Hilbert 90 for K_n^M

The main goal of the talk is to complete the inductive step of H90(n) K_n^M for ℓ-special field, which is a generalization for the classical Hilbert 90

Let's first state the Hilbert 90, which is totally different from $H90(n)$:

Suppose E/k is a cyclic extension of deg ℓ, $Gal(E/k) = \langle \sigma \rangle$, Classical Hilbert 90 states that:

$$E^\times \xrightarrow{1-\sigma} E^\times \xrightarrow{N} k^\times$$

is exact, i.e. $\ker(N) = \mathrm{im}(1-\sigma)$, now in our language, $k^\times = K_1^M(k)$, we expect to generalize this exact sequence to $K_n^M(k)$, i.e.

$$K_n^M(E) \xrightarrow{1-\sigma} K_n^M(E) \xrightarrow{N} K_n^M(k)$$

there are a few questions to be settled down:

1. Why we expect such N exists?
2. How to define this N?
3. How to use it in the norm-residue isomorphism? $\rightsquigarrow$ transfer argument

- Def of generalized norm map:

We would like to define $N: K_n^M(E) \to K_n^M(k)$ for $\forall n \geq 0$, obviously, we hope,

- When $n=0$: it is simply $\mathbb{Z} \xrightarrow{[E:k]} \mathbb{Z}$
- When $n=1$: it is the classical norm: $E^\times \xrightarrow{N_{E/k}} k^\times$

<u>Thm</u>: There exists a unique family of natural homomorphisms

$$N_{k'/k}: K_n^M(k') \to K_n^M(k)$$

associated with finite field extensions k'/k, s.t. $N_{k/k} = id$, and the reciprocity law holds:
Let $k(t)$ field of functions of one variable $/k$, then for $\forall x \in K_*^M(k(t))$

$$\sum_v N_{k(v)/k}(\partial_v(x)) = 0 \quad \rightsquigarrow \quad \sum_v [k(v):k]\, v(f) = 0,\ f \in k(t)^*$$

where v ranges over all discrete valuations of $k(t)$ over k

Here the ∂_v map is the following: suppose v is a d.v. on field K, $\mathcal{O}_v$, m_v the same meaning then $\forall x \in K_n^M(K)$ has the following form

$$\{u_1, u_2, \ldots, u_{n-1}, u\}, \quad u_i \in \mathcal{O}_v^*,\ u \in K^*,\ \mathcal{O}_v/m_v = k$$

$$\partial_v(\{u_1, \ldots, u_{n-1}, u\}) := v(u)\{\bar{u}_1, \ldots, \bar{u}_{n-1}\}: K_n^M(K) \to K_{n-1}^M(k),$$

Some examples: $K = k(t)$,

$$v = (t) \rightsquigarrow \partial_v\{t,\ t^2+1,\ t^{-2}\} = -\partial_v\{t^2+1,\ t,\ t^{-2}\} = 2\partial_v\{t^2+1,\ t,\ t\}$$
$$= 2\partial_v\{t^2+1,\ t,\ -1\}$$
$$= 2\partial_v\{-1,\ t^2+1,\ t\}$$
$$= 2\{-1,\ 1\} = 0$$

$v_\infty : f \mapsto -\deg f$, then $\mathcal{O}_{v_\infty} = k + (t^{-1})\mathcal{O}_{v_\infty}$

$$\partial_v\left\{\frac{2t}{t^2+1},\ t+1,\ t\right\} = \partial_v\left(\left\{\frac{2t^2}{t^2+1},\ t+1, t\right\} - \{t,\ t+1,\ t\}\right)$$
$$= \partial_v\left(\left\{\frac{2t^2}{t^2+1},\ \frac{t+1}{t},\ t\right\} + \left\{\frac{t^2}{t^2+1},\ t, t\right\} + \{t,\ t,\ t+1\}\right)$$
$$= -\{2,\ 1\} - \partial_v\left\{\frac{t^2}{t^2+1},\ -1,\ t\right\} + \partial_v\{t,\ -1,\ t+1\})$$
$$= -\underset{\substack{\| \\ 0}}{\{2, 1\}} + \underset{\substack{\| \\ 0}}{\{1, -1\}} + \partial_v\{-1,\ t+1, t\}$$
$$= \{-1,\ -1\}$$

$$\{-1,\ t+1, t\} = \left\{-1,\ \frac{t+1}{t},\ t\right\} + \{-1, t, t\}$$
$$\xrightarrow{\partial_v} -\{-1, 1\} + \partial_v\{-1, t, -1\}$$
$$= -\{-1, 1\} - \partial_v\{-1, -1, t\}$$
$$= -\{-1, 1\} + \{-1, -1\}$$
$$= \{-1, 1\} + \{-1, -1\} = \{-1, -1\}$$

there is a natural filtration on $K_n^M(k(t))$:

Fil^d: generated as $K_*^M(k)$-mod, by $\{\pi_1, \pi_2, \ldots, \pi_n\}$, with $0 < \deg\pi_1 < \deg\pi_2 < \cdots < \deg\pi_n \le d$, π_i monic irr poly $\in k[t]$

because if some place then are same degree: $\{t^2+t+1, t^2+2\} = \left\{\frac{t^2+t+1}{t^2+2},\ t^2+2\right\} = \left\{1 - \frac{1-t}{t^2+2},\ t^2+2\right\}$

$$= \left\{1 - \frac{1-t}{t^2+2},\ \frac{t^2+2}{1-t}\right\} + \left\{1 + \frac{t-1}{t^2+2},\ t-1\right\}$$
$$= 0 + \{t^2+t+1, t-1\} - \{t^2+2, t-1\}$$

$$\Rightarrow \mathrm{gr}^d \simeq \bigoplus_{\substack{v \\ [k(v):k] = d}} K_*^M(k(v))$$

$$\{\pi_1, \ldots, \pi_n\} \xmapsto{\partial_v} \underbrace{\{\bar\pi_1, \ldots, \bar\pi_{n-1}\}}_{\gamma(\pi_n)},\quad \partial_v\{\pi_1, \ldots, \pi_n\} \neq 0 \Leftrightarrow \pi_n = \pi_v$$

We also have an exact sequence

$$0 \to K_*^M(k) \to K_*^M(k(t)) \xrightarrow{(\partial_v)} \bigoplus_{v \neq v_\infty} K_{*-1}^M(k(v)) \to 0$$

it can be concluded from here that, $\exists!\ N_v : K_{n-1}^M(k(v)) \to K_{n-1}^M(k)$. for every v, s.t.

$$K_n^M(k(t)) \xrightarrow{(\partial_v)} \bigoplus_v K_{n-1}^M(k(v)) \xrightarrow{\sum N_v} K_{n-1}^M(k)$$

the composite is 0, i.e. $(\sum N_v) \circ (\partial_v) = 0$ & $N_{v_\infty} = id$

we can just take $\left(\sum_{v \neq \infty} N_v\right)\left(\partial_v(x)\right) = -\partial_\infty(x)$, since ∂_v is surj, we only need

to show, different choice of x makes no difference:

suppose $\partial_v(x) = \partial_v(x')$, for $v \neq \infty \Rightarrow x - x' \in K_*^M(k) \Rightarrow \partial_\infty(x - x') = 0$

examples of N:

deg 0 part: $0 \longrightarrow k^\times \longrightarrow k(t)^\times \longrightarrow \bigoplus_{v \neq v_\infty} \mathbb{Z} \to 0$

$$(f) \longmapsto (v_v(f))$$

i.e. $\partial_v : f \longmapsto [k(v):k]\, v_v(f)$

$N_v : K_0^M(k(v)) = \mathbb{Z} \longrightarrow K_0^M(k) = \mathbb{Z}$ & $\sum_v [k(v):k]\, v_v(f) = 0$

$\Rightarrow N_v$ = multiplication by $[k(v):k]$

deg 1 part: we should know $\partial_v : K_2^M(k(t)) \to K_1^M(k(v))$

$$\{\pi, \pi_v\} \longmapsto \bar{\pi}$$

we need to show

$$\left(\sum_{v \neq \infty} Nm_v\right)\left(\partial_v \{\pi_1, \pi_2\}\right) = -\partial_\infty \{\pi_1, \pi_2\} \in K_1^M(k)$$

$$\|$$

$$Nm_{v_1}\left(\partial_{v_1}\{\pi_1, \pi_2\}\right) + Nm_{v_2}\left(\partial_{v}\{\pi_1, \pi_2\}\right) = -\partial_\infty \{\pi_1, \pi_2\}$$

$$Nm_{v_1}\left(-\partial_{v_1}\{\pi_2, \pi_1\}\right) = Nm_{v_1}\left(-\{\bar{\pi}_2\}\right), \quad Nm_{v_2}\left(\{\bar{\pi}_1\}\right)$$

$$= -\{Nm_{v_1}(\bar{\pi}_2)\} + \{Nm_{v_2}(\bar{\pi}_1)\} = \left\{\frac{Nm_{v_2}(\bar{\pi}_1)}{Nm_{v_1}(\bar{\pi}_2)}\right\}$$

$$\left.\begin{aligned} \pi_1(t) &= (t-a_1)\cdots(t-a_{d_1}) \\ \pi_2(t) &= (t-b_1)\cdots(t-b_{d_2}) \\ Nm_{v_2}\bar{\pi}_1 &= \prod_{i=1}^{d_2}(b_i-a_1)\cdots(b_i-a_{d_1}) \\ Nm_{v_1}\bar{\pi}_2 &= \prod_{j=1}^{d_1}(a_j-b_1)\cdots(a_j-b_{d_2}) \end{aligned}\right\} \Rightarrow (-1)^{d_1 d_2}$$

$$\begin{aligned} \{\pi_1, \pi_2\} &= \left\{\frac{\pi_1}{t^{d_1}}, \pi_2\right\} + \{t^{d_1}, \pi_2\} \\ &= \left\{\frac{\pi}{t^{d_1}}, \pi_2\right\} + d_1\left\{t, \frac{\pi_2}{t^{d_2}}\right\} + d_1\{t, t^{d_2}\} \\ &= \left\{\frac{\pi}{t^{d_1}}, \pi_2\right\} - d_1\left\{\frac{\pi_2}{t^{d_2}}, t\right\} - d_1 d_2\{-1, t\} \end{aligned}$$

$$\downarrow$$

$$-d_2\{1\} + d_1\{1\} + d_1 d_2\{-1\} = \{(-1)^{d_1 d_2}\}$$

An abstract intepretation of $N_{E/k}$

$K^M_n(k) = H^n(k, \mathbb{Z}(n))$ $\qquad$ $K^M_n(k)_{(\ell)} \simeq H^n(k, \mathbb{Z}_{(\ell)}(n)) \simeq H^n_{et}(k, \mathbb{Z}_{(\ell)}(n))$

not true without this localization !

$$
\begin{aligned}
K^M_n(E) &= H^n(E, \mathbb{Z}(n)) \\
&= H^n(E, C_*\mathbb{Z}_{tr}(\mathbb{G}_{m/E}^{\wedge n})[-n]) \\
&= H^0(E, C_*\mathbb{Z}_{tr}(\mathbb{G}_{m/E}^{\wedge n})) \\
&= \mathbb{Z}_{tr}(\mathbb{G}_{m/E}^{\wedge n})(\operatorname{Spec} E) \Big/ \operatorname{im} \mathbb{Z}_{tr}(\mathbb{G}_{m/E}^{\wedge n} \times \Delta^1)(\operatorname{Spec} E)
\end{aligned}
$$

$$
\begin{tikzcd}
\mathbb{Z}_{tr}(\mathbb{G}_{m/E}^{\wedge n})(\operatorname{Spec} E) \arrow[r, phantom, "="] \arrow[d, squiggly] & \operatorname{Coker}\Big(\bigoplus_{i=1}^{n-1} \mathbb{Z}_{tr}(\overbrace{\mathbb{G}_{m/E}\times\cdots\times\mathbb{G}_{m/E}}^{(n-1)\text{-copy}}) \arrow[r] \arrow[d, "\text{natural}"] & \bigoplus \mathbb{Z}_{tr}(\overbrace{\mathbb{G}_{m/E}\times\cdots\times\mathbb{G}_{m/E}}^{n\text{-copy}})\Big)(\operatorname{Spec} E) \arrow[d, "\text{natural}"] \\
\mathbb{Z}_{tr}(\mathbb{G}_{m/k}^{\wedge n})(\operatorname{Spec} k) \arrow[r, phantom, "="] & \operatorname{Coker}\Big(\bigoplus_{i=1}^{n-1} \mathbb{Z}_{tr}(\mathbb{G}_{m/k}\times\cdots\times\mathbb{G}_{m/k}) \arrow[r] & \bigoplus \mathbb{Z}_{tr}(\mathbb{G}_{m/k}\times\cdots\times\mathbb{G}_{m/k})\Big)(\operatorname{Spec} k)
\end{tikzcd}
$$

$$Sm/E \rightsquigarrow Sm/k$$
$$(X \to E) \mapsto (X \to E \to k)$$

$$
\begin{tikzcd}
\mathbb{Z}_{tr}(Y)(\operatorname{Spec} E) \arrow[r] \arrow[d, equal] & \mathbb{Z}_{tr}(Y)(\operatorname{Spec} k) \arrow[d, equal] \\
Cor_k(\operatorname{Spec} E, Y) \arrow[r] & Cor_k(\operatorname{Spec} k, Y)
\end{tikzcd}
$$

$$\mathbb{Z}_{tr}(Y_E)(X_E) \longrightarrow \mathbb{Z}_{tr}(Y)(X_k)$$

$$
\begin{tikzcd}
W \arrow[r] \arrow[dr, "\text{fin. surj}"'] & Y_E \times_E X \simeq Y \times_k X \arrow[d] & \rightsquigarrow & W \arrow[r] \arrow[dr] & Y \times_k X \arrow[d] \\
& X & & & X
\end{tikzcd}
$$

$$
\begin{tikzcd}
W \arrow[r] \arrow[d, "\text{fin surj}"'] & Y \arrow[dl] & \rightsquigarrow & N_{E/k}W = \Sigma \sigma W \arrow[r] \arrow[d] & Y \arrow[dl] \\
E & & & \operatorname{Spec} k &
\end{tikzcd}
$$

$\rightsquigarrow$ is there any natural map ?

$$F(\operatorname{Spec} E) \xrightarrow{?} F(\operatorname{Spec} k)$$

Now, for $\forall$ finite extension k'/k, suppose $k' = k(a)$, $a \rightsquigarrow \pi_v$, monic irr, then define

$$N_{k'/k, a} = N_{k(v)/k}.$$

i.e. independence of a, s.t. $k' = k(a)$

Generally, the norm map can be shown to be independent of the isomorphism $k' \simeq k(v)$, and the general norm map can be computed successively, i.e. suppose $k = k_0 - k_1 \to \cdots \to k_n = k'$ the intermedean extensions are all simple extensions, then

$$N_{k'/k} = N_{k'/k_{n-1}} \circ \cdots \circ N_{k_1/k}$$

Some properties of Norm homomorphism: • deg 0 part: $N_{k'/k}$ = multiplication by $[k':k]$

deg 1 part: $N_{k'/k}(x) = N_m(x)$

• Projection formula ($K_*^M(k)$- linearity)

$$N_{k'/k}\left(x \cdot j_{k'/k}(y)\right) = N_{k'/k}(x) \cdot y \qquad , \forall\, x \in K_*^M(k'),\ y \in K_*^M(k)$$

especially, $N_{k'/k}(j_{k'/k}(y)) = N_{k'/k}(1) \cdot y = [k':k] \cdot y$

• Commutative diagram:

$$\begin{array}{ccc} K_n^M(k') & \xrightarrow{\sum e_i j_{L_i/k'}} & \bigoplus_i K_n^M(L_i') \\ {\scriptstyle N_{k'/k}} \downarrow & & \downarrow {\scriptstyle \sum N_{L_i'/L}} \\ K_n^M(k) & \xrightarrow{j_{L/k}} & K_n^M(L) \end{array}$$

where $k' \overset{\pi_v}{\simeq} k(a)$ simple extension

L/k is finite extension

$\pi_v = \prod \pi_i^{e_i}$

$\Rightarrow$ if k'/k is Galois, then $j_{k'/k} \circ N_{k'/k} = \sum \sigma_*$ $\quad (L = k')$

Def: We say a field satisfies Hilbert 90, if for every Galois extension of deg $\overset{?}{l}$, and $G = \langle \sigma \rangle$

$$K_n^M(E) \xrightarrow{1-\sigma} K_n^M(E) \xrightarrow{N} K_n^M(k)$$

is an exact sequence

$$\underbrace{N(x) = N(\sigma x)}$$

$$\begin{array}{ccc} E & \xrightarrow{\sigma} & E \\ & \nwarrow \quad \nearrow & \\ & k & \end{array}$$

quick conclusion: this sequence is exact modulo l-torsion, i.e.

$K_n^M(E)_G \xrightarrow{N} K_n^M(k)$ has coker & ker l-torsion

pf: consider $K_n^M(k) \xrightarrow{j} K_n^M(E)^G$

<u>Claim</u>: ker & Coker of this homo is l-torsion

$x \in \ker(j) \Rightarrow lx = Nj(x) = 0$

$\bar{x} \in \operatorname{Coker}(j) \Rightarrow jN(x) = \sum \sigma x = lx \Rightarrow l\bar{x} = 0$

$\Rightarrow K_n^M(k) \underset{N}{\overset{j}{\rightleftarrows}} K_n^M(E)^G$ up to l torsion

$K_n^M(E)_G \xrightarrow{\sim} K_n^M(E)^G$ up to l torsion $\Rightarrow$ $K_n^M(E)_G \underset{\sim}{\xrightarrow{lN}} K_n^M(k) \Rightarrow K_n^M(E)_G \xrightarrow{N} K_n^M(k)$

$\bar{x} \longmapsto \sum \sigma x$ $\qquad$ up to l-torsion $\qquad$ up to l-torsion

Hence to prove the exactness, we only need to show exactness at 1, i.e.

$$K_n^M(E)_{(l)} \xrightarrow{1-\sigma} K_n^M(E)_{(l)} \xrightarrow{N} K_n^M(k)_{(l)} \quad \text{is exact}$$

<u>Thm</u>: If $BL(n)$ holds, then any k with char $k \neq l$ satisfies Hilbert 90 for K_n^M

pf: From Hung's lecture, we know

$$K_n^M(E)_{(l)} \underset{BL(n)}{\simeq} H^n_{et}(E, \mathbb{Z}_{(l)}(n)) \simeq H^n(Gal(\Omega/E), \mathbb{Z}_{(l)}(n))$$
$$\simeq H^n(Gal(\Omega/k), ind_k^E \mathbb{Z}_{(l)}(n)) \simeq H^n(Gal(\Omega/k), \mathbb{Z}_{(l)}[G](n))$$

And we have natural map:

$$0 \to I \to \mathbb{Z}[G] \xrightarrow{tr} \mathbb{Z} \to 0$$

also

$$0 \to \mathbb{Z} \to \mathbb{Z}[G] \to I \to 0, \quad 1 \mapsto (1-\sigma)$$

$\Rightarrow \otimes_{\mathbb{Z}} \mathbb{Z}_{(l)}(n)$, we will obtain

$$H^n_{et}(k, \mathbb{Z}_{(l)}[G](n)) \to H^n_{et}(k, I_{(l)}(n)) \to H^{n+1}_{et}(k, \mathbb{Z}_{(l)}(n)) = 0$$

$\Rightarrow$ this map is onto, now consider the following diagram:

$$\begin{tikzcd}
K_n^M(E)_{(l)} = H^n_{et}(k, \mathbb{Z}_{(l)}[G](n)) \arrow[r, "1-\sigma"] \arrow[d, "1-\sigma"] & K_n^M(E)_{(l)} \arrow[r, "N"] \arrow[d, "\wr\|"] \arrow[dr, phantom, "?"] & K_n^M(k)_{(l)} = H^n_{et}(k, \mathbb{Z}_{(l)}(n)) \arrow[d, "\wr\|"] \\
H^n_{et}(k, I_{(l)}(n)) \arrow[r] \arrow[d] & H^n_{et}(k, \mathbb{Z}_{(l)}[G](n)) \arrow[r, "tr"] & H^n_{et}(k, \mathbb{Z}_{(l)}(n)) \\
0 & &
\end{tikzcd}$$

maybe follow from some abstract construction?

<u>Technical lemma</u>: Suppose k is l-special, E/k deg l, then $K_n^M(E)$ is spanned by elements of the form

$$\{a_1, a_2, \dots, a_n\}, \text{ with } a_n \in E^\times, a_i \in k^\times$$

pf: only need to show the case $n=2$: , $K_2^M(E)$ is spanned by $\quad 1-x, x^{-1}$

$\{u_1, u_2\}$

$u_i \in E^*$

u_i : deg $< l$

$u_i(t) = \prod \pi_i$, the π_i must be linear (l-special)

$$\Rightarrow \{a-u_1, a-u_2\} = \left\{\frac{a-u_1}{a-u_2}, a-u_2\right\} + \{a-u_2, -1\} \checkmark$$
$$= \left\{1+\frac{u_2-u_1}{a-u_2}, a-u_2\right\}$$
$$= \underbrace{\left\{1+\frac{u_2-u_1}{a-u_2}, \frac{a-u_2}{u_1-u_2}\right\}}_{=0} + \left\{1+\frac{u_2-u_1}{a-u_2}, u_1-u_2\right\} \checkmark$$

Now let's come to a crucial proposition: inductive step

Prop: Suppose k is l-special, and $\forall$ finite extension of k satisfies Hilbert 90 for K^M_{n-1},
Then for $\forall$ deg l field E of k, s.t.

$$N_{E/k}: K^M_{n-1}(E) \to K^M_{n-1}(k) \text{ is onto}$$

then Hilbert 90 for K^M_n is true for k:

$$K^M_n(E) \xrightarrow{1-\sigma} K^M_n(E) \xrightarrow{N} K^M_n(k)$$

pf: For $\forall\ \underline{a} = \{a_1,\dots,a_{n-1}\} \in K^M_{n-1}(k)$, $\exists\ \underline{b} \in K^M_{n-1}(E)$, s.t. $N(\underline{b}) = \underline{a}$

now consider $\{a,\underline{a}\} \in K^M_n(k) \Rightarrow N(\{a,\underline{b}\}) = \{a, N(\underline{b})\} = \{a,\underline{a}\}$

hence N is surjective

now for a different choice of $\underline{b} \in K^M_{n-1}(E)$, s.t. $N(\underline{b}) = \underline{a}$, say $\underline{b}'$, we know

$$\underline{b}' - \underline{b} = (1-\sigma)c, \text{ for some } c \in K^M_{n-1}(E)$$

then $\{a,\underline{b}'\} - \{a,\underline{b}\} = (1-\sigma)\{a,c\}$

therefore we have a well-defined map:

$$k^\times \otimes K^M_{n-1}(k) \xrightarrow{\phi} \overline{K^M_n(E)}/(1-\sigma)K^M_n(E) \xrightarrow{N} K^M_n(k)$$
$$(a,\underline{a}) \longmapsto \overline{(a,\underline{b})}$$

our goal is to show that ϕ descends to $K^M_n(k)$, then $N\circ\bar\phi = id \Rightarrow \bar\phi$ is injective.

and since $K^M_n(E)$ is generated by $\{a_1,a_2,\dots,a_n\}$ where $a_n \in E^\times$, $a_1,\dots,a_{n-1} \in k^\times$

then consider $a_1 \otimes N\{a_2,\dots,a_n\} \in k^\times \otimes K^M_{n-1}(k)$, it is mapped to $\{a_1,a_2,\dots,a_n\} \Rightarrow \bar\phi$ is surjective

remember $K^M_n(k) = k^\times \otimes K^M_{n-1}(k) \big/ 1-a \otimes \{a,\underline{a}'\}$ hence, we only need to show

$$\phi(1-a \otimes \{a,\underline{a}'\}) = 0,\ \forall\ \underline{a}' \in K^M_{n-2}(k)$$

• set $\alpha = \sqrt[l]{1-a}$, ① if $\alpha \in E$, then $N_{E/k}(\underbrace{1-\alpha}_{T}) = a$, thus $\phi(1-a\otimes\{a,\underline{a}'\}) = \{\alpha^l, 1-\alpha, a_3,\dots,a_{n-1}\}$
$= l\{\alpha, 1-\alpha, a_3,\dots,a_{n-1}\} = 0$

↑ k is l-special $\Rightarrow \mu_l \in k^\times$

$(1-T)^l = 1-a$

$T^l \cdots - a = 0$

② if $\alpha \notin E$, choose $\underline{b} \in K^M_{n-1}(E)$, s.t. $N_{E/k}(\underline{b}) = \{a,\underline{a}'\}$, and consider the diagram

$$\begin{tikzcd}
K^M_n(E) \arrow[r, "j"] \arrow[d, "N"'] & K^M_n(E(\alpha)) \arrow[d, "N"] \\
K^M_n(k) \arrow[r, "j"] & K^M_n(k(\alpha))
\end{tikzcd}$$

$N_{E(\alpha)/k(\alpha)}(j_{E(\alpha)/E}(\underline{b})) = j_{k(\alpha)/k} N_{E/k}(\underline{b}) = j_{k(\alpha)/k}\{a,\underline{a}'\}$

now $N_{E(\alpha)/k(\alpha)}(\{1-\alpha, a_3,\dots,a_{n-1}\}) = j_{k(\alpha)/k}\{a,\underline{a}'\}$

$\Rightarrow j_{E(\alpha)/E}(\underline{b}) - \{1-\alpha, a_3,\dots,a_{n-1}\} = (1-\sigma)c$, $c \in K^M_{n-1}(E(\alpha))$

suppose $\phi(1-a\otimes\{a,\underline{a}'\}) = \{1-a,\underline{b}\} = N_{E(\alpha)/E}\{\alpha,\underline{b}\} = \underbrace{N_{E(\alpha)/E}\{\alpha, 1-\alpha, a_3,\dots,a_{n-1}\}}_{=0} + (1-\sigma)N_{E(\alpha)/E}(\alpha,c)$

Galois cohomology sequence

The main theorem of this section is the following:

<u>Thm</u>: Assume BL(n) holds, and k contains ℓ^{th} roots of unity. if E/k is a cyclic extension of deg ℓ, then the following sequence is exact

$$H^n_{\acute{e}t}(E, \mathbb{Z}/\ell) \xrightarrow{tr} H^n_{\acute{e}t}(k, \mathbb{Z}/\ell) \xrightarrow{\cup [E]} H^{n+1}_{\acute{e}t}(k, \mathbb{Z}/\ell) \longrightarrow H^{n+1}_{\acute{e}t}(E, \mathbb{Z}/\ell)$$

Now we come to the main theorem: Hilbert 90 for ℓ-special field

<u>Thm</u>: Suppose that k is ℓ-special, $K_n^M(k)/\ell = 0$, and $BL(n-1)$ holds, then

a) $H^n_{\text{ét}}(k, \mathbb{Z}/\ell) = 0$

b) $H^{n+1}_{\text{ét}}(k, \mathbb{Z}_{(\ell)}(n)) = 0$ $\quad H90(n)$

moreover, for every finite field extension E/k

c) $K^M_{n-1}(E) \xrightarrow{N} K^M_{n-1}(k)$ is onto

d) $K_n^M(E)/\ell = 0$

pf: We first prove c), consider the following commutative diagram

$$\begin{array}{ccccc} K^M_{n-1}(E) & \xrightarrow{N} & K^M_{n-1}(k) & \xrightarrow{x[a]} & K^M_n(k)/\ell = 0 \\ \downarrow & & \downarrow & & \downarrow \\ H^{n-1}_{\text{ét}}(E, \mathbb{Z}/\ell) & \xrightarrow{tr} & H^{n-1}_{\text{ét}}(k, \mathbb{Z}/\ell) & \xrightarrow{U[E]} & H^n_{\text{ét}}(k, \mathbb{Z}/\ell) \end{array}$$

here, we use two simplifications:

1) $[E:k] = \ell$, because we can do it inductively

2) $E = k(\sqrt[\ell]{a})$, since k contains ℓ^{th} root of unity

onto because: $K^M_{n-1}(E)_{(\ell)} \simeq H^{n-1}_{\text{ét}}(E, \mathbb{Z}_{(\ell)}(n-1)) \twoheadrightarrow H^{n-1}_{\text{ét}}(E, \mathbb{Z}/\ell)$ $\Big(\to H^n_{\text{ét}}(E, \mathbb{Z}_{(\ell)}(n-1)) = 0$ by $H90(n-1)$, since $BL(n-1) \Rightarrow H90(n-1)\Big)$

now onto property follows from exactness of the bottom line

because for $\forall x \in K^M_{n-1}(k)$, by diagram chase. $\exists x' \in K^M_{n-1}(E)$, $\Rightarrow$ $x - N(x') \in \ell K^M_{n-1}(k)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cap$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad Nj\, K^M_{n-1}(k)$

From c) & prop, we know that

$$K_n^M(E) \xrightarrow{1-\sigma} K_n^M(E) \xrightarrow{N} K_n^M(k) \quad \text{is exact}$$

we can also get $K_n^M(E) \xrightarrow{N} K_n^M(k)$ is onto, since $\ell K_n^M(k) \subset \operatorname{im} N \Rightarrow \operatorname{im} N = K_n^M(k)$

therefore modulo $\ell \Rightarrow$ $K_n^M(E)/\ell \xrightarrow{1-\sigma} K_n^M(E)/\ell \to 0$ is exact $\Rightarrow$ $1-\sigma$ is onto

now $K = K_n^M(E)/\ell$ is exponent ℓ $\operatorname{Gal}(E/k) \simeq \mathbb{Z}/\ell\mathbb{Z}$ - mod. then $(1-\sigma)^\ell$ must be 0!

because $(1-\sigma)^\ell K = (1 - \ell(\cdots) - \sigma^\ell)K = (1-1)K = 0 \Rightarrow K_n^M(E)/\ell = 0$

now let's prove (a) & (b)

(a): choose $x \in H^n_{ét}(k, \mathbb{Z}/\ell)$ which is nonzero, (supposedly),

then x is actually: $G_k \times \cdots \times G_k \to \mathbb{Z}/\ell$, then $\exists\, E/k$, s.t. $x_E = 0$

suppose E is the minimum one satisfying this condition, E/k is ℓ-extension, then $\exists\, k'/k$, s.t. E/k' is deg ℓ

so $x_{k'}$ is nonzero $\in H^n_{ét}(k', \mathbb{Z}/\ell)$, suppose $E = k'(\sqrt[\ell]{a})$, $a \in k'$, we have

$$\begin{tikzcd}
K^M_{n-1}(k')/\ell \arrow[r, "\cup\{a\}"] \arrow[d] & K^M_n(k')/\ell \arrow[r] \arrow[d] & K^M_n(E)/\ell \arrow[d] \\
H^{n-1}_{ét}(k', \mathbb{Z}/\ell) \arrow[r, "\cup\{E\}"] & H^n_{ét}(k', \mathbb{Z}/\ell) \arrow[r] & H^n_{ét}(E, \mathbb{Z}/\ell) \\
 & x_{k'} \arrow[r, mapsto] & 0
\end{tikzcd}$$

$\Rightarrow \exists\, x' \in H^{n-1}_{ét}(k', \mathbb{Z}/\ell)$, $x' \cup \{E\} = x_{k'}$

$\uparrow \qquad \uparrow$

$BL(n-1) \rightsquigarrow y \cup \{a\} = \{y, a\} = 0$ by (d) $\Rightarrow x_{k'} = 0$, contradiction

$\Rightarrow H^n_{ét}(k, \mathbb{Z}/\ell) = 0$

(b) $H90(n)$ $\quad 0 \to \mathbb{Z}_{(\ell)}(n) \to \mathbb{Q}(n) \to \mathbb{Q}/\mathbb{Z}_{(\ell)}(n) \to 0$

$\Rightarrow H^i_{ét}(k, \mathbb{Z}_{(\ell)}(n)) \to H^i_{ét}(k, \mathbb{Q}(n)) \to H^i_{ét}(k, \mathbb{Q}/\mathbb{Z}_{(\ell)}(n)) \to H^{i+1}_{ét}(k, \mathbb{Z}_{(\ell)}(n))$ is exact

for $i > n$, $H^i_{ét}(k, \mathbb{Q}(n)) = 0$ $\left(\simeq H^i(k, \mathbb{Q}(n))\right.$, vanishing thm for MC$\left.\right)$

$\Rightarrow H^i_{ét}(k, \mathbb{Z}(n))$ is torsion, ℓ-th torsion is simply $H^i_{ét}(k, \mathbb{Z}_{(\ell)}(n))$

$$0 \to \mathbb{Z}_{(\ell)}(n) \xrightarrow{\times \ell} \mathbb{Z}_{(\ell)}(n) \to \mathbb{Z}/\ell(n) \to 0$$

$\Rightarrow H^n_{ét}(k, \mathbb{Z}/\ell(n)) \to H^{n+1}_{ét}(k, \mathbb{Z}_{(\ell)}(n)) \xrightarrow{\times \ell} H^{n+1}_{ét}(k, \mathbb{Z}_{(\ell)}(n)) \to H^{n+1}_{ét}(k, \mathbb{Z}/\ell(n))$

$\quad \parallel$

$K^M_n(k)/\ell = 0$, hence $H^{n+1}_{ét}(k, \mathbb{Z}_{(\ell)}(n)) = 0$