme, we denote by $\mathrm{Pic}_{A/S}$

A very important open subscheme is $\mathrm{Pic}^{\tau}_{A/S}$: intuitively, it is the union of components of finite order

For Abelian schemes, we have the following nice results

- $\mathrm{Pic}^{\tau}_{A/S}$ is itself an Abelian scheme$/S$. ($\mathrm{Pic}^{\tau}_{A/S} = \mathrm{Pic}^{0}_{A/S}$?)

$$1 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

$$1 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to \mathrm{Pic}(X) \to \underset{\sim}{H^2(X,\mathbb{Z})}$$

$\Lambda^2 H^1(X,\mathbb{Z})$ — torsion free

We denote this scheme by $X^\vee$, the dual Abelian scheme

- From an invertible sheaf to a homomorphism

Now we choose $\mathcal{L}$ is an invertible sheaf on X, we consider

$$\mu^*(\mathcal{L}) \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1} \quad \text{on} \quad X \times_S X = X_X$$

then this is equivalent to define a morphism: $X \xrightarrow{\psi} \mathrm{Pic}_{X/S}$, now $\psi \circ \varepsilon = \text{identity}$.

We know that ψ is a homomorphism between group schemes by rigidity, now fibrally, X is connected.

We get the following homomorphism, which is a restriction ψ to the open subgroup scheme

$$\Lambda(\mathcal{L}) : X \longrightarrow X^\vee$$

Some basic facts:

- $\Lambda(\mathcal{L}_1 \otimes \mathcal{L}_2) = \Lambda(\mathcal{L}_1) + \Lambda(\mathcal{L}_2)$ as homomorphisms
- When S is connected, and for some fiber $X_{\bar{s}}$, $\mathcal{L}_{\bar{s}}$ algebraically equivalent to 0, then

$$\Lambda(\mathcal{L})_{\bar{s}} = \Lambda(\mathcal{L}_{\bar{s}}) = 0, \text{ hence } \Lambda(\mathcal{L}) = 0 \text{ by rigidity}$$

Moreover, $\Lambda(\mathcal{L})$ is an isogeny iff $\mathcal{L}$ is relatively ample over S, otherwise it is not an isogeny over some fiber, then by rigidity, it is not an isogeny globally

Note: in the case of Abelian schemes,

$\mathcal{L}$ on A/S is relatively ample

$\Leftrightarrow$ $\mathcal{L}_s$ is ample over $A_s/_{k(s)}$

Now let's give the definition of polarization:

Definition 6.3. Let $\pi: X \to S$ be a projective abelian scheme. A *polarization* of X is an S-homomorphism:

$$\lambda: X \to \hat{X}$$

such that, for all geometric points of S, the induced $\bar{\lambda}: \bar{X} \to \hat{\bar{X}}$ is of the form $\Lambda(\bar{L})$, for some *ample* invertible sheaf $\bar{L}$ on $\bar{X}$.

Basic facts
- λ is finite flat & surjective $\rightsquigarrow$ can be checked over geometric fibers
- Why we impose "projective"? because such λ exists only if X is projective: Zariski locally $\exists$ ample line bundle

Question: How far from λ to $\Lambda(\mathcal{L})$ for some relatively ample line bundle $\mathcal{L}$?

We consider the following line bundle on X

$$X \xrightarrow{(1_X, \lambda)} X \times_S X^\vee$$ on the RHS, we have Poincaré line bundle $\mathcal{P}$ normalized along identity section of $X \times_S X^\vee$

Now we define $\mathcal{L}^\Delta(\lambda) := (1_X, \lambda)^* \mathcal{P}$. then we have the following pleasing result:

$\Rightarrow \varepsilon^* \mathcal{L}^\Delta(\lambda) \simeq \mathcal{O}_X$, by the normalization of $\mathcal{P}$

Prop: $\Lambda(\mathcal{L}^\Delta(\lambda)) = 2\lambda$, i.e. although λ may not come from a global line bundle, but 2λ does!

- Cohomology for relatively ample line bundles

Proposition 6.13. Let $\pi: X \to S$ be a projective abelian scheme. Let L be an invertible sheaf on X, relatively ample for π. Then

(i) $R^i\pi_*(L) = (0)$, if $i > 0$.

(ii) $\pi_*(L)$ is a locally free sheaf on S. Let r be its rank.

(iii) Let $\Lambda(L): X \to \hat{X}$ be the finite flat morphism defined by L. Then the degree of $\Lambda(L)$, i.e. the rank of $\Lambda(L)_*(\mathcal{O}_X)$, is r^2.

(iv) If $n \geq 2$, then the sections of $\pi_*(L^n)$ have no common zeroes in X. Therefore there is a morphism

$$\phi_n: X \to \mathbf{P}(\pi_* L^n).$$

If $n \geq 3$, ϕ_n is a closed immersion.

<u>Deformations</u>

We state and give some key ingredients for the following GROTHENDIECK's theorem:

Theorem 6.14. Let S be a connected, locally noetherian scheme. Let $\pi: X \to S$ be a smooth projective morphism, and let $\varepsilon: S \to X$ be a section of π. Assume that for one geometric point s of S, the fibre X_s of π is an abelian variety with identity $\varepsilon(s)$. Then X is an abelian scheme over S with identity ε.

roughly it tells us, Abelian scheme structure is a "fibral structure", it is both open & closed.
Sketch of the proof:

Basic step:

Proposition 6.16. Let F be the functor on the category of locally noetherian S-schemes defined by:

$F(T) = \{$set of all structures of abelian scheme on $X \underset{S}{\times} T$ over T with identity $(\varepsilon \circ f, 1_T): T \to X \underset{S}{\times} T$, where $f: T \to S$ is the given morphism$\}$.

Then F is represented by an open set $U \subset S$.

Sketch: An AS structure on X_T with $\eta: T \to X_T$ fixed as identity section is equivalent to

- A morphism $\mu: X_T \times_T X_T \longrightarrow X_T$, with some additional equalities

<u>Key input</u>: there exists a scheme $\mathrm{Hom}_T(Y \times_T Y, Y)$ representing the family of morphisms from $Y \times_T Y$ to Y

Sketch: roughly, a morphism determines a graph in $Y \times_T Y \times_T Y$, this graph Γ is a closed subscheme,
Now we need the assumption of projectivity of $X/_S$, hence $Y = X_T/_T$, Γ satisfies:

$$\begin{tikzcd} \Gamma \arrow[r, "pr"] \arrow[d, "\wr"] & Y \\ Y \times_T Y & \end{tikzcd}$$

i.e. we need those flat closed subschemes of $Y \times_T Y \times_T Y$, which induces by pr_1 isomorphic to $Y \times_T Y$. (first we find the Hilbert scheme associated to the Hilbert polynomial of $Y \times_T Y$)

The reason of openness is based on the following deformation-theoretic lemma

Proposition 6.15. Let $S = \mathrm{Spec}(A)$, where A is an Artin local ring. Let $\mathfrak{m} \subset A$ be the maximal ideal, and let $I \subset A$ be an ideal such that $\mathfrak{m} \cdot I = (0)$. Let $\pi: X \to S$ be a smooth proper morphism, and let $\varepsilon: S \to X$ be a section. Let $S_0 = \mathrm{Spec}(A/I)$ and let $X_0 = X \underset{S}{\times} S_0$. Assume that X_0 is an abelian scheme over S_0 with identity $\varepsilon \mid S_0$. Then X is an abelian scheme over S with identity ε.

Now the argument goes as follows: 6.15 implies the moduli scheme Z of F is smooth over S moreover, it is also geometrically injective, i.e. $Z_{\bar{s}} = F(\bar{s})$ consists of $\underbrace{\text{at most one element!}}_{\text{rigidity}}$

The final step is U is actually closed,

The third step is a Theorem of KOIZUMI ([19] p. 377) to the effect that when, in the above situation, $S = \mathrm{Spec}(R)$, R a discrete, rank 1 valuation ring, and the generic fibre of X is an abelian variety, then X is an abelian scheme over S. This, combined with Proposition 6.16 implies that the U of that Proposition is closed. Hence the Theorem follows. *QED*.

Level Structure

We make the following definition

Definition 7.1. Let $\pi : X \to S$ be an abelian scheme whose fibres have dimension g. Assume that the characteristics of the residue fields of all $s \in S$ do not divide n. Then if $n \geq 2$, a ***level n structure*** on X/S consists of $2g$ sections $\sigma_1, \sigma_2, \ldots, \sigma_{2g}$ of X over S, such that i) for all geometric points s of S, the images $\sigma_i(s)$ form a basis for the group of points of order n on the fibre $\bar{X}_s$, and ii) $\psi_n \circ \sigma_i = \varepsilon$, where $\psi_n : X \to X$ is multiplication by n, and ε is the identity. In order to state our theorems uniformly, without special cases, it is convenient to call X/S by itself a ***level 1 structure.***

Comparison with Drinfeld level structure: (for Elliptic curves, i.e. $g = 1$)

A $\Gamma(N)$-structure on E/S (also called a "full level N structure", or a "Drinfeld basis of E[N]") is a group homomorphism

$$\phi : (\mathbb{Z}/N\mathbb{Z})^2 \to E[N](S) \tag{3.1.1}$$

which is a "generator" of E[N] in the sense of Chapter 1, 1.5. Explicitly, this means that we have an equality of effective Cartier divisors in E :

$$E[N] = \sum_{a,b \bmod N} [\phi(a,b)] \tag{3.1.2}$$

or equivalently that the N^2 sections $\phi(a, b)$ of $E[N](S)$ form a "full set of sections." The points

$$P = \phi(1,0), \qquad Q = \phi(0,1)$$

in $E[N](S)$ are the corresponding "Drinfeld basis" of $E[N]$.

this definition easily generalize to arbitrary dimension g, just replace $(\mathbb{Z}/N\mathbb{Z})^2$ by $(\mathbb{Z}/N\mathbb{Z})^{2g}$

The equivalence of these two definitions only works for $g=1$, for higher g, still a problem

PL Moduli problem

We consider the following moduli problem which only deals with polarization & level structure

Definition 7.2. If S is any locally noetherian scheme, let $\mathcal{A}_{g,d,n}(S)$ be the set of triples:

i) an abelian scheme X over S of dimension g,

ii) a polarization $\overline{\omega}: X \to \hat{X}$ of degree d^2, i.e., $\overline{\omega}_*(\mathcal{O}_X)$ is locally free of rank d^2, (cf. lemma 6.12),

iii) a level n structure $\sigma_1, \ldots, \sigma_{2g}$ of X over S, all up to isomorphism.

Note that the collection of sets $\mathcal{A}_{g,d,n}(S)$ forms a contravariant functor from the category of locally noetherian schemes to the category of sets in the obvious way: this functor is written $\mathcal{A}_{g,d,n}$.

The basic idea to study this moduli problem is to embed X into some "fixed" projective space, then use the argument from Hilbert schemes, (compare with $\mathcal{M}_g$!)

· Linear Rigidification

Definition 7.5. Let $\pi: X \to S$ be an abelian scheme of dimension g, and let $\overline{\omega}: X \to \hat{X}$ be a polarization of X of degree d^2. Consider:

$$\mathcal{E} = \pi_*\left(L^{\Delta}(\overline{\omega})^3\right).$$

According to Propositions 6.10 and 6.13, this is a locally free sheaf on X of rank $6^g \cdot d^*$. Put

$$m = 6^g \cdot d - 1.$$

Then a *linear rigidification* of X/S is an S-isomorphism

$$\phi: \boldsymbol{P}(\mathcal{E}) \xrightarrow{\sim} \boldsymbol{P}_m \times S.$$

A nontrivial fact is that linear rigidification is functorial:

It is not quite trivial that a linear rigidification of a polarized abelian scheme X/S induces a linear rigidification of every polarized abelian scheme $X \times_S T/T$ obtained by base extension. If $\overline{\omega}: X \to \hat{X}$ is the polarization of X, then $\overline{\omega} \times 1_T : X \times_S T \to \hat{X} \times_S T = \widehat{X \times_S T}$ is the polarization of $X \times_S T$. And

$$L^{\Delta}(\omega)^3 \otimes_{\mathcal{O}_S} \mathcal{O}_T \cong L^{\Delta}(\overline{\omega} \times 1_T)^3.$$

However, we must prove that π_* commutes with base extension in this case, i.e.

$$\pi_*\left[L^{\Delta}(\overline{\omega})^3 \otimes_{\mathcal{O}_S} \mathcal{O}_T\right] \cong \pi_* \left[L^{\Delta}(\overline{\omega})^3\right] \otimes_{\mathcal{O}_S} \mathcal{O}_T.$$

This follows from Proposition 6.13, part (i) and the criterion that we have used before (Ch. 0, § 5, (a)). Consequently, it does follow that

$$\boldsymbol{P}[\pi_*(L^{\Delta}(\overline{\omega})^3)] \times_S T = \boldsymbol{P}[\pi_*(L^{\Delta}(\overline{\omega} \times 1_T)^3)],$$

hence linear rigidifications are functorial.

Now we consider the following moduli problem

$$\mathcal{H}_{g,d,n}(S) = \{ (A, \bar{\omega}, (\alpha_1, \dots, \alpha_{2g}), \phi \mid \text{where } (A, \bar{\omega}, (\alpha_1, \dots, \alpha_{2g})) \in \mathcal{A}_{g,d,n}(S), \ \phi \text{ is a l.r} \}$$

it works as the role of $\widetilde{\mathcal{M}}_g$, we have a natural map of Stacks: (Q: over which site of $\mathbb{Z}[\frac{1}{n}]$?)

$$\mathcal{H}_{g,d,n} \longrightarrow \mathcal{A}_{g,d,n}$$

Just as $\mathcal{M}_g \simeq [\widetilde{\mathcal{M}}_g / G]$, we will see that $\mathcal{A}_{g,d,n} \simeq [\mathcal{H}_{g,d,n} / PGL(m+1)]$

Theorem: $\mathcal{H}_{g,d,n}$ is representable by schemes

Sketch: Firstly we forget the Abelian scheme structure, only care the flat closed subscheme structure of X as closed subscheme of $\mathbb{P}^m_S$, then we will identify those AS locus.

First step:

I claim that the whole structure defining $\alpha \in \mathcal{H}_{g,d,n}(S)$ is determined only by

(a) the embedding $\quad I: X \hookrightarrow (\boldsymbol{P}_m \times S)$,

(b) the $2g+1$-sections of X/S: $\varepsilon, \sigma_1, \sigma_2, \dots, \sigma_{2g}$.

i.e. we can recover all the information $(X/S, \bar{\omega}, (\sigma_1, \dots, \sigma_{2g}), \phi)$ via the data of a) & b)

- Abelian scheme structure: if it has, then it's totally determined by ε
- $\bar{\omega}$: We know that $(p_1 \circ I)^*(\mathcal{O}_{\mathbb{P}^m_S}(1))$ & $\mathcal{L}^\Delta(\bar{\omega})^3$ can be viewed as the "same" relatively very ample line bundle on X over S, then since $\mathcal{L}^\Delta(\bar{\omega})^3$ is normalized already, we have

$$(p_1 \circ I)^* \mathcal{O}_{\mathbb{P}^m_S}(1) \simeq \mathcal{L}^\Delta(\bar{\omega})^3 \otimes \pi^* \varepsilon^* [(p_1 \circ I)^* \mathcal{O}_{\mathbb{P}^m_S}(1)]$$

then we see that $\mathcal{L}^\Delta(\bar{\omega})^3$ is determined by I & $\varepsilon \Rightarrow 6\bar{\omega} = \Lambda(\mathcal{L}^\Delta(\bar{\omega})^3)$ is determined by I & ε

Now $\mathrm{Hom}_S(X, X^\vee)$ is torsion free, we get $\bar{\omega}$ is determined by I & ε

- ϕ: no torsion, hence $\bar{\omega}$ is determined. Putting $M = \varepsilon^*[(p_1 \circ I)^*(\mathcal{O}_{\boldsymbol{P}_m}(1))]$, it follows that the embedding I gives a homomorphism:

$$\begin{array}{c} H^0(\boldsymbol{P}_m, \mathcal{O}_{\boldsymbol{P}_m}(1)) \otimes M^{-1} \to \pi_* \{(p_1 \circ I)^* (\mathcal{O}_{\boldsymbol{P}_m}(1)) \otimes \pi^*(M^{-1})\} \\ \wr\!\| \text{ via } (*) \\ \pi_*(L^\Delta(\bar{\omega})^3) \\ \| \\ \mathcal{E} \end{array}$$

which induces the isomorphism ϕ:

$$\boldsymbol{P}(\mathcal{E}) \xrightarrow[\phi]{\sim} \boldsymbol{P}[H^0(\boldsymbol{P}_m, \mathcal{O}_{\boldsymbol{P}_m}(1)) \otimes M^{-1}] = \boldsymbol{P}_m \times S.$$

Therefore now we only focus on the data of $(X/S, (\varepsilon, \sigma_1, \dots, \sigma_{2g}))$, since $X \hookrightarrow \mathbb{P}^m_S$ is flat over S, it's Hilbert polynomial is:

$$P(X) = 6^g \cdot d \cdot X^g$$

therefore X is essentially a S-point of $\mathrm{Hilb}^{P(X)}_{\mathbb{P}^m}$, i.e. we have Cartesian diagrams

$$\begin{tikzcd}
 & \mathbb{P}^m_S \arrow[rr] & & \mathbb{P}^m_{\mathrm{Hilb}^{P(X)}_{\mathbb{P}^m}} \\
X \arrow[ur] \arrow[rr] \arrow[d] & & Z \arrow[ur] \arrow[d] & \\
S \arrow[rr] & & \mathrm{Hilb}^{P(X)}_{\mathbb{P}^m} &
\end{tikzcd}$$

We also have additional $2g+1$ sections $(\varepsilon, \sigma_1, \dots, \sigma_{2g})$, there is also a moduli scheme corresponding

$$\mathrm{Hilb}^{P(X),k}_{\mathbb{P}^m} = Z \times_{\mathrm{Hilb}} \cdots \times_{\mathrm{Hilb}} Z \quad (k\text{-factors})$$

there are k "universal" projections: $Z^{(k)} := Z \times_{\mathrm{Hilb}} \mathrm{Hilb}^{P(X),k}_{\mathbb{P}^m} \underset{\tau_i,\ 1 \le i \le k}{\overset{p_2}{\rightleftarrows}} \mathrm{Hilb}^{P(X),k}_{\mathbb{P}^m}$, then

$(X/S, (\varepsilon, \sigma_1, \dots, \sigma_{2g}))$ corresponds to a S-point of $\mathrm{Hilb}^{P(X),2g+1}_{\mathbb{P}^m}$, i.e.

$$\begin{tikzcd}
 & \mathbb{P}^m_S \arrow[rr] & & \mathbb{P}^m_{\mathrm{Hilb}^{P(X)}_{\mathbb{P}^m}} \arrow[rr] & & \mathbb{P}^m_{\mathrm{Hilb}^{P(X)}_{\mathbb{P}^m}} \\
X \arrow[ur] \arrow[rr] \arrow[d] & & Z^{(k)} \arrow[ur] \arrow[rr] \arrow[d] & & Z \arrow[ur] \arrow[d] & \\
S \arrow[u, bend right, "\sigma_i"'] \arrow[rr] & & \mathrm{Hilb}^{P(X),2g+1}_{\mathbb{P}^m} \arrow[u, bend right, "\tau_i"'] \arrow[rr] & & \mathrm{Hilb}^{P(X)}_{\mathbb{P}^m} &
\end{tikzcd}$$

To sum up, actually we get an injective map of Stacks (although essentially schemes)

$$\mathcal{H}_{g,d,n} \longrightarrow \mathcal{H}ilb^{P(X),2g+1}_{\mathbb{P}^m}$$

Proposition 7.3. There is a locally closed subscheme

$$H_{g,d,n} \subset Hilb^{P(X),2g+1}_{\mathbf{P}_m}$$

such that an S-valued point of $Hilb^{P(X),2g+1}_{\mathbf{P}_m}$ is in the image of Φ if and only if it is an S-valued point of $H_{g,d,n}$. Therefore $H_{g,d,n}$ represents $\mathcal{H}_{g,d,n}$.

A key point is to identify Abelian scheme locus, which has been shown by GROTHENDIECK's theorem to be a "connected component" of some suitable scheme

need the base scheme to be locally Noetherian :

Prop : $\mathcal{A}_{g,d,n} \simeq [\, H_{g,d,n} / PGL(m+1) \,]$

Note first that there is a natural action of $PGL(m+1)$ on $H_{g,d,n}$

$$PGL(m+1) \times_S H_{g,d,n} \longrightarrow H_{g,d,n}$$

$$(g,\ (X/_T, \bar{\omega}, (\sigma_1, \ldots, \sigma_{2g}), \phi)) \longmapsto (X/_T, \bar{\omega}, (\sigma_1, \ldots, \sigma_{2g}), \phi \circ g)$$

We also has a natural map:

$$H_{g,d,n} \longrightarrow \mathcal{A}_{g,d,n}$$

When we consider the base change to S, then

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & S \arrow[d] \\ H_{g,d,n} \arrow[r] & \mathcal{A}_{g,d,n} \end{tikzcd}$$

$S \to \mathcal{A}_{g,d,n}$ corresponds to $(X/S, \bar{\omega}, (\sigma_1, \ldots, \sigma_{2g}))$, then for $\forall\, T \to S$,

$$?(T) = \{ (\underbrace{T \to H_{g,d,n}}_{(X'/_T, \bar{\omega}', (\sigma_1', \ldots, \sigma_{2g}'), \phi)},\ T \to S,\ \gamma) \mid \gamma : (X_T/_T, \bar{\omega}, (\sigma_1, \ldots, \sigma_{2g})) \xrightarrow{\sim} (X'/_T, \bar{\omega}', (\sigma_1', \ldots, \sigma_{2g}')) \}$$

$$= \{ \phi : \mathbb{P}(\mathcal{L}^{\Delta}(\bar{\omega}_T)^3) \simeq \mathbb{P}(\mathcal{L}^{\Delta}(\bar{\omega})^3) \times_S T \xrightarrow{\sim} \mathbb{P}^m_T \}$$

obviously, $? \to H_{g,d,n}$ is a $PGL(m+1)$-torsor, because once $?(T) \neq \emptyset$, $?(T) \simeq PGL(m+1)(T)$!
and since $PGL(m+1)$ is affine, by descent theory, $?$ is a scheme, and just a $PGL(m+1)$-torsor

Thm : When $n \geq 3$, this quotient is represented by a scheme

pf: Step 1 : When $n \gg 0$, use stable points

Step 2 : When $n \geq 3$, chose two large prim $p, q > 0$ $p, q \nmid dN$, $p \neq q$
exists over $\mathbb{Z}[\frac{1}{pN}]$ & $\mathbb{Z}[\frac{1}{qN}]$, try to glue them together !

Naive Level Structure & General Level Structure

Naive one: $\sigma_1, \dots, \sigma_{2g}$ are $2g$ sections: $S \to A$, it makes up a basis for $A[N]$

i.e. there is an isomorphism: $\left(\mathbb{Z}/N\mathbb{Z}\right)_S^{2g} \xrightarrow{\sim} A[N]$ as finite étale group scheme $/S$

General one: $\eta: T(A_{\bar{s}}) \xrightarrow{\sim} L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ as a $\pi_1(S, \bar{s})$-invariant $\hat{\Gamma}(N)$-orbit

$\Rightarrow$ $\pi_1(S, \bar{s})$ acts trivially on $A_{\bar{s}}[N] = A[N]_{\bar{s}}$ $\left(A[N]_{\bar{s}} \simeq L/NL \simeq \frac{1}{N}L/L \right)$

$\Rightarrow$ there is an isomorphism $\left(\mathbb{Z}/N\mathbb{Z}\right)_S^{2g} \simeq A[N]$ as finite étale group scheme $/S$

PEC Moduli Varieties

General Set-up

F has to be a field because if $F = F_1 \times F_2 \Rightarrow D = D_1 \times D_2$

D : finite dim'l simple $\mathbb{Q}$-alg & center F $\rightsquigarrow$ (unr)

$*$: positive involution on D, i.e.

$$\mathrm{Tr}_{D_{\mathbb{R}}/\mathbb{R}}(x x^*) > 0 \quad \text{for all } 0 \neq x \in D \otimes_{\mathbb{Q}} \mathbb{R}$$

there is a reduced trace on D : $\mathrm{Tr}_{D/F} : D \to F$.

$$\mathrm{Tr}_{D/\mathbb{Q}} = \mathrm{Tr}_{F/\mathbb{Q}} \circ \mathrm{Tr}_{D/F} \sim \mathrm{Tr}_{D_{\mathbb{R}}/\mathbb{R}} = \mathrm{Tr}_{D/\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$$

$F_0 = F^{*=1} \Rightarrow F_0$ is totally real because $\mathrm{Tr}_{F_0/\mathbb{Q}}(x^2) > 0$ for all $x \in F_0^\times$

$(D, *)$ is of the first kind if $F = F_0$

the second kind if F/F_0 imaginary quadratic extension

(unr): S : places of F over p, we have

$$D_p = D \otimes_{\mathbb{Q}} \mathbb{Q}_p = D \otimes_F \underbrace{F \otimes_{\mathbb{Q}} \mathbb{Q}_p}_{= \prod_{\mathfrak{p} \in S} F_{\mathfrak{p}}} \simeq \prod_{\mathfrak{p} \in S} M_d(F_{\mathfrak{p}}), \ \& \ F_{\mathfrak{p}}/\mathbb{Q}_p \text{ unramified for } \mathfrak{p} \in S$$

assume $\mathcal{O}_D$ is a maximal order, stable under involution, the above isomorphism induces

$$\mathcal{O}_{D,p} = \mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \prod_{\mathfrak{p} \in S} M_d(\mathcal{O}_{F,\mathfrak{p}})$$

D-mod V : V is a left D-mod & finite dim'l, we have a non-degenerate alternating form

$$\langle \, , \, \rangle : V \times V \to \mathbb{Q}$$

s.t. • $\langle bv, w \rangle = \langle v, b^* w \rangle$, $\forall b \in D$

assume $L \subset V$ is a $\mathcal{O}_D$-submod of V, s.t.

- $L \otimes_{\mathbb{Z}} \mathbb{Q} = V$
- $\langle \cdot , \cdot \rangle$ on $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is self-dual

The algebraic group

$C = \mathrm{End}_D(V)$ is a semi-simple $\mathbb{Q}$-alg with natural involution induced by $\langle \cdot, \cdot \rangle$

$$\langle cv, w \rangle = \langle v, c^\iota w \rangle$$

<u>Check:</u> $c^\iota \in C$

$$\langle v, c^\iota d w \rangle = \langle d^* cv, w \rangle = \langle c d^* v, w \rangle = \langle v, dc^\iota w \rangle \Rightarrow dc^\iota = c^\iota d$$

We have $\mathbb{Q}$-groups

$G(R) = \{ x \in C \otimes_{\mathbb{Q}} R \mid x x^\iota \in R^\times \}$

$U(R) = \{ x \in G(R) \mid x x^\iota = 1 \}$

$GU(R) = \{ x \in C \otimes_{\mathbb{Q}} R \mid x x^\iota \in (F_0 \otimes_{\mathbb{Q}} R)^\times \}$

$1 \to U \to G \xrightarrow{\nu} \mathbb{G}_m \to 1$ over $\mathbb{Q}$

$1 \to U \to GU \xrightarrow{\nu} \mathrm{Res}_{F_0/\mathbb{Q}} \mathbb{G}_m \to 1$

$G_1 = G^{der} = GU^{der} \begin{cases} \text{first kind, } G_1 = U \\ \text{second kind, may not, write } SU \end{cases}$

(sc) G_1 is simply connected with non-compact $G_1(\mathbb{R})$ (in case A, C, not necessarily in D)

<u>Examples:</u>

- $D = F$ totally real, $* = \mathrm{id}$; $V = F^2$, $\langle (x,y), (x',y') \rangle = xy' - x'y$ $\Big\} \Rightarrow C = M_2(F)$ $\iota: x \to x^\iota = \det(x) \cdot x^{-1}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$G(R) = \{ x \in M_2(F \otimes_{\mathbb{Q}} R) \mid \det x \in R^\times \}$

$GU(R) = \mathrm{Res}_{F/\mathbb{Q}} GL(2) \to G_1 = \mathrm{Res}_{F/\mathbb{Q}} SL(2) \rightsquigarrow$ Hilbert modular surface

- $D = \mathbb{Q}$, $V = \mathbb{Q}^{2g}$, $\langle \cdot, \cdot \rangle$ is induced by J_g $\Big\} \Rightarrow C = M_{2g}(\mathbb{Q})$ $\iota: \langle gx, y \rangle = \langle x, g^\iota y \rangle$

$$x^t g^t J_g y = x^t J_g g^\iota y \Rightarrow g^\iota = J_g^{-1} g^t J_g$$

$G = GSp(2g)$

$G_1 = U = Sp(2g)$ $\rightsquigarrow$ Siegel modular semi-simple algebra + involution

Classification of $(D, *)$ over algebraically closed field

(A) $M_n(k) \times M_n(k)$, $(a, b)^* = (b^t, a^t)$

(C) $M_n(k)$, $b^* = b^t$ orthogonal type

(BD) $M_n(k)$, $b^* = J b^t J^{-1}$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ symplectic type

simple algebra / $\mathbb{R}$ + positive involution

(A) $M_n(\mathbb{C})$ $a^* = \bar{a}^t$ $\rightsquigarrow$ $M_n(\mathbb{C}) \times M_n(\mathbb{C})$

(C) $M_n(\mathbb{R})$ $a^* = a^t$ $\rightsquigarrow$ $M_n(\mathbb{C})$

(D) $M_n(\mathbb{H})$ $a^* = \bar{a}^t$ $\rightsquigarrow$ $M_n(M_2(\mathbb{C})) \simeq M_{2n}(\mathbb{C})$

$$M_n(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} \simeq M_n(\mathbb{C}) \times M_n(\mathbb{C})$$

$$a \otimes 1 \longmapsto (a, \bar{a})$$

$$a^* \otimes 1 \longmapsto (a^*, \bar{a^*}) \quad \bar{a}^t, a^t$$

$$(x + yi + zj + wk)(x - yi - zj - wk) = x^2 + y^2 + z^2 + w^2$$

the corresponding algebra $C_{\mathbb{R}}$ are:

(A) corresponds to V^m, where $V = \mathbb{C}^n$ $\rightsquigarrow$ $C_{\mathbb{R}} \simeq M_m(\mathbb{C})$ $\rightsquigarrow$ G corresponds to unitary groups

$*$ on $C_{\mathbb{R}}$ is induced by symplectic form!

(C) corresponds to V^m, where $V = \mathbb{R}^n$ $\rightsquigarrow$ $C_{\mathbb{R}} \simeq M_m(\mathbb{R})$ $\rightsquigarrow$ G correspond to GSp_m $\Rightarrow$ m is even

(D) corresponds to V^m, where $V = \mathbb{H}^n$ $\rightsquigarrow$ $C_{\mathbb{R}} \simeq M_m(\mathbb{R})$ $\rightsquigarrow$ G correspond to orthogonal groups

$$V \otimes_{\mathbb{R}} W \times V \otimes_{\mathbb{R}} W \longrightarrow \mathbb{R}$$

$$\langle \cdot, \cdot \rangle = \underbrace{\phi_V}_{\text{symplectic}} \otimes \psi_W \Rightarrow \psi \text{ symmetric}$$

when we have h (complex structure, C only allows m to be even)

(Lan's note P54, Milne's note Example 5.5, 5.6)

Complex structure & PEL Shimura data

Goal: give a complex structure on $V_\infty = V \otimes_{\mathbb{Q}} \mathbb{R}$

method: give $h: \mathbb{C} \to C_\infty = C \otimes_{\mathbb{Q}} \mathbb{R}$, s.t.

1. $h(\bar{z}) = h(z)^\iota$
2. $(v, w) = \langle v, h(i) w \rangle$ on V_∞ is positive-definite

Claim: h induces $h: \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$ of algebraic groups over $\mathbb{R}$

pf: $h(z) \cdot h(z)^\iota = h(z) \cdot h(\bar{z}) = h(|z|^2) =$ multiplication by $|z|^2$ on V_∞

Proposition (ZINK): Suppose $(D, *)$ is either of type A or C. There exists a homo

$$h: \mathbb{S} \to G_{\mathbb{R}}$$

s.t. h satisfies 1, 2 & 3: (V_∞, h) is of type $(-1, 0)$ & $(0, -1)$

moreover, h is unique up to conjugation by an element of $G(\mathbb{R})$

$\rightsquigarrow$ (G, X) is a (simple PEL) Shimura datum what about case BD ?

We can form Shimura varieties,

$\mathrm{Sh}_K(G, X)(\mathbb{C})$ classifies:

THEOREM 8.17. *Let (G,X) be PEL Shimura datum, as above, and let K be a compact open subgroup of $G(\mathbb{A}_f)$. Then $\mathrm{Sh}_K(G,X)(\mathbb{C})$ classifies the isomorphism classes of quadruples $((A,i),s,\eta K)$, where*

- *A is a complex abelian variety,*
- *$\pm s$ is a polarization of the Hodge structure $H_1(A,\mathbb{Q})$,* $V = H_1(A, \mathbb{Q})$; $a \in \mathbb{Z}_{>0}$, $b \to \phi = a^{-1}\varphi$, $\varphi \in \mathrm{End}(A)$
- *i is a homomorphism $B \to \mathrm{End}^0(A)$, and* $\rightsquigarrow$ gives t_{b_i} the ab induces $\underset{\sim}{\mathrm{End}_{\mathbb{Q}}(V)}$ (ϕ induces $\mathrm{End}_{\mathbb{Q}}(V)$)
- *ηK is a K-orbit of $B \otimes \mathbb{A}_f$-linear isomorphisms $\eta\colon V(\mathbb{A}_f) \to V_f(A)$ sending ψ to an $\mathbb{A}_f^\times$-multiple of s,*

satisfying the following condition:

B-mod by $i: B \to \mathrm{End}^0(A)$

t_{b_i}

polarization condition (pol) $\Leftarrow$ (**) *there exists a $\underset{\sim}{B\text{-linear isomorphism } a\colon H_1(A,\mathbb{Q})}$ $\to V$ sending s to a $\mathbb{Q}^\times$-multiple of ψ, and for such an isomorphism $a \circ h_A \circ a^{-1} \in X$.*

then $\{s_i\}$ in 7.4 just pullback of t_{b_i} ! defined a,

PROOF. In view of the dictionary $b \leftrightarrow t_b$ between endomorphisms and tensors (8.16), this follows from Theorem 7.4 □

p-integral moduli problem

reflex field: serveral defs

- Def 1: $V_{\mathbb{C}} = \underbrace{V_1 \oplus V_2}_{}$ (complex v.s.) $\quad$ Now $h(\mathbb{C})$ commutes with D-action

  $h(z): z \quad \bar{z}$ $\quad$ We got D preserves V_1 & V_2

  hence we have $\rho_1 : D \to \mathrm{End}_{\mathbb{C}}(V_1)$

  E = fixed field of $\{\sigma \in \mathrm{Aut}(\mathbb{C}) \mid \rho_1^{\sigma} \simeq \rho_1\}$

  How to understand this ?
  consider a $\mathbb{C}$-basis of V_1, e,
  then $\rho_1(d) \in M_n(\mathbb{C})$
  $\rho_1^{\sigma}(d) = (\rho_1(d))^{\sigma}$ applied to matrix coefficient
  for another basis, $(f) = (e) M$
  $\Rightarrow (\rho_1)_f = M (\rho_1)_e M^{-1} \Rightarrow (\rho_1)_f^{\sigma} \simeq (\rho_1)_e^{\sigma}$

- Def 2: E = generated by $\mathrm{Tr}(\rho_1(b))$ for $b \in D$

  point: D is simple, D-mod is determined by Tr (since we are working over char 0)

Claim: E is a finite extension of $\mathbb{Q}$

pf: only need to show $\{\rho_1^{\sigma} \mid \sigma \in \mathrm{Aut}(\mathbb{C})\}$ is a finite set
but ρ_1 are repns of D_{∞}, which is semi-simple, $\dim V_1$ is $< +\infty$, hence there are only finitely many iso classes of $\dim V_1$, D_{∞}-repn

Claim: (unr) $\Rightarrow$ p is unramified in $E/\mathbb{Q}$

We denote $V = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$: semi-local Dedekind domain
Our p-integral moduli problem will be defined over Sch/V, why? Since the over $\mathbb{C}$ moduli problem tells us E is the field of rationality of the repn of D on $\mathrm{Lie}(A)$ $\Rightarrow$ (A_h, ι) is defined over E at least

$\mathcal{E}^{D}_{K^{(p)}}(S)$ is a groupoid consisting of (S is a scheme over V)

$(A, \lambda, i, \bar{\eta}^{(p)})_{/S}$:
- A is a proj AS $/S$
- $\lambda: A \to {}^{t}A$ is a polarization of deg prime to p
- $i: \mathcal{O}_{D,(p)} = \mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \hookrightarrow \mathrm{End}_S(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$

  s.t. $\lambda \circ i(\alpha^*) = {}^{t}i(\alpha) \circ \lambda$

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & {}^{t}A \\ {\scriptstyle \alpha^*}\downarrow & & \downarrow{\scriptstyle i^{t}(\alpha)} \\ A & \xrightarrow{\lambda} & {}^{t}A \end{array}$$

①给出一个 ample line bundle
②将 A & ᵗA 在该意义下等同起来
(对于 elliptic curve, 存在 canonical 的 isomorphism)

- $\bar{\eta}^{(p)}$ is a $\pi_1(S, \bar{s})$-invariant, $K^{(p)}$-orbit of skew Hermitian D-modules

$$V^{p}(A_{\bar{s}}) \xrightarrow{\sim} V(A^{p}_{f})$$

- (det)

Explain: λ & $\bar{\eta}^{(p)}$ & (det)

- For every line bundle $\mathcal{L}$ on A/S, we have the following isogeny:

$$\Lambda(\mathcal{L}) : A \longrightarrow {}^tA$$
$$\left(T \xrightarrow{x} A\right) \longmapsto T_x^*\mathcal{L}_T \otimes \mathcal{L}_T^{-1} \in {}^tA(T)$$

this is an isogeny iff $\mathcal{L}$ is ample ($\Leftrightarrow$ ample on every geometric fiber)

Def: Polarization: $\lambda : A \to {}^tA$ isogeny & fiber-by-fiber induced by an ample line bundle $\Rightarrow$ λ is symmetric

Question: Does λ come from an global ample line bundle ?

$\mathcal{L}$: Poincaré bundle

Prop: if λ is a polarization, then $2\lambda = \Lambda(L^\Delta(\lambda))$, where $L^\Delta(\lambda) = (1_X, \lambda)^*\mathcal{L}$, $\quad X \xrightarrow{(1_X,\lambda)} X \times_S {}^tX$

and by rigidity of homo between AS, λ is totally determined by 2λ

giving $\lambda \Rightarrow$ ample line bundle $\Rightarrow$ giving an isogeny (polarization)
so roughly speaking, giving $\lambda \Leftrightarrow$ giving an ample line bundle on A/S

- $\bar{\eta}^{(p)}$ is a $\pi_1(S,\bar{s})$-invariant $K^{(p)}$-orbit of isomorphism of symplectic $\mathcal{O}_{D,(p)}$-modules

$$\eta^{(p)} : V(\mathbb{A}_f^p) \xrightarrow{\sim} V^p(A_{\bar{s}}) \quad + \quad \nu(\eta^{(p)}) : \mathbb{A}_f^p \xrightarrow{\sim} V^p\mathbb{G}_{m,\bar{s}}$$

$$V(\mathbb{A}_f^p) = L \otimes_{\mathbb{Z}} \mathbb{A}_f^p$$

$$\begin{tikzcd}
V^p(A_{\bar{s}}) \times V^p(A_{\bar{s}}) \arrow[r, "\lambda\text{-Weil}"] & V^p\mathbb{G}_{m,\bar{s}} \\
V(\mathbb{A}_f^p) \times V(\mathbb{A}_f^p) \arrow[u, "\eta^{(p)}"] \arrow[r, "\langle\cdot,\cdot\rangle"] & \mathbb{A}_f^p \arrow[u, "\nu(\eta^{(p)})"']
\end{tikzcd}$$

A way it comes: chose any symplectic $\mathcal{O}_{D,(p)}$-equivariant isomorphism

$$L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p \xrightarrow{\sim} \underbrace{T^pA_{\bar{s}}}_{\pi_1(S,\bar{s})\text{-invariant}}$$

then tensoring with $\otimes_{\mathbb{Z}} \mathbb{Q}$, + take K^p-orbit, then find isomorphism $T^p\mathbb{G}_{m,\bar{s}} \leftarrow \hat{\mathbb{Z}}^p$

Later we will show that all level structure essentially come from this way

examples of $K^{(p)}$ & K_p: $\hat{\Gamma} = \{ x \in G(\mathbb{A}_f) \mid x\hat{L} = \hat{L} \}$, $\hat{\Gamma}^{(p)} = \{ x \in \hat{\Gamma} \mid x_p = 1 \}$

$\hat{\Gamma}(N) = \{ x \in \hat{\Gamma} \mid x\ell \equiv \ell \mod N\hat{L}, \forall \ell \in \hat{L} \}$ for $p \nmid N$

K_p is called maximal if $K_p = \hat{\Gamma}_p$

Kottwitz determinant condition

We choose $\mathbb{Z}_{(p)}$-base $\{\alpha_j\}_{1\le j\le r}$ of $\mathcal{O}_{D,(p)}$ and consider a homogeneous polynomial

$$f(t_1, t_2, \dots, t_r) = \det(\alpha_1 t_1 + \cdots + \alpha_r t_r \,|_{V_1}) \in \underset{\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}}{\overset{\shortparallel}{\mathcal{V}}} [t_1, \dots, t_r]$$

$\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ $\rightsquigarrow$ because of the def of E

We consider $\mathrm{Lie}(A) = e^* \Omega^1_{A/S}$ is a locally free $\mathcal{O}_S$-mod of rank $= \dim_S A$
Since $\mathcal{O}_D$ acts on A/S linearly, we get

$$\mathrm{Lie}(A) \text{ is an } \mathcal{O}_{D,(p)} \otimes_{\mathbb{Z}} \mathcal{O}_S\text{-module}$$

then consider $g(t_1, \dots, t_r) = \det(t_1\alpha_1 + \cdots + t_r\alpha_r \,|_{\mathrm{Lie}(A)}) \in \mathcal{O}_S[t_1, \dots, t_r]$

(det) condition says that $\left(S \text{ over } \mathcal{V} \Rightarrow \mathcal{V} \xrightarrow{j} \mathcal{O}_S \right)$

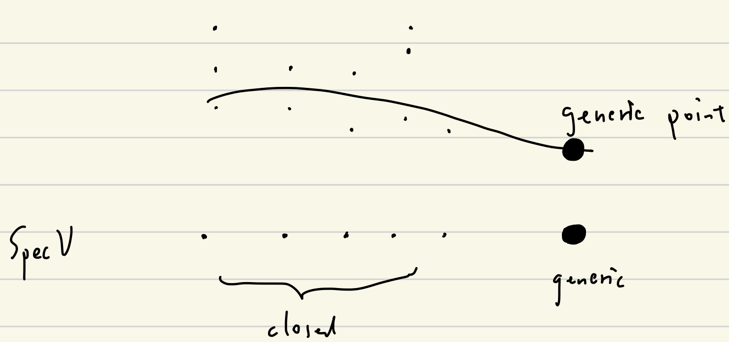
$$j(f(t_1, \dots, t_r)) = g(t_1, \dots, t_r)$$

Meaning of this condition

consistency
V_1 is a $\mathbb{C}$ (char 0) repn of $\mathcal{O}_{D,(p)}$
$\mathrm{Lie}(A)$ may be a $\underbrace{\mathcal{O}_{D,(p)} \otimes_{\mathcal{V}} \mathcal{V}/\mathfrak{m}}_{\text{semi-simple alg}}$-mod (positive char)

then "trace" along doesn't fix the isomorphism type of $\mathrm{Lie}(A)$ as $\mathcal{O}_{D,(p)} \otimes_{\mathcal{V}} \mathcal{V}/\mathfrak{m}$-mod
we should also impose the condition of determinant to fix the module type

if no (det) condition, then



Let's recall the groupoid $\mathcal{E}^D_K(S)$

$\mathcal{E}^D_{K^{(p)}}(S)$ is a groupoid consisting of (S is a scheme over V)

$(A, \lambda, i, \bar{\eta}^{(p)})_{/S}$: • A is a proj AS $/S$

• $\lambda : A \to {}^tA$ is a polarization of deg prime to p

• $i : \mathcal{O}_{D,(p)} = \mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \hookrightarrow \mathrm{End}_S(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$

s.t. $\lambda \circ i(\alpha^*) = {}^ti(\alpha) \circ \lambda$

$$\begin{tikzcd} A \arrow[r, "\lambda"] \arrow[d, "\alpha^*"'] & {}^tA \arrow[d, "i^t(\alpha)"] \\ A \arrow[r, "\lambda"] & {}^tA \end{tikzcd}$$

+ (det)

rational version

• $\bar{\eta}^{(p)}$ is a $\pi_1(S, \bar{s})$-invariant, $K^{(p)}$-orbit of skew Hermitian D-modules

$$V^p(A_{\bar{s}}) \xrightarrow{\sim} V(\mathbb{A}^p_f)$$

two objects $(A, \lambda, i, \bar{\eta}^{(p)})$, $(A', \lambda', i', \bar{\eta}'^{(p)})$ are isomorphic to each other if

$\exists$ $\mathbb{Z}^\times_{(p)}$-isogeny $\phi : A \to A'$, s.t.

much simpler when we use "integral" version

1. $p \nmid \deg \phi$
2. $\phi^*\lambda' = {}^t\phi \circ \lambda' \circ \phi = c\lambda$ for some $c \in \mathbb{Z}^\times_{(p)}$
3. $\phi \circ i = i' \circ \phi$, i.e.

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "i(b)"'] & A' \arrow[d, "i'(b)"] \\ A \arrow[r, "\phi"] & A' \end{tikzcd}$$

4. $\bar{\eta}'^{(p)} = \phi \circ \bar{\eta}^{(p)}$

$$\begin{tikzcd} V(\mathbb{A}^p_f) \arrow[r, "\bar{\eta}'^{(p)}", "\sim"'] \arrow[dr, "\bar{\eta}^{(p)}"', "\sim"] & V(A'_{\bar{s}}) \\ & V(A_{\bar{s}}) \arrow[u, "\phi"'] \end{tikzcd}$$

Thm: If the open compact subgp $K \subset G(\mathbb{A}_f)$ is maximal at p, and is sufficiently small, then the functor $\mathcal{E}_K^{(p)}$ is representable by a quasi-proj smooth scheme $Sh_K^{(p)}$ over $\mathcal{V}$.
For any K maximal at p, the coarse moduli scheme $Sh_K^{(p)}$ of $\mathcal{E}_K^{(p)}$ exists as a quasi-projective scheme of finite type over $\mathcal{V}$. If D is division & $V = D$, $Sh_K^{(p)}$ is proj / $\mathcal{V}$

Construction of the Moduli:

Step 1: Modification of the groupoids

We consider the $\underset{\text{full}}{\text{subcategory}}$ consisting of $\underline{A} = (A, \lambda, i, \bar{\eta}^{(p)})$ where $\eta^{(p)}$ gives an isomorphism between $T^p(A_{\bar{s}})$ and $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}^p$ & $i : \mathcal{O}_p \to End_S(A)$

We use $\mathcal{E}_K$ denote this modified groupoid, then

Claim: $\mathcal{E}_K(S) \to \mathcal{E}_K^{p}(S)$ is a category equivalence
pf: Only need to show essential surjectivity, i.e. every $\underline{A} \in \mathcal{E}_K^{p}(S)$ can be modified nicely this can be easily done since we only care $\mathbb{Z}_{(p)}^{\times}$-isogeny class of A

Then let's see the morphism set in $\mathcal{E}_K(S)$

$\underline{A} \xrightarrow{\phi} \underline{A}'$ is a prime-to-p quasi-isogeny, then $\exists N$, $p \nmid N$, $\underset{\psi}{N\phi}$ is a prime-to-p isogeny

then $\eta'^{(p)} = \eta^{(p)} \circ \phi \Rightarrow \phi$ gives an isomorphism $T^pA_{\bar{s}} \xrightarrow[\phi]{\sim} T^pA'_{\bar{s}} \Rightarrow \psi$ factor through multiplication by N

then ϕ is a isogeny! also $(\deg \phi, p) = 1$ + $T^pA_{\bar{s}} \overset{\phi}{\simeq} T^pA'_{\bar{s}} \Rightarrow \phi$ is an isomorphism!

$\Rightarrow \mathcal{E}_K \simeq \mathcal{E}_K^{(p)}$ is a DM stack (basically automorphism group is finite)

Step 2: We have a natural morphism:

$$\mathcal{E}_K^{D} \longrightarrow \mathcal{E}_{\tilde{K}}^{\mathbb{Q}}$$

$\tilde{K}$ is a suitable compact open subgp of $GSp(V)(\mathbb{A}_f^{(p)})$, $K = \tilde{K} \cap G(\mathbb{A}_f^{(p)})$

$$\underline{A} = (A, \lambda, i, \bar{\eta}^{(p)}) \longmapsto (A, \lambda, \bar{\eta}^{(p)})$$

$\uparrow$ don't require D-linearity

Thm: When $\tilde{K}$ is sufficiently small, $\mathcal{E}_{\tilde{K}}^{\mathbb{Q}}$ is representable by a quasi-projective smooth scheme M.

Therefore we only need to prove the relative representability of the above morphism

Step 3: relative representability

We must show that ? is a scheme, for $\forall S \to \mathcal{E}^{\mathbb{Q}}_{\tilde{K}}$

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & S \arrow[d] \\ \mathcal{E}^{D}_{K} \arrow[r] & \mathcal{E}^{\mathbb{Q}}_{\tilde{K}} \end{tikzcd}$$

By definition, $S \to \mathcal{E}^{\mathbb{Q}}_{\tilde{K}}$ gives a triple $(A, \lambda, \bar{\eta}^{(p)})$

$$?(T) = \{ (A_{/T}, \lambda, i, \bar{\eta}^{(p)}) \mid i : \mathcal{O}_D \to \mathrm{End}(A) \text{ satisfying (det) } \& \ {}^t i(b) \circ \lambda = \lambda \circ i(b^*),\ \forall b \in D \}$$

Let's first consider

$$\mathcal{M}(T) = \{ (A_{/T}, \lambda, i, \bar{\eta}^{(p)}) \mid i : \mathcal{O}_D \to \mathrm{End}(A) \}$$

Claim: $\mathcal{M}$ is representable by a scheme M_D

pf: serveral ingredients

- For proj AS $A/_S$ (with each connected comp projective over S), $\mathrm{End}_S(A)$ is represented by a scheme over S, i.e.

$$T/_S \mapsto \mathrm{End}_T(A_T) \text{ is rep by a scheme over } S$$

Key inputs: a) For proj scheme $/S$, $T \to \mathrm{Mor}_T(X_T, Y_T)$ is representable (using Hil schemes)

b) for AS, any morphism preserving 0 section is automatically a homo

$$\begin{tikzcd} & \mathrm{End}(A) \arrow[dl] \arrow[dr] & \\ \mathrm{Mor}(A, A) \arrow[dr] & & S \arrow[dl, "e"] \\ & A & \end{tikzcd}$$

$$\phi \mapsto \phi \circ e$$

- For $\mathcal{O}_D$ we consider the constant ring scheme $\underline{\mathcal{O}}$ over S,

$$T \to \mathrm{Mor}_T(\underline{\mathcal{O}}_T, E) \text{ representable by } X/_S$$

algebra homomorphism requires:

a) $X \overset{①}{\underset{②}{\rightrightarrows}} X \times_S X$ ①: $\phi \mapsto \phi \circ +$ ②: $\phi \mapsto + \circ (\phi \times \phi)$

taking the diagonal subscheme!, same for

b), $0 \to 0$

c), multiplication

d). $1 \to 1$

$\Rightarrow \exists\, E^{\mathcal{O}}$ representing $T \mapsto \mathrm{AlgHom}_T(\underline{\mathcal{O}}_T, E)$ ($E^{\mathcal{O}} = \mathcal{M}_D$ we are looking for) lastly, ${}^*i(b) \circ \lambda = \lambda \circ i(b^*)$ can also be shown in the same way that it is representable condition $\Big\} \Rightarrow \mathcal{M}$ is smooth, by rigidity lemma, we only need to care $\mathrm{Def}_{(A,\lambda,i,\alpha)}$, ignore (det) over V

A

$\downarrow$

$\mathcal{M}$

· What about (det)?

Note here that $\mathcal{M}$ is smooth!

Claim: $\mathcal{M}$ + (det) = union of some irr components of $\mathcal{M}$

pf: For irr component I, and generic point $\eta \in I$, we have $\mathcal{O}_D$ acts on $\mathrm{Lie}(A_\eta)$

$$f(t_1, \dots, t_r) \in \mathcal{O}_I[t_1, \dots, t_r]$$

then at $\forall x \in I$, $\mathcal{O}_D$ acts on $\mathrm{Lie}(A_x)$ by f mod m_x

why, because $\mathrm{Lie}(A)$ is a locally free $\mathcal{O}_I$-mod of rank $= \dim_M A$

$\Rightarrow$ each connected component of ? is proj over S

Our last statement is that this is a finite morphism, since we have projectivity already, we only need to show quasi-finiteness

For a geometric point $\mathrm{Spec}\, k \to \mathcal{E}^{\mathbb{Q}}_{\tilde{K}}$, it corresponds to an AV $(A/_k, \lambda, \bar{\eta}^{(p)})$

We consider all the possibilities for $i: \mathcal{O}_D \to \mathrm{End}_k(A)$ satisfying (det) + Rosati involution condition

$$\Big\{ \text{determined by}$$

$$D \to \mathrm{End}^\circ(A)$$

up to inner automorphism of $\mathrm{End}^\circ(A)$, there are only finitely many choices for $D \to \mathrm{End}^\circ(A)$

$\mathrm{Spec}\, k \times_{\mathcal{E}^{\mathbb{Q}}_{\tilde{E}}} \mathcal{E}^D_K$ consists of $(A/_k, \lambda, i, \bar{\eta}^{(p)})$

different i are conjugate by $\alpha \in \mathrm{End}^\circ(A)$, i.e. $i' = \alpha i \alpha^{-1}$

now conjugation-by-α has to satisfy it preserve λ, i.e. a positive involution on $\mathrm{End}^\circ(A)$

$\Rightarrow$ only finitely many possibilities for the inner automorphism!

hence the morphism is quasi-finite + projective $\Leftrightarrow$ finite

Relation with generic fiber

We consider the following moduli problem $\mathcal{E}_K^{(\emptyset)}$

$\mathcal{E}_K^{(\emptyset)}$: for $\forall$ E-scheme S, $\mathcal{E}_K^{(\emptyset)}(S)$ is the following category

objects : $(A, \lambda, i, \bar{\eta})$

- A is AS over S, up to isogeny
- λ is a $\mathbb{Q}_+^*$-class of polarization
- $i: D \to \mathrm{End}_S(A) \otimes_{\mathbb{Z}} \mathbb{Q}$
- $\bar{\eta}$ is K-orbit of D-linear symplectic isomorphism

$$V(A_{\bar{s}}) \xrightarrow{\sim} V \otimes_{\mathbb{Q}} \mathbb{A}_f$$

essential surjectivity :

$+$ (det) $+$ (pol)

remember there is always an integral version!

- A up to isomorphism
- λ is a $\mathbb{Q}_+^*$-polarization
- $D \to \mathrm{End}_S(A) \otimes_{\mathbb{Z}} \mathbb{Q}$

fully faithfulness

Question : Do we have $\mathcal{E}_K^{(p)} \times_{\mathrm{Spec}\, V} \mathrm{Spec}\, E \simeq \mathcal{E}_K^{(\emptyset)}$?

the only thing we need to worry about : (pol) condition

(pol) There exists a D-linear isomorphism $f : V \cong H_1(A, \mathbb{Q})$ such that $(f \otimes 1_{\mathbb{A}^{(\infty)}}) \in (\eta \circ K)$, $f^{-1} \circ h_A \circ f \in X_G$, and $\langle f(x), f(y) \rangle_\lambda = \alpha \langle x, y \rangle_0$ up to $\alpha \in F_0^\times$,

for an object in $\mathcal{E}_K^{(p)}$ we only have isomorphism passing to $\mathbb{A}_f^p$, to get (pol), we need

(1) $(V_{\mathbb{R}}, \langle \cdot, \cdot \rangle_0) \cong (V_A \otimes_{\mathbb{Q}} \mathbb{R}, \langle \cdot, \cdot \rangle_\lambda)$, and for any two ι-homomorphisms $\mathbb{C} \hookrightarrow C_\infty$ are conjugates under $G(\mathbb{R})$;
(2) $(V_p, \langle \cdot, \cdot \rangle_0) \cong (V_A \otimes_{\mathbb{Q}} \mathbb{Q}_p, \langle \cdot, \cdot \rangle_\lambda)$;
(3) the Hasse principle for the alternating form $\langle \cdot, \cdot \rangle_0$.

(A) or (C)

Theorem 7.5 *Suppose* (unr) *and one of the conditions* (B1–2)*. Then the p-integral model $Sh^{(p)}(GU, X)$ representing $\mathcal{E}_1^{A^{(p)}}$ is smooth over $O_{E,(p)}$, and we have $Sh^{(p)}(GU, X) \otimes_{O_{E,(p)}} E = Sh(GU, X)/GU(\mathbb{Z}_p)_{/E}$, where $GU(\mathbb{Z}_p) = \{x \in GU(\mathbb{Q}_p) | x L_p = L_p\}$.*