CONSTRUCTION OF KM-CYCLES AND EXAMPLES

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Contents

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1. Introduction and Notations

Suppose F is a totally real number field of degree d, we denote its real embeddings by $\{\tau_1, \tau_2, \cdots, \tau_d\}$. V is a finite dimensional vector space over $\mathbb{R}$ with a nondegenerate symmetric bilinear form $(\ ,\ )$. We know that

$$V \otimes_{\mathbb{Q}} \mathbb{R} \simeq \bigoplus_{i=1}^{d} V \otimes_{F,\tau_i} \mathbb{R}$$

and the bilinear form $(\ ,\ )$ decomposes into

$$(\ ,\ ) \otimes_{\mathbb{Q}} \mathbb{R} \simeq \bigoplus_{i=1}^{d} (\ ,\ )_{\tau_i}$$

We assume $(\ ,\ )_{\tau_i}$ is postive definite for $i \geqslant 2$, and has signature $(p, 2)$ for $(\ ,\ )_{\tau_1}$. We consider the following Shimura datum: $G = Res_{F/\mathbb{Q}} GSpin(V)$, then

$$G_{\mathbb{R}} \simeq \prod_{i=1}^{d} GSpin(V_{\tau_i})$$

for $i \geqslant 2$, we define $h_i : \mathbb{S} \to GSpin(V_{\tau_i})$ to be the trivial map. Next we choose an orthogonal basis of V_{τ_1} and define

$$\begin{aligned} u : U_1 &\longrightarrow SO(V_{\tau_1}) \\ e^{i\theta} &\longmapsto \begin{pmatrix} 1_p & & \\ & cos(2\theta) & sin(2\theta) \\ & -sin(2\theta) & cos(2\theta) \end{pmatrix} \end{aligned}$$

then u will lift to the double cover $Spin(V_{\tau_1})$ of $SO(V_{\tau_1})$, we denote it by $\tilde{u}$, then by the following exact sequence,

$$1 \to \mathbb{G}_m \to \mathbb{S} \to U_1 \to 1$$

we get a homomorphism $h_1 : \mathbb{S} \to GSpin(V_{\tau_1})$, put all h_i together, we get

$$h : \mathbb{S} \xrightarrow{\prod_{i=1}^{d} h_i} G_{\mathbb{R}}$$

we use X to denote the conjugacy class of h, then (G, X) is a Shimura datum. The associated Hermitain symmetric domain can be described as follows

$$D_V = \{U \subset V_{\tau_1} |\ U \text{ is an oriented negative definite 2-plane}\}$$

D_V has two connected components, we will use D_V^+ to denote one of it. At a point $z \in D_V^+$, it corresponds to a decomposition,

$$V = z + z^{\perp} \tag{1}$$

z is a negative 2-plane, and $z^{\perp}$ is a positive p-plane. By Cartan decomposition, we have

$$SO(V_{\tau_1})^+ = exp(\mathfrak{p}) \cdot K_{\infty}$$

where K_{∞} is the stabilizer of the decomposition (1), or equivalently, the stabilizer of z. $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k} = Lie(K)$ w.r.t the Killing form on $\mathfrak{so}(V_{\tau_1})$, and we have a diffeomorphism,

$$D_V^+ \simeq SO(V_{\tau_1})/K \xleftarrow{exp} \mathfrak{p}$$

$\mathfrak{p}$ can be viewed as the tangent space at z of D_V^+. On the other hand, since D_V^+ is an open subset of the Grassmannian, we can identify the tangent space at z to be,

$$T_z D_V^+ \simeq Hom_{\mathbb{R}}(z, V/z) \simeq z^* \otimes z^{\perp} \simeq z \otimes z^{\perp} \tag{2}$$

we use $\{z_1, z_2\}$ to denote an oriented orthonormal basis of z, and $\{w_1, w_2, \cdots, w_p\}$ to denote an oriented orthonormal basis of $z^{\perp}$, then by (2), $\{v_{ij} = w_i \otimes z_j\}_{i,j}$ make up a basis of $T_z D_V^+$, and we use ω_{ij} to denote the coodinate function on $\mathfrak{p} \simeq T_z D_V^+$ w.r.t this basis.

Our main goal is to construct a harmonic form on the Shimura variety $Sh_K(G, X)$. Recall that for an open compact subgroup $K \subset G(\mathbb{A}_f)$,

$$Sh_K(G, X) = G(\mathbb{Q})\backslash D_V \times G(\mathbb{A}_f)/K \simeq G(\mathbb{Q})_+\backslash D_V^+ \times G(\mathbb{A}_f)/K$$

Our method is first constructing a harmonic $2n-$form $\varphi_V^{(n)}$ on the Hermitian symmetric domain D_V^+ satisfying some good properties, then take "average" of it by theta distribution, i.e.

$$\Theta(g) = \sum_{X \in V(F)^n} \omega(g)\varphi_V^{(n)}(X) \cdot \varphi_f(X) \tag{3}$$

here

$$\varphi_V^{(n)} \in \mathcal{S}(V(F \otimes_{\mathbb{Q}} \mathbb{R})^n) \bigotimes \Omega^{2n}(D_V) \tag{4}$$

we can do the "average" operation because we have the Weil representation of $G(\mathbb{A})$ on the Schwartz space $\mathcal{S}(V(\mathbb{A}_F)^n)$ which is trivial on rational points $G(\mathbb{Q})$, then (3) will turn out to be a harmonic form on the Shimura variety. We can also include the Metaplectic action on (3), we denote $G' = Mp(2n, \mathbb{R})$, this is not an algebraic group, but we can still define $G'(\mathbb{A}_F) = Mp(2n, \mathbb{A}_F)$, which is a double cover of $Sp(2n, \mathbb{A}_F)$, we use g' to denote an element in $G'(\mathbb{A}_F)$, then

$$\Theta(g', g) = \sum_{X \in V(F)^n} \omega(g', g)\varphi^{(n)}(X) \cdot \varphi_f(X) \tag{5}$$

This will also turn out to be a smooth form on the Shimura variety, which only depends on the following double coset,

$$Sp(2n, \mathbb{Q})\backslash G'(\mathbb{A}_F)/K'$$

where K' stabilize $\varphi_f \in \mathcal{S}(V(\mathbb{A}_{F,f}))$. In this note, we will study the construction and uniqueness property of the element (4).

In section 2 and 3, we make a summary of all the properties we want for this element, and then study the cohomological class of (5), we compute its Fourier expansion w.r.t. the metaplectic group.

In section 4, we explain two ways of construction. The first construction assumes the known results of $p = 1$, and gives an explicit formula (27) and (28). The second construction uses Howe operator and works more generally. These two constructions coincides with each other.

In section 5, we prove the uniqueness of the element (4) satisfying all the properties listed in section 2. Then we prove that (4) will be an eigenform under the action of the maximal compact subgroup K' of $Mp(2n, \mathbb{R})$.

2. Sum up of properties we want

Note that we have the following isomorphism,

$$\mathcal{S}((V(F \otimes_{\mathbb{Q}} \mathbb{R})^n)) \simeq \bigotimes_{i=1}^{d} \mathcal{S}(V_{\tau_i})$$

$$\mathcal{S}((V(F \otimes_{\mathbb{Q}} \mathbb{R})^n)) \bigotimes \Omega^{2n}(D_V) \simeq \big(\mathcal{S}(V_{\tau_1}) \bigotimes \Omega^{2n}(D_V)\big) \otimes \bigotimes_{i=2}^{d} \mathcal{S}(V_{\tau_i})$$

these two isomorphisms are compatible with the $G(\mathbb{R}) \times G'(\mathbb{R})$ action. In the rest of this note, we will mainly focus on the first term $\mathcal{S}(V_{\tau_1}) \bigotimes \Omega^{2n}(D_V)$. To simplify notation, we will still use V to denote a finite dimensional vector space over $\mathbb{R}$ with a non-degenerate symmetric bilinear form of signature $(p, 2)$ for some

$p \geqslant 1$. Our main goal is to find a system of forms $\{\varphi_V^{(n)}\}_{V,n}$, such that

1. $\varphi_V^{(n)} \in \mathcal{S}(V^n) \bigotimes \Omega^{2n}(D_V)$ is closed.
2. $\{\varphi_V^{(n)}\}_{V,n}$ satisfy the restriction rule.
3. $\{\varphi_V^{(n)}\}_{V,n}$ satisfy the product rule.

In the rest of this section, we will explain the meanings of $1, 2, 3$ in detail, especially 2 and 3.

Let's explain the closedness condition, since the coefficients of $\varphi_V^{(n)}$ lie in the Schwartz function space, the d operator will commute with the summation in (5), then the closedness of $\varphi_V^{(n)}$ implies the closedness of $\Theta(g', g)$ as a smooth form on D_V.

2.1. **Restriction rule.** It's convenient to introduce the following category $\mathfrak{Q}_2$

Object : Finite dimensional vector space over $\mathbb{R}$ with a non-degenerate symmetric bilinear form of signature $(p, 2)$ for

Morphism : Linear homomorphisms preserving the forms on target and source

By the non-degeneracy of the bilinear form, it's easy to show the following

Lemma 2.1.1. *Suppose V_1 and V_2 are objects in $\mathfrak{Q}_2$, and $\phi \in Mor_{\mathfrak{Q}_2}(V_1, V_2)$, then ϕ must be injective.*

A system of forms $\{\varphi_V^{(n)}\}_{V,n}$ can be viewed as a functor

$$\mathcal{F} : \mathfrak{Q}_2 \longrightarrow \mathbf{Rings}_{\blacksquare}$$

$$V \longmapsto \left(\prod_{n=0}^{p} \mathcal{S}(V^n) \bigotimes \Omega^{2n}(D_V), (\varphi_V^{(n)}) \right) \tag{6}$$

where we extend the definition that

$$n = 0,\ \mathcal{S}(V^0) \bigotimes \Omega^0(D_V) = C^\infty(D_V) \text{ and } \varphi_V^{(0)} = 1$$

$$n \geqslant p,\ \mathcal{S}(V^0) \bigotimes \Omega^0(D_V) = 0 \text{ and } \varphi_V^{(n)} = 0$$

Suppose $V_1, V_2 \in Ob(\mathfrak{Q}_2)$, and $\phi \in Mor_{\mathfrak{Q}_2}(V_1, V_2)$ then ϕ is necessarily injective, therefore we get a natural map which is actually a closed immersion,

$$D_{V_1} \xrightarrow{\phi} D_{V_2}$$

this induces $\phi^* : \Omega^{2n}(D_{V_2}) \to \Omega^{2n}(D_{V_1})$, we also have

$$\mathcal{S}(V_2^n) \xrightarrow{\phi^*} \mathcal{S}(V_1^n)$$

Definition 2.1.1. *We say $\mathcal{F}$ (or $\{\varphi_V^{(n)}\}_{V,n}$) satisfy the restriction rule if*

$$\mathcal{F}(\phi)\left((\varphi_{V_1}^{(n)})\right) = (\phi^* \otimes \phi^*)(\varphi_{V_1}^{(n)}) = \left(\varphi_{V_2}^{(n)} \cdot \varphi^{+}_{(V_2^\perp)^n}\right) \tag{7}$$

Here $\varphi^0_{(V_2^\perp)^n}$ means the Gaussian function corresponding to the positive definite space $V_2^\perp$, and $\phi^* \otimes \phi^*$ means the pullback map on both the Schwartz function space and form space.

Let's explain (7) further, here $V_2^\perp$ means the orthogonal complement of V_2 inside V_1 under the linear map ϕ. Every $X \in V_1^n$ has the following decompostion,

$$X = \phi(X') + Y$$

where $X' \in V_2^n$, and $Y \in (V_2^\perp)^n$, then (7) means,

$$(\phi^* \otimes \phi^*)\left(\varphi_{V_1}^{(n)}(X)\right) = \varphi_{V_2}^{(n)}(X') \cdot \varphi^{+}_{(V_2^\perp)^n}(Y)$$

when ϕ is an isomorphism, we have

$$(\phi^* \otimes \phi^*)\varphi_{V_1}^{(n)} = \varphi_{V_2}^{(n)} \tag{8}$$

This should be viewed as naturality or functoriality property of the Schwartz class we want. Especially, when ϕ is an linear automorphism preserving the symmetric form on V, i.e. $\phi \in SO(V)$, then this implies

Lemma 2.1.2. *If $\mathcal{F}$ satisfies the restriction rule, then*

$$\varphi^{(n)} \in \left(S(V^n)\bigotimes \Omega^{2n}(D_V)\right)^{SO(V)} \simeq \left(S(V^n)\bigotimes \Omega^{2n}(D_V^+)\right)^{SO(V)^+}$$

Remark 2.1.1. *This lemma enables us to construct a form locally at a point, to be more precisely, we have the following isomorphism, for any $z \in D_V^+$, K_∞ is the stabilizer*

$$\left(S(V^n)\bigotimes \Omega^{2n}(D_V)\right)^{SO(V)} \simeq \left(S(V^n)\bigotimes \Omega^{2n}(D_V^+)\right)^{SO(V)^+} \simeq \left(S(V^n)\bigotimes \wedge^{2n}\mathfrak{p}^*\right)^{K_\infty}$$

We will also refer to this as invariance property.

Next we study how restriction rule affects the harmonic form we get via (5). When $U \subset V$ is an $F-$subspace of dimension p' and $U(\mathbb{R})^\perp \subset V(\mathbb{R})$ is positive definite. Then $U \hookrightarrow V$ gives us,

$$G_U \hookrightarrow G_V$$

together with the closed embedding

$$D_U \xrightarrow{i} D = D_V \tag{9}$$

we get a compatible system of closed immersion of Shimura varieties (when $K \subset G_V(\mathbb{A}_f)$) is sufficiently small)

$$(M_U)_{K'} \xrightarrow{i} (M_V)_K \tag{10}$$

here $K' = K \cap G_U(\mathbb{A}_f)$. We use superscript V in φ_f^V to indicate

$$\varphi_f^V \in \mathcal{S}(V(\mathbb{A}_f))$$

by the isomorphism,

$$\mathcal{S}(V(\mathbb{A}_f)) \simeq \mathcal{S}(U(\mathbb{A}_f))\bigotimes \mathcal{S}(U^\perp(\mathbb{A}_f))$$

we assume

$$\varphi_f^V(X,Y) = \varphi_f^U(X)\cdot\varphi_f^{U^\perp}(Y)$$

when $g \in G_U(\mathbb{A}_f)$, and $n \leqslant p'$, (7) and the definition of the theta distribution (5) implies

$$\begin{aligned}
i^*\Theta_V(g',g) &= \sum_{X\in V(F)^n} i^*(\omega(g',g)\varphi_V^{(n)}(X)\cdot\varphi_f^V(X))\\
&= \sum_{Y\in U^\perp(F)^n}\sum_{Z\in U(F)^n} \omega(g')i^*\varphi_V^{(n)}(Y+Z)\cdot\varphi_f^V(Y+g^{-1}Z)\\
&= \sum_{Y\in U^\perp(F)^n}\sum_{Z\in U(F)^n} \omega(g')(\varphi_U^{(n)}(Z)\varphi^0_{(U^\perp)^n}(Y))\cdot\varphi_f^U(g^{-1}Z)\varphi_f^{U^\perp}(Y)\\
&= \left(\sum_{Z\in U(F)^n}\omega(g')\varphi_U^{(n)}(Z)\cdot\varphi_f^U(g^{-1}Z)\right)\cdot\left(\sum_{Y\in U^\perp(F)^n}\omega(g')\varphi^0_{(U^\perp)^n}(Y)\cdot\varphi_f^{U^\perp}(Y)\right)\\
&= \Theta_U(g',g)\cdot\theta_{U^\perp}(g') \qquad (11)
\end{aligned}$$

(11) is the analytic version of the pullback formula appeared in [5] (Proposition 3.1).

2.2. **Product rule.**

Definition 2.2.1. *We say $\mathcal{F}$ (or $\{\varphi_V^{(n)}\}_{V,n}$) satisfy the product rule if for any $X = (X_1, ..., X_n) \in V^n$, we have*

$$\varphi^{(n)}(X) = \varphi_1^{(1)}(X_1)\wedge\varphi_2^{(1)}(X_2)\wedge\cdots\wedge\varphi_n^{(1)}(X_n)$$

Here the subscript of $\varphi_i^{(1)}$ corresponds to the $i-$th component of the left hand side of the following map

$$\mathcal{S}(V)\otimes\mathcal{S}(V)\otimes\cdots\otimes\mathcal{S}(V)\longrightarrow\mathcal{S}(V^n)$$

Although every component looks like the same, but actually they are not the same! Recall the construction of the Weil representation, we choose a standard symplectic space W of dimension $2n$ over F, suppose $e_1, \cdots, e_n, f_1, \cdots, f_n$ is a standard basis, denote $W' = span_{\mathbb{R}}\{e_1, \cdots, e_n\}$, the Weil representation is realized on the following space,

$$\mathcal{S}(V \otimes W')) \simeq \mathcal{S}(V^n)$$

Therefore the $i-$th component is actually $\mathcal{S}(V \otimes e_i)$.

Let's see how the product rule affects the Harmonic form we get via (5). We have the following isomorphism,

$$\mathcal{S}(V(\mathbb{A}_{F,f})^n) \simeq \bigotimes_{i=1}^{n} \mathcal{S}(V(\mathbb{A}_{F,f})_i)$$

here the subscript i means the $i-$th copy of V^n. Now we pick $\varphi_f \in \mathcal{S}(V(\mathbb{A}_{F,f})^n)$ satisfying,

$$\varphi_f(X_1, X_2, \cdots, X_n) = \varphi_{f,1}(X_1)\varphi_{f,2}(X_2)\cdots\varphi_{f,n}(X_n)$$

where $\varphi_{f,i} \in \mathcal{S}(V(\mathbb{A}_{F,f})_i)$. We also denote ι_i to be the $i-$th embedding of metaplectic groups,

$$\iota_i : Mp(2, \mathbb{A}_F) \hookrightarrow Mp(2n, \mathbb{A}_F)$$

Then by definition (5), we have

$$\begin{aligned}\Theta(\prod_{i=1}^{n}\iota(g_i'), g, \varphi_f) &= \sum_{X\in V(F)^n} \omega(\prod_{i=1}^{n}\iota(g_i'), g)\varphi^{(n)}(X)\cdot\varphi_f(X)\\ &= \sum_{X_1\in V(F)}\sum_{X_2\in V(F)}\cdots\sum_{X_n\in V(F)} \omega(g_1')\varphi^{(1)}(X_1)\varphi_{f,1}(g^{-1}X_1)\wedge\\ &\omega(g_2')\varphi^{(1)}(X_2)\varphi_{f,2}(g^{-1}X_2)\wedge\cdots\wedge\omega(g_n')\varphi^{(1)}(X_n)\varphi_{f,n}(g^{-1}X_n)\end{aligned}$$

$$= \Theta(g_1', g, \varphi_{f,1}) \wedge \Theta(g_2', g, \varphi_{f,2}) \wedge \cdots \wedge \Theta(g_n', g, \varphi_{f,n}) \tag{12}$$

here we add the extra variable φ_f to indicate the dependence of the Harmonic form Θ on the finite Schwartz function space. For the rest of this note, we won't seriously consider this dependence, so we will omit it.

(12) is not only true for the embedding of $Mp(2, \mathbb{A}_F)$ in $Mp(2n, \mathbb{A}_F)$. It's also true for any decomposition of the symplectic space

$$W = W_1 + W_2$$

where W_1 and W_2 are themselves symplectic and orthogonal to each other. Then we get

$$\iota_i : \ Mp(W_i, \mathbb{A}_F) \hookrightarrow Mp(W, \mathbb{A}_F)$$

and

$$\Theta(\iota_1(g_1')\cdot\iota_2(g_2'), g, \varphi_{f,1}\cdot\varphi_{f,2}) = \Theta(g_1', g, \varphi_{f,1}) \wedge \Theta(g_2', g, \varphi_{f,2}) \tag{13}$$

(13) is the analytic version of the product formula appeared in [5] (Theorem 1.1).

2.3. **Theta distribution.** In this subsection we explain (5), we will prove the following proposition,

Proposition 2.3.1. *when $\varphi_f \in \mathcal{S}(V(\mathcal{A}_{F,f})^n)^K$, then $\Theta(g', g)$ is a harmonic $2n-$form on the Shimura variety $Sh_K(G, X)$.*

Proof: We will prove this result both globally and locally. We abbreviate D_V as D.

$\circ$*Globally,* $\varphi^{(n)} \in \left(\mathcal{S}(V(F\otimes_{\mathbb{Q}}\mathbb{R})^n)\bigotimes\Omega^{2n}(D)\right)^{G(\mathbb{R})} \simeq \left(\mathcal{S}(V(F\otimes_{\mathbb{Q}}\mathbb{R})^n)\bigotimes\Omega^{2n}(D^+)\right)^{G(\mathbb{R})^+}$

In this case we require $g \in G(\mathbb{A}_f)$. We can easily verify that for fixed $g' \in G'(\mathbb{A}_F)$, $\Theta(g', g)$ only depends on the image of g in the following double coset decomposition,

$$G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K = \bigsqcup_j G(\mathbb{Q})g_j K$$

Each $\Theta(g', g_j)$ gives rise to a differntial form on D, which is invariant under the action of $\Gamma_j = G(\mathbb{Q}) \cap g_j K g_j^{-1}$, i.e. $\Theta(g', g_j) \in H^{2n}(\Gamma_j\backslash D, \mathbb{C})$, and we know that

$$M_K = \bigsqcup_j \Gamma_j\backslash D$$

This is an explanation of the meaning of $\Theta(g', g) \in \mathcal{A}^{2n}(M_K, \mathbb{C})$ by explicitly given a global differential form. It's better to denote this form simply by $\Theta(g')$, we will sometimes use this notation in the rest of this note.

$\circ$*Locally,* $\varphi^{(n)} \in \left(\mathcal{S}(V(F \otimes \mathbb{R})^n) \bigotimes \bigwedge^{2n}(\mathfrak{p}^*)\right)^{K_\infty}$

In this case $g = g_\infty g_f \in G(\mathbb{A})$. As we already stated in previous section, $\varphi^n \in (S(V(F \otimes \mathbb{R})^n) \bigotimes \bigwedge^{2n}(\mathfrak{p}^*))^{K_\infty}$ is the restriction of the global differenrial form on D to the point $z \in D$ corresponding to the maximal compact group K_∞. Then $\Theta(g', g_j)$ is the restriction of the global form we just explained to $[g_\infty z, g_f] \in M_K$.

3. Computation of Fourier expansion

3.1. **Simplification.** In this section we calculate the Fourier expansion of $\Theta(g')$. By definition,

$$\Theta(g') = \sum_{\beta \in Sym_n(F)} \int_{N(F)\backslash N(\mathbb{A}_F)} \Theta(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} g')\psi_F(-\frac{1}{2}tr(\beta \cdot b))db \tag{14}$$

We denote $\Theta_\beta(g') = \int_{N(F)\backslash N(\mathbb{A}_F)} \Theta(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} g')\psi_F(-\frac{1}{2}tr(\beta \cdot b))db$. It's easy to see that $\Theta_\beta(g')$ also gives rise to a differential form on M_K. In (14) and the definition of Θ_β, we can add the variable $g \in G(\mathbb{A}^\infty)$ to indicate which component (sightly different with connected component!) are we in. Moreover, it can be computed by explicit formula of Weil representation that,

$$\Theta_\beta(g', g) = \sum_{\substack{X \in V(F)^n \\ (X,X)=\beta}} \omega(g', g)\varphi^{(n)}(X) \cdot \varphi_f(X)$$

Define $\Omega_\beta(F) = \{X \in V(F)^n | (X, X) = \beta\}$, it is stable under the action of $G(\mathbb{Q})$, we consider the $G(\mathbb{Q})_+-$orbits of $\Omega_\beta(F)$,

$$\Omega_\beta(F) = \bigsqcup_i \mathcal{O}_i = \bigsqcup_i G(\mathbb{Q})_+ X_i$$

and define

$$\Theta_{\mathcal{O}_i}(g', g) = \sum_{X \in \mathcal{O}_i} \omega(g', g)\varphi^{(n)}(X) \cdot \varphi_f(X)$$

It's easy to see that $\Theta_{\mathcal{O}_i}(g', g)$ is invariant on the second variable under the double coset decomposition $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$, hence it is a closed $2n-$form on

$$G(\mathbb{Q})_+\backslash D^+ \times G(\mathbb{A}_f)/K \simeq G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K = M_K$$

Therefore we have, as closed $2n-$forms on M_K,

$$\Theta(g') = \sum_{\beta \in Sym_n(F)} \Theta_\beta(g') = \sum_{\beta \in Sym_n(F)} \sum_i \Theta_{\mathcal{O}_i}(g')$$

Now we focus on the orbit $\mathcal{O}_i = G(\mathbb{Q})_+ X_i$. Define $U_i = Span_F\{X_i\}$, the F-subspace of V spanned by the component of X_i. There are two possibilities,

(a). $\mathrm{rank}(\beta) = \dim_F U_i$, we call $\mathcal{O}_i$ a non-degenerate orbit.
(b). $\mathrm{rank}(\beta) < \dim_F U_i$, we call $\mathcal{O}_i$ a degenerate orbit.

Suppose $\mathcal{O}_i$ is a non-degenerate orbit, we have a decomposition,

$$V = U_i + U_i^{\perp}$$

The stabilizer of this decomposition is

$$H_i = G_{U_i} \times G_{U_i^{\perp}}$$

where G_{U_i} (resp. $G_{U_i^{\perp}}$) is the pointwise stabilizer of $U_i^{\perp}$(resp. U_i). Then $\Theta_{\mathcal{O}_i}(g')$ can be further decomposed as,

$$\begin{aligned}
\Theta_{\mathcal{O}_i}(g',g) &= \sum_{X\in\mathcal{O}_i} \omega(g',g)\varphi^{(n)}(X) \\
&= \sum_{\gamma\in G_{X_i}(\mathbb{Q})_+\backslash G(\mathbb{Q})_+} \omega(g',g)\varphi^{(n)}(\gamma^{-1}X_i)\cdot\varphi_f(\gamma^{-1}X_i) \\
&= \sum_{\eta\in H(\mathbb{Q})_+\backslash G(\mathbb{Q})_+} \sum_{\gamma\in G_{X_i}(\mathbb{Q})_+\backslash H(\mathbb{Q})_+} \omega(g',\eta g)\varphi^{(n)}(\gamma^{-1}X_i)\cdot\varphi_f(\gamma^{-1}X_i) \\
&= \sum_{\eta\in H(\mathbb{Q})_+\backslash G(\mathbb{Q})_+} \sum_{X\in U_i^n\cap\mathcal{O}_i} \omega(g',\eta g)\varphi^{(n)}(X)\cdot\varphi_f(X)
\end{aligned}$$

We define

$$\Theta_{U_i}(g',g) = \sum_{X\in U_i^n\cap\mathcal{O}_i} \omega(g',g)\varphi^{(n)}(X)$$

It's easy to see that $\Theta_{U_i}(g',g)$ is invariant on the second variable under the double coset decomposition $H(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$, hence it is a closed $2n-$form on

$$E_K^{H_i} := H_i(\mathbb{Q})_+\backslash D^+ \times G(\mathbb{A}_f)/K$$

and

$$\Theta_{\mathcal{O}_i}(g',g) = \sum_{\eta\in H(\mathbb{Q})_+\backslash G(\mathbb{Q})_+} \Theta_{U_i}(g',\eta g)$$

Now we assume further that U_i is positive definite everywhere, i.e. $\beta = (X_i, X_i)$ is positive semi-definite everywhere and $\mathrm{rank}\beta = \dim_F U_i$. Then there is a closed immersion,

$$D_{U_i^{\perp}} \xrightarrow{i} D$$

hence,

$$M_K^{H_i} := H_i(\mathbb{Q})_+\backslash D^+_{U_i^{\perp}} \times G(\mathbb{A}_f)/K \xrightarrow{i} G(\mathbb{Q})_+\backslash D^+ \times G(\mathbb{A}_f)/K = M_K$$

We have the following diagram,

$$\begin{array}{ccc}
E_K^{H_i} & & \\
\downarrow{\scriptstyle pr} & \searrow & \\
M_K^{H_i} & \xrightarrow[i]{} & M_K
\end{array}$$

Where pr is induced by $pr : D^+ \to D^+_{U_i^{\perp}}$, and the fiber of this map can be described explicitly as

$$pr^{-1}(z) = \{oriented, negative\ 2-planes\ contained\ in\ z + U\otimes_{\tau_1}\mathbb{R}\} \cap D^+$$

By our notation, $pr^{-1}(z) = D^+_{z+U_1}$ (we abbreviate U_1 as $U\otimes_{\tau_1}\mathbb{R}$), it's also easy to show that $pr : E_K^{H_i} \to M_K^{H_i}$ also has fiber isomorphic to $D^+_{z+U_1}$ over the point $[z,g]\in M_K^{H_i}$.

One of the most important theorem in [2] (Theorem 3.1) is the following,

Theorem 3.1.1. *Suppose $\mathcal{O} = G(\mathbb{Q})_+X$ is a non-degenerate and positive definite orbit of rank t, and $\varphi^{(n)}$ satisfies closedness property and invariance property, then* *(i) $\Theta_{U_i}(g')$ is integrable along the fibers of $pr: E_K^{H_i} \to M_K^{H_i}$, so the fiber integral $(pr_*)\Theta_{U_i}(g')$ is a well-defined closed $2n-2t$ form on $M_K^{H_i}$. Especially, when $t = n$, this fiber integral is a number.*

(ii) Moreover, as cohomology classes on M_K,

$$[\Theta_{\mathcal{O}_i}(g')] = i_*[(pr)_*\Theta_{U_i}(g')]$$

Remark 3.1.1. *When $t = n$, the fiber of pr is a complex manifold of dimension n, the fiber integral $(pr_*)\Theta_{U_i}(g')$ is 0 if $\Theta_{U_i}(g')$ is not type (n,n). So we will only focus on $\varphi^{(n)}$ of type (n,n), i.e.,*

$$\varphi^{(n)} \in (S(V(F\otimes\mathbb{R})^n) \bigotimes \Omega^{n,n}(D))^{G(\mathbb{R})} \simeq (S(V(F\otimes\mathbb{R})^n) \bigotimes \Lambda^{n,n}(\mathfrak{p}^*))^{K_\infty}$$

Remark 3.1.2. *U Anisotropic???*

3.2. **Simplification of the fiber integral.** We have reduced the problem of computing the Fourier expansion of $\Theta(g')$ to computing the fiber integral $(pr_*)\Theta_U(g')$, at least for those positive definite orbit. In this section we compute explicitly this fiber integral for positive definite orbit of full rank, we assume $\mathcal{O} = G(\mathbb{Q})_+X$ is a positive definite orbit of rank n, and $U = Span_F\{X\}$, $U_1 = U \otimes_{\tau_1} \mathbb{R}$, $\beta = (X, X)$. By definition, for $z \in D_{U^\perp}$, $g \in G(\mathbb{A}_f)$,

$$\begin{aligned} pr_*[\Theta_U(g')]([z,g]) &= pr_*[\sum_{Y\in U^n\cap\mathcal{O}} \omega(g')\varphi^{(n)}(Y)\cdot\varphi_{f,Y}]([z,g]) \\ &= \int\limits_{pr^{-1}([z,g])} \sum_{Y\in U^n\cap\mathcal{O}} \omega(g')\varphi^{(n)}(Y)\cdot\varphi_{f,Y} \\ &= \sum_{Y\in U^n\cap\mathcal{O}} \int\limits_{D_{z+U_1}} \omega(g')\varphi^{(n)}(Y)\cdot\varphi_{f,Y}(g) \end{aligned}$$

Let's explain these equalities by the following diagram,

$$\begin{array}{ccc} E_K^H = H(\mathbb{Q})_+\backslash D^+ \times G(\mathbb{A}_f)/K & = & \bigsqcup_i \Gamma_i\backslash D^+ \\ \downarrow pr & & \downarrow pr \\ M_K^H = H(\mathbb{Q})_+\backslash D^+_{U^\perp} \times G(\mathbb{A}_f)/K & = & \bigsqcup_i \Gamma_i\backslash D^+_{U^\perp} \end{array}$$

where the disjoint union ranges over the double coset decomposition,

$$G(\mathbb{A}_f) = \bigsqcup_i H(\mathbb{Q})_+ g_i K$$

$\Theta_U(g')$ is a $(n,n)-$form on E_K^H, $\Theta_U(g', g)$ is this form on the component corresponding to $g \in H(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$, it becomes the $g-$component of the $(0,0)-$form, i.e. a function, $pr_*\Theta_U(g')$ after doing fiber integral.

Let's now deal with the fiber integral,

$$\kappa_{\varphi^{(n)}}(g', Y) := \int\limits_{D^+_{z+U_1}} \omega(g')\varphi^{(n)}(Y) \tag{15}$$

We can only consider the case $g' \in G'(\mathbb{R}) = Mp(2n,\mathbb{R}) \times Mp(2n,\mathbb{R}) \times \cdots \times Mp(2n,\mathbb{R})$($d$ copies corresponding to every real place of F), because the finite adele part of g' only acts on φ_f, hence doesn't affect the fiber integral. We know that,

$$\varphi^{(n)}(Y) = \varphi^{(n)}_{V_{\tau_1}}(Y_{\tau_1}) \cdot \prod_{i\geqslant 2} \varphi^0_{V_{\tau_i}}(Y_{\tau_i})$$

here $\varphi^0_{V_{\tau_i}}$ is the Gaussian function of the positive definite space V_{τ_i},

$$\varphi^0_{V_{\tau_i}}(Y_{\tau_i}) = exp(-\pi(Y_{\tau_i}, Y_{\tau_i})_{\tau_i}) = exp(-\pi \cdot tr\beta_i)$$

In (15), we are actually integrating the restriction of $\omega(g')\varphi^{(n)}(Y)$ to D_{z+U_1}, then by the Restriction rule, since $Y_{\tau_1} \in z + U_1$

$$i^*(\omega_{V_{\tau_1}}(g')\varphi^{(n)}_{V_{\tau_1}}(Y_{\tau_1})) = \omega_{z+U_1}(g')\varphi^{(n)}_{z+U_1}(Y_{\tau_1})$$

By Product rule, suppose $Y_{\tau_1} = (Y_{\tau_1,1}, Y_{\tau_1,2}, \cdots, Y_{\tau_1,n})$, $Y_{\tau_1,i} \in U_1$,

$$\varphi^{(n)}_{z+U_1}(Y_{\tau_1}) = \varphi^{(1)}_{z+Y_{\tau_1,1}}(Y_{\tau_1,1}) \wedge \varphi^{(1)}_{z+Y_{\tau_1,2}}(Y_{\tau_1,2}) \wedge \cdots \varphi^{(1)}_{z+Y_{\tau_1,n}}(Y_{\tau_1,n})$$

[2] (Proposition 6.1) proves that, (growth control!! I guess it's true for $K_\infty-$finite vectors)

$$\int_{D^+_{z+U_1}} \varphi^{(n)}_{z+U_1}(Y_{\tau_1}) = \prod_{i=1}^{n} \int_{D^+_{z+Y_{\tau_1,i}}} \varphi^{(1)}_{z+Y_{\tau_1,i}}(Y_{\tau_1,i})$$

Restriction rule and Product rule helps us reduce the computation of of type (n,p) ($(n,n)-$form on dimension p complex manifold) integral into the computation of the type (n,n) and then to type $(1,1)$. To make a small summary, we get

$$\kappa_{\varphi^{(n)}}(1,Y) = exp(-\pi \sum_{j\geqslant 2} tr\beta_j) \cdot \prod_{i=1}^{n} \int_{D^+_{z+Y_{\tau_1,i}}} \varphi^{(1)}_{z+Y_{\tau_1,i}}(Y_{\tau_1,i}) \tag{16}$$

Remark 3.2.1. *(16) holds only when we have the Product rule. However, the product rule is not known to $\omega(g')\varphi^{(n)}(Y)$. That's why the left hand side is $g' = 1$.*

Remark 3.2.2. *Although we don't have a nice decomposition of (16) type, it's simply by definition that,*

$$\kappa_{\varphi^{(n)}}(g', Y) = \kappa_{\omega(g')\varphi^{(n)}}(1, Y)$$

Our hope is that $\varphi^{(n)}$ behaves well under Weil representation, so we can relate $\varphi^{(n)}$ and $\omega(g')\varphi^{(n)}$ by some simple formula, hence for their fiber integral.
Actually this hope is already true for parabolic subgroup of $G'(\mathbb{R})$. Since $G'(\mathbb{R})$ is generated by the parabolic subgroup and

$$w = (\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, 1)$$

whose action on Schwartz function is Fourier transformation. Therefore we expect the Schwartz function coefficients of $\varphi^{(n)}$ to be Hermite polynomials which are invariant under the Fourier transformation.

3.3. **Computation of the fiber integral in the case (1,2).** In this subsection we explained the computation of the fiber integral

$$\int_{D^+_{z+X}} \varphi^{(1)}_{z+X}(X)$$

$z + X$ is real vector space with a non-degenerate quadratic form $(\ ,\ )$ of signature $(1,2)$, X is a positive vector. By definition,

$$D^+_{z+X} = \{oriented, negative\ 2-planes\ contained\ in\ z+X\} \cap D^+$$

In this case, it's better to describe D^+_{z+X} in the following way,

$$D^+_{z+X} = a\ connected\ component\ of\ \{oriented, positive\ 1-lines\ contained\ in\ z+X\}$$

We choose a basis for $z + X$ as follows,

$$x = (X,X)^{-\frac{1}{2}}X, z_1 \in z, z_2 \in z$$

s.t., $v = x_0 x + x_1 z_1 + x_2 z_2$ has norm

$$(v,v) = x_0^2 - x_1^2 - x_2^2$$

then we have a clearer description of D^+_{z+X},

$$D^+_{z+X} \simeq \{(x_0,x_1,x_2) \,|\, x_0^2 - x_1^2 - x_2^2 = 1,\ x_0 > 0\}$$

this is the hyperboloid model $\mathbf{H}_1^2$, we can also identify the tangent space at $z' \in D_{z+X}^+$ as,

$$T_{z'}D_{z+X}^+ \simeq z'^{\perp}$$

the canonical metric on this tangent space is $-(\ ,\ )$. $(\mathbf{H}_1^2, -(\ ,\ ))$ is isometric to upper half plane $(\mathcal{H}_1,\ y^{-2}(dx^2 + dy^2))$. Next we will try to compute the fiber integral of a form in the following space,

$$\varphi^{(1)} \in (S(z+X) \bigotimes \Omega^{1,1}(D_{z+X}))^{SO(1,2)} \simeq (S(z+X) \bigotimes \Omega^{1,1}(D_{z+X}^+))^{SO(1,2)^+} \tag{17}$$

$$\simeq (S(z+X) \bigotimes \bigwedge^{1,1}(\mathfrak{p}^*))^{SO(2)} \tag{18}$$

here the second isomorphism is restricting the global form to a point $z' \in D_{z+X}^+$, $\mathfrak{p} \simeq T_{z'}D_{z+X}^+$ is a Lie subalgebra of $\mathfrak{so}(1,2)$, and exponential map gives a differeomorphism between $\mathfrak{p}$ and D_{z+X}^+. With the basis we have already chosen, $\mathfrak{so}(1,2)$ can be identified as,

$$\mathfrak{so}(1,2) = \{ \begin{pmatrix} 0 & x_1 & x_2 \\ x_1 & 0 & z \\ x_2 & -z & 0 \end{pmatrix} \mid x_2,\, x_2,\, z \in \mathbb{R}\}$$

we choose the base point $z' = z = (1,0,0)$, then $\mathfrak{p}$ can be identified as

$$\mathfrak{p} = \{ \begin{pmatrix} 0 & x_1 & x_2 \\ x_1 & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix} \mid x_1,\, x_2 \in \mathbb{R}\}$$

and the isomorphism $\mathfrak{p} \simeq T_{z'}D_{z+X}^+$ is,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longmapsto (0,1,0) \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \longmapsto (0,0,1)$$

hence we can identify $\mathfrak{p}^* = Span_{\mathbb{R}}\{x_1, x_2\}$, then a general element in the space (18) is of the following form,

$$\varphi_z^{(1)}(x_0 x + x_1 z_1 + x_2 z_2) = f(x_0^2) \cdot g(x_1^2 + x_2^2) \cdot exp(-\pi(x_0^2 + x_1^2 + x_2^2)) \cdot dx_1 \wedge dx_2$$

where f and g are one variable polynomials. Since D_{z+X} has complex dimension 1, this form is automatically closed. Particularly, if we evaluate this at X, we get,

$$\varphi_z^{(1)}(X) = f((X,X)) \cdot exp(-\pi(X,X)) \cdot dx_1 \wedge dx_2 = (\varphi^{(1)}(X))_z$$

For other points in D_{z+X}, we have the following

Lemma 3.3.1. *For $z' \in D_{z+X}^+$ a positive vector with norm 1, suppose $gz = z'$ for $g \in SO(1,2)^+$, then we have,*

$$(\varphi^{(1)}(X))_{z'} = f((X,z')) \cdot g((X,z') - (X,X)) \cdot exp(-\pi(X,X)_{z'}) \cdot (g^*)^{-1}(dx_1 \wedge dx_2)$$

here $(X,X)_{z'} = 2(X,z') - (X,X)$ and $(g^*)^{-1}(dx_1 \wedge dx_2)$ is the volume form of D_{z+X}^+ at z'.

Consider the following coordinate change,

$$(0,+\infty) \times S^1 \xrightarrow{e} D_{z+X}^+$$
$$(r,\theta) \longmapsto (ch(r), sh(r)cos(\theta), sh(r)sin(\theta))$$

this is actually $(0,+\infty) \times S^1 \longrightarrow \mathfrak{p} \xrightarrow{exp} D_{z+X}^+$. Under this coordinate change, it's easy to verify that

$$(e^*(\varphi^{(1)})(X))_{(r,\theta)} = f((X,X)ch(r)^2) \cdot g((X,X)sh(r)^2) \cdot e^{-2\pi(X,X)ch(r)^2 + \pi(X,X)} sh(r)drd\theta$$

then

$$\kappa_{\varphi^{(1)}}(1,X) = 2\pi e^{\pi(X,X)} \int_0^{+\infty} f((X,X)ch(r)^2) \cdot g((X,X)sh(r)^2) \cdot e^{-2\pi(X,X)ch(r)^2} sh(r)dr$$

take the change of variable $u \mapsto ch(r)$, we get,

$$\kappa_{\varphi^{(1)}}(1,X) = 2\pi e^{\pi(X,X)} \int_1^{+\infty} f((X,X)u^2) \cdot g((X,X)(u^2-1)) \cdot e^{-2\pi(X,X)u^2} du$$

these calculations suggest that the following function is important for our computation,

$$h(t) = \int_1^{+\infty} f(t^2u^2) \cdot g(t^2(u^2-1)) \cdot e^{-2\pi t^2 u^2} du \tag{19}$$

we have shown that $\kappa_{\varphi^{(1)}}(1,X) = 2\pi e^{\pi(X,X)} h((X,X)^{\frac{1}{2}})$ only depends on (X,X). Then (16) becomes,

$$\begin{aligned}\kappa_{\varphi^{(n)}}(1,Y) &= e^{-\pi \sum\limits_{j\geqslant 2} tr\beta_j} \cdot \prod_{i=1}^{n} 2\pi e^{\pi(Y_{\tau_1,i},Y_{\tau_1,i})} h((Y_{\tau_1,i},Y_{\tau_1,i})^{\frac{1}{2}}) \\ &= (2\pi)^n e^{-\pi \sum\limits_{j} tr\beta_j} \cdot \prod_{i=1}^{n} e^{2\pi\beta_{1,ii}} h((\beta_{1,ii})^{\frac{1}{2}})\end{aligned}$$

It's natural to hope that the function $h(t)e^{2\pi t^2}$ in the product of RHS is a constant, i.e., $h(t)$ is a (constant multiple of) Gaussian, if this is true, suppose $h(t) = \frac{1}{2\pi}e^{-2\pi t^2}$ (the constant is chose to be $\frac{1}{2\pi}$ to eliminate $(2\pi)^n$), then the Mellin transform of h is

$$\int_0^{+\infty} t^{s-1} h(t) dt = \frac{1}{2\pi} \frac{\Gamma(\frac{s}{2})}{2(2\pi)^{\frac{s}{2}}} \tag{20}$$

On the other hand, if we compute the Mellin transform directly form (19), we get

$$\int_0^{+\infty} t^{s-1} h(t) dt = \int_1^{+\infty}\int_0^{+\infty} v^{s-1} u^{-s} f(v^2) g(v^2(1-u^{-2})) e^{-2\pi v^2} du dv$$

For simplicity, we assume g is constant function 1, then this becomes,

$$\int_0^{+\infty} t^{s-1} h(t) dt = \int_1^{+\infty}\int_0^{+\infty} v^{s-1} u^{-s} f(v^2) e^{-2\pi v^2} du dv = \frac{1}{s-1} \int_0^{+\infty} v^{s-1} f(v^2) e^{-2\pi v^2} dv$$

Since f is a polynomial, it's easy to compute the following "single" integration,

$$\int_0^{+\infty} v^{s-1} v^{2n} e^{-2\pi v^2} dv = \frac{\Gamma(\frac{s}{2}+n)}{2(2\pi)^{\frac{s}{2}+n}} \tag{21}$$

we take linear combinations of (21) to eliminate $s-1$ and try to get the RHS of (20), then a simple calculation shows that the following function is the one we need,

$$f(T) = 2T - \frac{1}{2\pi}$$

and $h(t) = \frac{1}{2\pi}e^{-2\pi t^2}$ by this choice. Moreover, (16) becomes

$$\kappa_{\varphi^{(n)}}(1,Y) = exp(-\pi \sum_j tr\beta_j) \tag{22}$$

[need to be added, general formula for g']

4. Construction

Suppose V is real quadratic space with signature $(n,2)$, In this section, we will give the construction of a reasonable Schwartz class satisfying condition 1,2,3,4. Calculation from the last section tells us that, a reasonable Schwartz class in the case of signature $(1,2)$ should take the following form,

$$(\varphi^{(1)}(X))_z = (2(X,z) - \frac{1}{2\pi}) \cdot exp(-\pi(X,X)_z) \cdot dV_z \tag{23}$$

$$= ((X,X) + (X,X)_z - \frac{1}{2\pi})exp(-\pi(X,X)_z) \cdot dV_z \tag{24}$$

where dV is the volume form of D^+. We will give two constructions, the first one is based on this known case $(1,2)$, and uses $G(\mathbb{R})$ (or K_∞) invariant property, closeness property, the Restriction rule. The second one is the construction appeared in [1] using Howe operator. We will see finally that these two constructions coincide.

4.1. **Construction from the known case.** We don't have a very explicit description of the hermitian symmetric domain D_V^+ when $p > 1$, but a large part of the analysis of the case $p = 1$ can still be done.

As in the introduction, we fix a $z \in D_V^+$, and an oriented orthogonal normal basis $\{z_1, z_2\}$ of z, which is unique up to $SO(2)$ action. Then $z^\perp$ is a $p-$dimensional, positive definite subspace of V. We also fix an oriented orthogonal normal basis $\{w_1, w_2, \cdots, w_p\}$, which is unique up to $SO(p)$ action. We have the following identification,

$$\mathfrak{p} \simeq T_z D_V^+ \simeq Hom_{\mathbb{R}}(z, V/z) \simeq z^* \otimes z^\perp \simeq z \otimes z^\perp \tag{25}$$

here $\mathfrak{p} = \mathfrak{so}(p,2)/(\mathfrak{so}(p) \oplus \mathfrak{so}(2))$, with the basis we have chosen, it can be identified with the following matrices,

$$\mathfrak{p} = \{\begin{pmatrix} 0_p & A \\ A^t & 0_2 \end{pmatrix} \mid A \in M_{p\times 2}(\mathbb{R})\}$$

the isomorphism (25) can be made explicitly as follows,

$$z \otimes z^\perp \xrightarrow{\simeq} \mathfrak{p}$$

$$w_i \otimes z_j \longmapsto \begin{pmatrix} 0_n & E_{ij} \\ E_{ij}^t & 0_2 \end{pmatrix}$$

We use ω_{ij} to denote the coordinate function of $\mathfrak{p}$ w.r.t the basis $\{w_i \otimes z_j =: v_{ij}\}$, then ω_{ij} make up a basis of $\mathfrak{p}^*$ over $\mathbb{R}$. We also use x_1, x_2 to denote the coordinate function of z w.r.t the basis $\{z_1, z_2\}$, and y_i to denote the coordinate function of $z^\perp$ w.r.r the basis $\{w_i\}$.

Lemma 4.1.1. *The almost complex structure on $T_z D_V^+$ is given by*

$$Ad\left(\begin{pmatrix} I_p & \\ & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}\right)$$

Lemma 4.1.2. *The canonical Riemannian metric on $T_z D_V^+$ can be identified with the negative of the metric on $z \otimes z^\perp$ induced by $(\ ,\ )$ of V. The Chern class of the canonical line bundle on D_V^+ is*

$$\Omega = \sum_{i=1}^{p} \omega_{i1} \wedge \omega_{i2}$$

The Schwartz form we are finding is of the following form,

$$\varphi_V^{(1)}(X) = \sum_{i=1}^{p} f_i(X)\varphi_V^+(X)\omega_{i1} \wedge \omega_{i2} + \sum_{i\neq j} g_{ij}(X)\varphi_V^+(X)\omega_{i1} \wedge \omega_{j2} \tag{26}$$

$$+ \sum_{i\neq j} h_{ij}(X)\varphi_V^+(X)\omega_{i1} \wedge \omega_{j1} + \sum_{i\neq j} k_{ij}(X)\varphi_V^+(X)\omega_{i2} \wedge \omega_{j2} \tag{26'}$$

here $\varphi_V^+(X)$ is the Gaussian function on V w.r.t the quadratic form $(\ ,\ )_z$. Our main result in the section is the following,

Theorem 4.1.1. *Based on the case $p = 1$, and by $K_\infty-$invariance property and Restriction rule, we can obtain (up to a nonzero constant)*

$$f_i(X) = y_i^2 - \frac{1}{4\pi}$$
$$g_{ij}(X) = y_i y_j$$
$$(26') = 0$$

Moreover, with these choices, $\varphi_V^{(1)}(X)$ is closed.

This theorem tells us that the general case is determined by the case $p = 1$, and at point $z \in D_V^+$, it takes the following form,

$$\varphi_V^{(1)}(X) = \left(\sum_{i=1}^{p}(y_i^2 - \frac{1}{4\pi})\omega_{i1} \wedge \omega_{i2} + \sum_{i \neq j} y_i y_j \omega_{i1} \wedge \omega_{j2}\right) \cdot e^{-\pi(X,X)_z} \tag{27}$$

$$= \left(\left(\sum_{i=1}^{p} y_i \omega_{i1}\right) \wedge \left(\sum_{i=1}^{p} y_i \omega_{i2}\right) - \frac{1}{4\pi}\Omega\right) \cdot e^{-\pi(X,X)_z} \tag{28}$$

Proof: First we consider the subspace $V_i = z + \mathbb{R}w_i$, this is of signature $(1, 2)$, we have

$$D_{V_i}^+ \xhookrightarrow{j} D_V^+ \tag{29}$$

then by Restriction rule,

$$(y_i^2 - \frac{1}{4\pi}) \cdot \varphi_V^+(X) = j^*\varphi_V^{(1)}(X)(v_{i1}, v_{i2})$$
$$= f_i(X)\varphi_V^+(X)$$

then we get the result for f_i, but we can choose another base of $z^\perp$ by applying an element in $SO(p)$, suppose we have

$$(w_1', w_2', \cdots, w_p') = (w_1, w_2, \cdots, w_p) \cdot A$$

then for coordinate functions, we also have

$$(y_1', y_2', \cdots, y_p') = (y_1, y_2, \cdots, y_p) \cdot A$$

Now consider (29) for $V_i' = z + \mathbb{R}w_i'$, by restriction rule, we get

$$\left((\sum_{j=1}^{p} a_{ji} y_j)^2 - \frac{1}{4\pi}\right) \cdot \varphi_V^+(X) = j^*\varphi_V^{(1)}(X)(v_{i1}', v_{i2}')$$
$$= \varphi_V^{(1)}(X)(\sum_{j=1}^{p} a_{ji} v_{j1}, \sum_{k=1}^{p} a_{ki} v_{k2})$$
$$= \sum_{j,k} a_{ji} a_{ki} \varphi_V^{(1)}(X)(v_{j1}, v_{k2})$$
$$= \left(\sum_{l=1}^{p} a_{li}^2 (y_l^2 - \frac{1}{4\pi}) + \sum_{j \neq k} a_{ji} a_{ki} g_{jk}(X)\right) \cdot \varphi_V^+(X)$$

then it's trivial to see that we must have the expression for g_{ij}. Actually, this computation is equivalent to saying that $\varphi_V^{(1)}$ is $SO(p)$-invariant. Now let's check

$$\varphi(X) = \left(\sum_{i=1}^{p}(y_i^2 - \frac{1}{4\pi})\omega_{i1} \wedge \omega_{i2} + \sum_{i \neq j} y_i y_j \omega_{i1} \wedge \omega_{j2}\right) \cdot e^{-\pi(X,X)_z}$$

is $SO(2)-$invariant. Consider the following action of $SO(2)$ on z,

$$z_1' = cos(\alpha)z_1 + sin(\alpha)z_2$$
$$z_1' = -sin(\alpha)z_1 + cos(\alpha)z_2$$

then it's easy to see that,

$$w'_{i1} = cos(\alpha)w_{i1} + sin(\alpha)w_{i2}$$
$$w'_{i2} = -sin(\alpha)w_{i1} + cos(\alpha)w_{i2}$$

and

$$\omega'_{i1} \wedge \omega'_{j2} = cos(\alpha)^2\omega_{i1} \wedge \omega_{j2} - sin(\alpha)^2\omega_{i2} \wedge \omega_{j1} + sin(\alpha)cos(\alpha)\omega_{i2} \wedge \omega_{j2} - sin(\alpha)cos(\alpha)\omega_{i1} \wedge \omega_{j1}$$

under this action, $\varphi(X)$ becomes,

$$\begin{aligned}
\varphi(X)' &= \left(\sum_{i=1}^{p}(y_i^2 - \frac{1}{4\pi})\omega'_{i1} \wedge \omega'_{i2} + \sum_{i\neq j} y_i y_j \omega'_{i1} \wedge \omega'_{j2}\right) \cdot e^{-\pi(X,X)_z} \\
&= \left(\sum_{i=1}^{p}(y_i^2 - \frac{1}{4\pi})\omega_{i1} \wedge \omega_{i2} + \sum_{i\neq j} y_i y_j cos(\alpha)^2\omega_{i1} \wedge \omega_{j2} - sin(\alpha)^2\omega_{i2} \wedge \omega_{j1}\right) \cdot e^{-\pi(X,X)_z} \\
&+ \left(\sum_{i\neq j} sin(\alpha)cos(\alpha)\omega_{i2} \wedge \omega_{j2} - sin(\alpha)cos(\alpha)\omega_{i1} \wedge \omega_{j1}\right) \cdot e^{-\pi(X,X)_z} \\
&= \left(\sum_{i=1}^{p}(y_i^2 - \frac{1}{4\pi})\omega_{i1} \wedge \omega_{i2} + \sum_{i\neq j} y_i y_j \omega_{i1} \wedge \omega_{j2}\right) \cdot e^{-\pi(X,X)_z} + 0 = \varphi(X) \qquad (30)
\end{aligned}$$

Since $\varphi_V^{(1)}(X) = \varphi(X)$+(26') is $K_\infty = SO(p) \times SO(2)-$invariant, and $\varphi(X)$ is already $K_\infty-$invariant, (26') must be $K_\infty-$invariant. Under the same $SO(2)-$action,

$$\begin{aligned}
(26') &= \left(\sum_{i\neq j} h_{ij}(X')\omega'_{i1} \wedge \omega'_{j1} + k_{ij}(X')\omega'_{i2} \wedge \omega'_{j2}\right) \varphi_V^+(X) \\
&= \left(\sum_{i\neq j} h_{ij}(X')cos(\alpha)^2\omega_{i1} \wedge \omega_{j1} + h_{ij}(X')sin(\alpha)^2\omega_{i2} \wedge \omega_{j2}\right) \varphi_V^+(X) \\
&+ \left(\sum_{i\neq j}(h_{ij}(X') - h_{ji}(X'))sin(\alpha)cos(\alpha)\omega_{i1} \wedge \omega_{j2}\right) \varphi_V^+(X) \\
&+ \left(\sum_{i\neq j} k_{ij}(X')cos(\alpha)^2\omega_{i2} \wedge \omega_{j2} + k_{ij}(X')sin(\alpha)^2\omega_{i1} \wedge \omega_{j1}\right) \varphi_V^+(X) \\
&- \left(\sum_{i\neq j}(k_{ij}(X') - k_{ji}(X'))sin(\alpha)cos(\alpha)\omega_{i1} \wedge \omega_{j2}\right) \varphi_V^+(X) \qquad (31)
\end{aligned}$$

since (26') only involves $\omega_{i1} \wedge \omega_{j1}$ and $\omega_{i2} \wedge \omega_{j2}$, we get

$$h_{ij} = h_{ji}$$
$$k_{ij} = k_{ji}$$

this symmetry condition implies that the summation in (26') is 0. Then we get all the identifications in the theorem. Next we show $\varphi_V^{(1)}(X) = \varphi(X)$ is closed.

$$d = \sum_{i,j} \omega(v_{ij}) \otimes \omega_{ij} \wedge \qquad (32)$$

where $\omega_{ij}\wedge$ is left multiplication by ω_{ij} on the exterior part, $\omega(v_{ij})$ is the infinitesimal Weil representation on the Schwartz function space,

$$(\omega(v_{ij})f)(X) = df(X) \cdot \begin{pmatrix} 0_p & E_{ij} \\ E_{ij}^t & 0_2 \end{pmatrix} \cdot X$$

Now we use formula (28), and

$$d\left(\sum_{i=1}^{p} y_i\omega_{i1}\right) = \sum_{i,j} x_j\omega_{ij}\wedge\omega_{j1}$$

$$d\left(\sum_{i=1}^{p} y_i\omega_{i2}\right) = \sum_{i,j} x_j\omega_{ij}\wedge\omega_{j2}$$

then

$$\begin{aligned}
d\varphi_V^{(1)}(X) &= d\left(\left(\sum_{i=1}^{p} y_i\omega_{i1}\right)\wedge\left(\sum_{i=1}^{p} y_i\omega_{i2}\right)-\frac{1}{4\pi}\Omega\right)\cdot e^{-\pi(X,X)_z}\\
&+\left(\left(\sum_{i=1}^{p} y_i\omega_{i1}\right)\wedge\left(\sum_{i=1}^{p} y_i\omega_{i2}\right)-\frac{1}{4\pi}\Omega\right)\wedge d(e^{-\pi(X,X)_z})\\
&=\left(\left(\sum_{i,j} x_j\omega_{ij}\wedge\omega_{j1}\right)\wedge\left(\sum_{i=1}^{p} y_i\omega_{i2}\right)\right)\cdot e^{-\pi(X,X)_z}\\
&+\left(\left(\sum_{i=1}^{p} y_i\omega_{i1}\right)\wedge\left(\sum_{i,j} x_j\omega_{ij}\wedge\omega_{j2}\right)\right)\cdot e^{-\pi(X,X)_z}\\
&+\left(\left(\sum_{i=1}^{p} y_i\omega_{i1}\right)\wedge\left(\sum_{i=1}^{p} y_i\omega_{i2}\right)-\frac{1}{4\pi}\Omega\right)\wedge\left(\sum_{i,j} -4\pi x_j y_i\cdot\omega_{ij}\right)e^{-\pi(X,X)_z}
\end{aligned}$$

it can be shown that

$$\left(\left(\sum_{i=1}^{p} y_i\omega_{i1}\right)\wedge\left(\sum_{i=1}^{p} y_i\omega_{i2}\right)\right)\wedge\left(\sum_{i,j} -4\pi x_j y_i\cdot\omega_{ij}\right)=0$$

and

$$\left(\sum_{i,j} x_j\omega_{ij}\wedge\omega_{j1}\right)\wedge\left(\sum_{i=1}^{p} y_i\omega_{i2}\right)+\left(\sum_{i=1}^{p} y_i\omega_{i1}\right)\wedge\left(\sum_{i,j} x_j\omega_{ij}\wedge\omega_{j2}\right)=-\sum_{i,j} x_j y_i\cdot\Omega\wedge\omega_{ij}$$

4.2. **Construction by Howe operator.** In this section, we follow [1] closely to give another construction of the Schwartz class. In [1], Kudla and Millson use the Howe operator defined as follows,

$$\nabla = \left(\sum_{i=1}^{p}\left(\frac{\partial}{\partial y_i}-2\pi y_i\right)\otimes\omega_{i1}\wedge\right)\cdot\left(\sum_{i=1}^{p}\left(\frac{\partial}{\partial y_i}-2\pi y_i\right)\otimes\omega_{i2}\wedge\right) \tag{33}$$

this operator acts on $\mathcal{S}(z^\perp)$, and we can find that,

$$\varphi_V^{(1)} = \varphi_z\cdot\nabla\varphi_{z^\perp}$$

where φ_z (resp. $\varphi_{z^\perp}$) is the Gaussian function on z (resp. $z^\perp$). Our main goal in this section is to understand the construction of this operator in a much more general setting.

Our initial goal is to find a K_∞−invariant class in the following space

$$\mathcal{S}(V)\bigotimes\wedge^2\mathfrak{p}^*$$

where V is of signature $(p,2)$. Now we work more generally on V of signature (p,q), and fix a decomposition $V = V_+ + V_-$, where V_+ (resp. V_-) is positive definite (negative definite). We want to find the $K_\infty \simeq SO(V_+)\times SO(V_-)$-invariant inside the following space

$$\mathcal{S}(V^n)\bigotimes\wedge^{nq}\mathfrak{p}^*$$

or more generally, in the following algebra

$$\mathcal{S}(V^n)\bigotimes\wedge^{\bullet}\mathfrak{p}^*$$

here $\mathfrak{p}$ is the tangent space at $z = V_+$ of the associated symmetric space D_V,

$$\mathfrak{p} \simeq Hom_{\mathbb{R}}(V_-, V/V_-) \simeq V_-^* \otimes V_+ \simeq V_- \otimes V_+$$

∘ *Step 1.* We consider the Gaussian function $\varphi^0_{V_-^n}$ (resp. $\varphi^0_{V_+^n}$) associated to V_-^n (resp. V_-^n). We have the following injective map

$$i : Sym(V^{*n}) \bigotimes \wedge^{\bullet}\mathfrak{p}^* \hookrightarrow \mathcal{S}(V^n) \bigotimes \wedge^{\bullet}\mathfrak{p}^*$$
$$f \otimes v \mapsto f \cdot \varphi^0_{V_-^n}\varphi^0_{V_+^n} \otimes v$$

Since $\varphi^0_{V_-^n}\varphi^0_{V_+^n}$ is $K_\infty-$invariant, by multiplying it, we transfer the question of finding the invariant on the right hand side to finding invariant on the left hand side.

∘ *Step 2.* We consider the problem of finding the $SO(V_+)-$invariant first, since $SO(V_+) = K_+$ acts trivially on V_-, we focus on the following space,

$$Sym(V_+^{*n}) \bigotimes \wedge^{\bullet}\mathfrak{p}^* \simeq Sym(V_+^n) \bigotimes \wedge^{\bullet}\mathfrak{p}^* \tag{34}$$

Note that we get a trivial K_+-invariant element here, namely, $1 \otimes 1$. The idea is instead of finding an invariant element directly, we try to find invariant operator $\bigtriangledown$ on this space, then applying the operator on $1 \otimes 1$, what we get is also K_+-invariant element.

∘ *Step 3.* Now we use Theorem A.3.1, it gives us a system of K_+-invariant operators on the space (34). In the language of Appendix A.3, we make the following choice,

$$X = V_+,\ V_0 = V_-^* + V_-$$

W still denote a symplectic space of dimension $2n$, we use (51) as a standard basis. The symmetric form on V_0 is given as follows, recall that we choose $\{z_1, z_2\}$ as a basis of V_-, and $\{x_1, x_2\}$ as a basis of V_-, x_i is the coordinate function of z_i. Then,

$$(z_i, z_j) = (x_i, x_j) = 0,\ (z_i, x_j) = \delta_{ij}$$

this choice of symmetric form makes V_0 into a split orthogonal space of dimension 4 over $\mathbb{R}$. We also set $V_-^* = (V_0)_1$, $V_- = (V_0)_2$, under this choice,

$$\mathsf{W}_1 = W_1 \otimes_{\mathbb{R}} V_+ \simeq V_+^n,\ \mathsf{V}_1 = (V_0)_1 \otimes_{\mathbb{R}} V_+ \simeq V_-^* \otimes V_+ \simeq \mathfrak{p}$$

and the algebra (55) becomes

$$Sym(V_+^{*n}) \bigotimes \wedge^{\bullet}\mathfrak{p}^*$$

which is exactly (34). This algebra admits an action by the following Lie superalgebra

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^0 + \tilde{\mathfrak{g}}^1$$
$$\tilde{\mathfrak{g}}^0 = \mathfrak{sp}(\mathsf{W})_{\mathbb{C}} \oplus \mathfrak{o}(\mathsf{V})_{\mathbb{C}},\ \tilde{\mathfrak{g}}^1 = (\mathsf{W} \otimes \mathsf{V})_{\mathbb{C}}$$

and the centralizer of $\mathfrak{o}(V_+)$ inside $\tilde{\mathfrak{g}}$ is given by

$$\Gamma = \Gamma^0 + \Gamma^1$$
$$\Gamma^0 = \mathfrak{sp}(W)_{\mathbb{C}} \oplus \mathfrak{o}(V_0)_{\mathbb{C}},\ \Gamma^1 = (W \otimes V_0)_{\mathbb{C}}$$

Γ also generates the $SO(V_+)-$invariant operators in $\tilde{\mathcal{T}}_{2\pi i}(\mathsf{W}, \mathsf{V}) \subset End_{\mathbb{C}}(Sym(V_+^{*n}) \bigotimes \wedge^{\bullet}\mathfrak{p}^*)$.

The action of Γ is given by,

$$\mathfrak{sp}(W)_{\mathbb{C}} \oplus \mathfrak{o}(V_0)_{\mathbb{C}} \hookrightarrow \mathfrak{sp}(\mathsf{W}) \oplus \mathfrak{o}(\mathsf{V}) \xrightarrow{\sigma_{2\pi i} \bigoplus \sigma_{2\pi i}} End_{\mathbb{C}}(Sym(V_+^{*n}) \bigotimes \wedge^{\bullet}\mathfrak{p}^*) \tag{35}$$

$$(W_{\mathbb{C}} \otimes V_0)_{\mathbb{C}} \hookrightarrow (\mathsf{W} \otimes \mathsf{V})_{\mathbb{C}} \xrightarrow{\sigma_{2\pi i}} End_{\mathbb{C}}(Sym(V_+^{*n}) \bigotimes \wedge^{\bullet}\mathfrak{p}^*) \tag{36}$$

We focus on the action of (36), more precisely,

$$(W \otimes V_0)_{\mathbb{C}} = (W_{\mathbb{C}} \otimes (V_-^*)_{\mathbb{C}}) \oplus (W_{\mathbb{C}} \otimes (V_-)_{\mathbb{C}}) \tag{37}$$

$$(\mathsf{W} \otimes \mathsf{V})_{\mathbb{C}} = (\mathsf{W}_{\mathbb{C}} \otimes (\mathsf{V}_1)_{\mathbb{C}}) \oplus (\mathsf{W}_{\mathbb{C}} \otimes (\mathsf{V}_2)_{\mathbb{C}}) \tag{38}$$

Every element in the second summand of (37) and (38) raises the degree in the wedge part by 1, by the formula (54), and

$$W_{\mathbb{C}} \otimes (V_-)_{\mathbb{C}} = (W_1 \otimes (V_-)_{\mathbb{C}}) \otimes (W_2 \oplus (V_-)_{\mathbb{C}})$$

and the action of $W_1 \otimes (V_-)_{\mathbb{C}}$ is 0 by the construction of oscillator representation. Therefore the only interesting part of (37) is $W_2 \otimes (V_-)_{\mathbb{C}}$, it has complex dimension nq, therefore raise the "obvious" $SO(V_+)-$invariant element $1 \otimes 1$ to the degree nq in the wedge part, which is exactly what we want.

$W_2 \otimes (V_-)_{\mathbb{C}}$ has the following basis,

$$w_{2i} \otimes z_j,\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant q \tag{39}$$

it is mapped to the following element in $(\mathsf{W} \otimes \mathsf{V})_{\mathbb{C}}$

$$\sum_{k=1}^{p} (w_{2i} \otimes w_k) \otimes (z_j \otimes w_k) = \sum_{k=1}^{p} (w_{2i} \otimes w_k) \otimes v_{kj}$$

this element acts on the Schrödinger model $\mathcal{S}(V_+^n) \bigotimes \wedge^{\bullet}\mathfrak{p}^*$ by

$$\sum_{k=1}^{p} (\frac{\partial}{\partial y_{ki}} - 2\pi y_{ki}) \otimes \omega_{kj} \wedge$$

here k indicates the $k-$th copy of V_+^n. Then we define the Howe operator of type (p, q) to be

$$\triangledown_{p,q}^{n} = \prod_{i=1}^{n} \prod_{j=1}^{q} \sum_{k=1}^{p} (\frac{\partial}{\partial y_{ki}} - 2\pi y_{ki}) \otimes \omega_{kj} \wedge \tag{40}$$

This operator corresponds to the action of the one-dimensional space

$$\wedge^{nq}(W_2 \otimes (V_-)_{\mathbb{C}}) \tag{41}$$

especially when $q = 2$, $n = 1$, (40) agrees with (33).

Finally, at the point $z = V_- \in D_V^+$, the Schwartz form we want can be expressed by

$$\tilde{\varphi}_V^{(n)} = \varphi_{V_-}^0 \cdot \triangledown_{p,q}^n \varphi_{V_+}^0 \tag{42}$$

We already know that this is a $nq-$form, and invariant under the action of $SO(V_+)$. Now we consider the action of $SO(V_-)$,

Proposition 4.2.1. *$\varphi_V^{(n)}$ is invaraint under the action of $SO(V_-)$. Therefore,*

$$\tilde{\varphi}_V^{(n)} \in (\mathcal{S}(V^n) \bigotimes \wedge^{nq}\mathfrak{p}^*)^{K_\infty}$$

Proof: $\varphi_{V_-}^0$ is obviously invariant under the action of $SO(V_-)$, so $\varphi_V^{(n)}$ is $SO(V_-)$-invariant is equivalent to $\triangledown_{p,q}^n \varphi_{V_+}^0$ is $SO(V_-)$-invariant, we prove this by arguing that $\triangledown_{p,q}^n$ is $SO(V_-)$-invariant. But $SO(V_-)$ acts on $W_2 \otimes (V_-)_{\mathbb{C}}$ by changing an oriented basis of V_-, hence leaves (41) invariant, hence leaves the operator $\triangledown_{p,q}^n$ invariant.

Lemma 4.2.1. *When $q = 2$,$n = 1$, the Schwartz form (42) constructed via Howe operator equals the form constructed in Theorem 4.1 up to a nonzero constant, i.e.*

$$\tilde{\varphi}_V^{(1)} = \varphi_{V_-}^0 \cdot \triangledown_{p,2}^1 \varphi_{V_+}^0 = 16\pi^2 \varphi_V^{(1)}$$

5. Uniqueness of Schwartz form

5.1. **Uniqueness theorem.** Continue with previous notations, we assume V is a finite dimensional vector space over $\mathbb{R}$, equipped with a non-degenerate bilinear symmetric form of signature $(p, 2)$, $p \geqslant 1$. For every such V, we consider the following space

$$\mathcal{S}(V^n) \bigotimes \Omega^{2n}(D_V^+)$$

Our main goal is to find a system of elements in this space satisfying $1, 2, 3, 4$ in section 2, because then this element will give rise to a Harmonic form on the Shimura varieties via theta distribution (5) by $1, 2$, compatible with the natural closed immersion (11) by 3, and wedge product (13) by 4.

In section 2, we consider the case $p = 1$ and $n = 1$, and find that in this case if we choose the form to be

$$\varphi^{(1)} = (y^2 - \frac{1}{4\pi})\omega_1 \wedge \omega_2$$

Then the fiber integral (19) is easy to compute and we get a clean result (22).

In section 3.1, we construct the Schwartz form for general p and $n = 1$, we first use restriction rule and the known case $p = 1$ to determine some coefficients of the form, and then use $G(\mathbb{R})-$invariance (i.e. $K_\infty-$invariance) to determine the others. Then in section 3.2, we construct the Schwartz form by Howe operator, and the result coincides with 3.1

It's a natural question to ask whether there exists other systems of forms satisfying $1, 2, 3, 4$. We will prove in this section that there are essentially no other forms satisfying all these four conditions. Firstly, The product rule tells us that we only need to find a form satisfying $1, 2, 3$ in the case $n = 1$. We consider the following dense subspace of Schwartz functions,

$$S(V) = \{P(X) \cdot \varphi_V^+(X) \mid P \textit{ is a polynomial function on } V\} \subset \mathcal{S}(V)$$

Theorem 5.1.1. *There is a unique (up to a nonzero scalar) system of elements* $\{\varphi_V^{(n)}\}_{V,n}$

$$\varphi_V^{(n)} \in S(V^n) \otimes \Omega^{2n}(D_V^+)$$

s.t. 1,2,3 holds for these elements.

Or equivalently, There is a unique (up to a nonzero scalar) system of elements $\{\varphi_V^{(1)}\}_V$

$$\varphi_V^{(1)} \in S(V) \otimes \Omega^2(D_V^+)$$

s.t. 1,2 holds for these elements.

Proof: We prove the second statement. As before, we fix a $z \in D_V^+$, an oriented orthogonal normal basis $\{z_1, z_2\}$ of z and coordinate function $\{x_1, x_2\}$, and an oriented orthogonal normal basis $\{w_1, w_2, \cdots, w_p\}$ of $z^\perp$ and coordinate function $\{y_1, y_2, \cdots, y_p\}$, using the isomorphism,

$$(S(V) \bigotimes \Omega^2(D_V^+))^{G(\mathbb{R})^+} \simeq (S(V) \bigotimes \wedge^2 \mathfrak{p}^*)^{K_\infty}$$

From now on we work in the space on the right hand side (as we did before). We use ω_{ij} to denote the coordinate function of $\mathfrak{p}$ w.r.t the basis $\{w_i \otimes z_j =: v_{ij}\}$, then ω_{ij} make up a basis of $\mathfrak{p}^*$ over $\mathbb{R}$. The Schwartz form we are looking for is of the following form,

$$\varphi_V^{(1)}(X) = \sum_{i=1}^{p} f_i(X)\varphi_V^+(X)\omega_{i1} \wedge \omega_{i2} + \sum_{i \neq j} g_{ij}(X)\varphi_V^+(X)\omega_{i1} \wedge \omega_{j2} \tag{43}$$

$$+ \sum_{i \neq j} h_{ij}(X)\varphi_V^+(X)\omega_{i1} \wedge \omega_{j1} + \sum_{i \neq j} k_{ij}(X)\varphi_V^+(X)\omega_{i2} \wedge \omega_{j2} \tag{37'}$$

Now we consider restricting this form to the subspace $V_i = w_i + z$, this subspace has an oriented orthonormal basis $\{v_{i1}, v_{i2}\}$,

$$\iota_i : D^+_{w_i+z} \hookrightarrow D^+_V$$

then the restriction rule (7) tells us,

$$\iota_i^* \varphi_V^{(1)}(x_1, x_2, y) = \varphi^{(1)}_{w_i+z}(x_1, x_2, y_i) \cdot \varphi^+_{V_i^\perp}(y_1, y_2, \cdots, \hat{y_i}, \cdots, y_p)$$

By the $SO(z)-$invariance, $\varphi^{(1)}_{w_i+z}(x_1, x_2, y_i)$ should take the following form,

$$\varphi^{(1)}_{w_i+z}(x_1, x_2, y_i) = f(y_i) g(x_1^2 + x_2^2) \cdot \varphi^+_V(x_1, x_2, y_i)$$

here f, g are both polynomials, and f is independent of i because of the naturality property (8) following from the restriction rule. Now we consider the value of both sides at vector pair (v_{i1}, v_{i2}), we get

$$\begin{aligned} f_i(x_1, x_2, y) &= (\varphi^{(1)}_{w_i+z}(x_1, x_2, y_i)(v_{i1}, v_{i2})) \cdot \varphi^+_V(x_1, x_2, y_i)^{-1} \\ &= f(y_i) g(x_1^2 + x_2^2) \end{aligned}$$

Now we consider the $SO(V_+)$ action on the form (43), under the basis $\{w_i\}$, we choose $A = (a_{ij}) \in SO(V_+)$, then

$$w_i' = \sum_{k=1}^p a_{ki} w_k, \;\; v_{ij}' = \sum_{k=1}^p a_{ki} v_{kj}, \;\; y_i' = \sum_{k=1}^p a_{ki} y_k$$

$A \in SO(z^\perp)$ acts on (43) by

$$\begin{aligned} A^* \varphi_V^{(1)}(X) &= \sum_{i=1}^p f(AX) g(AX) \varphi^+_V(AX) \omega_{i1}' \wedge \omega_{i2}' + \sum_{i \neq j} g_{ij}(AX) \varphi^+_V(AX) \omega_{i1}' \wedge \omega_{j2}' \\ &+ \sum_{i \neq j} h_{ij}(AX) \varphi^+_V(AX) \omega_{i1}' \wedge \omega_{j1}' + \sum_{i \neq j} k_{ij}(AX) \varphi^+_V(AX) \omega_{i2}' \wedge \omega_{j2}' \end{aligned} \tag{44}$$

By $SO(z^\perp)-$invariance, we have

$$A^* \varphi_V^{(1)} = \varphi_V^{(1)}$$

evaluate both sides at the vector pair (v_{i1}', v_{i2}'), we get

$$\begin{aligned} f(\sum_{k=1}^p a_{ki} y_k) \cdot g(x_1^2 + x_2^2) &= \varphi_V^{(1)}(X)(\sum_{k=1}^p a_{ki} v_{k1}, v \sum_{k=1}^p a_{ki} v_{k2}) \\ &= \sum_{k=1}^p a_{ki}^2 f(y_k) \cdot g(x_1^2 + x_2^2) + \sum_{j \neq l} a_{ji} a_{li} g_{jl}(x, y) \end{aligned}$$

divide both sides by $g(x_1^2 + x_2^2)$, and denote $g_{jl}' = \frac{g_{jl}}{g(x_1^2 + x_2^2)}$, we have

$$f(\sum_{k=1}^p a_{ki} y_k) = \sum_{k=1}^p a_{ki}^2 f(y_k) + \sum_{j \neq l} a_{ji} a_{li} g_{jl}'(x, y)$$

for each pair $j \neq l$, it's easy to find an element in $SO(z^\perp)$ such that only g_{jl}' survives in the second term of the right side (e.g. rotate by $\frac{\pi}{4}$ by the plane spanned by w_j and w_l). Then we see that g_{jl}' depends only on y, and it's a polynomial functions in $y_1, y_2, \cdots, y_n$. The previous equation becomes

$$f(\sum_{k=1}^p a_{ki} y_k) = \sum_{k=1}^p a_{ki}^2 f(y_k) + \sum_{j \neq l} a_{ji} a_{li} g_{jl}'(y) \tag{45}$$

then since f and g_{jl}' are polynomial functions, by Lemma (5.1.1), up to a constant multiple,

$$f(T) = T^2 + a, \;\; g_{jl}'(T_1, T_2, \cdots, T_n) = T_j T_l$$

for some $a \in \mathbb{C}$. Therefore we have simplified (43) to the following form,

$$\varphi_V^{(1)}(x,y) = \left(\sum_{i=1}^{p}(y_i^2+a)\omega_{i1}\wedge\omega_{i2} + \sum_{i\neq j} y_iy_j\omega_{i1}\wedge\omega_{j2}\right)\cdot g(x_1^2+x_2^2)\varphi_V^+(X) \tag{46}$$

$$+\sum_{i\neq j} h_{ij}(X)\varphi_V^+(X)\omega_{i1}\wedge\omega_{j1} + \sum_{i\neq j} k_{ij}(X)\varphi_V^+(X)\omega_{i2}\wedge\omega_{j2} \tag{46'}$$

Now we consider the $SO(V_-)-$invariance, the same computation and argument in (30) and (31) implies (46) is $SO(V_-)-$invariant, and

$$h_{ij} = h_{ji}$$
$$k_{ij} = k_{ji}$$

i.e. (46')=0. Therefore (43) takes the following form,

$$\varphi_V^{(1)}(x,y) = \left(\sum_{i=1}^{p}(y_i^2+a)\omega_{i1}\wedge\omega_{i2} + \sum_{i\neq j} y_iy_j\omega_{i1}\wedge\omega_{j2}\right)\cdot g(x_1^2+x_2^2)\varphi_V^+(X) \tag{47}$$

Now we consider the closedness condition, we rewrite (47) as

$$\varphi_V^{(1)}(x,y) = \left(\phi + a\Omega\cdot\varphi_V^+\right)\cdot g$$

where

$$\phi = \left(\sum_{i=1}^{p} y_i\omega_{i1}\right)\wedge\left(\sum_{i=1}^{p} y_i\omega_{i2}\right)$$

and

$$\Omega = \sum_{i=1}^{p}\omega_{i1}\wedge\omega_{i2}$$

we know that Ω is the Kähler form of D_V^+, it's closed. Then by applying the $d-$operator (32),

$$d\varphi_V^{(1)} = (a+\frac{1}{4\pi})\left(\Omega\wedge d\varphi_V^+\right)\cdot g + \left(\phi + a\Omega\cdot\varphi_V^+\right)\wedge dg \tag{48}$$

the closedness condition is reduced to the computation of $d\varphi_V^+$ and dg.

$$\begin{aligned} d\varphi_V^+ &= \sum_{j=1}^{p}\omega(v_{j1})\varphi_V^+\cdot\omega_{j1} + \sum_{j=1}^{p}\omega(v_{j1})\varphi_V^+\cdot\omega_{j1} \\ &= -4\pi\varphi_V^+\left(\sum_{j=1}^{p} x_1y_j\cdot\omega_{j1} + \sum_{j=1}^{p} x_2y_j\varphi_V^+\cdot\omega_{j2}\right) \end{aligned}$$

$$\begin{aligned} dg &= \sum_{j=1}^{p}\omega(v_{j1})g\cdot\omega_{j1} + \sum_{j=1}^{p}\omega(v_{j1})g\cdot\omega_{j1} \\ &= g'\cdot\left(\sum_{j=1}^{p} x_1y_j\cdot\omega_{j1} + \sum_{j=1}^{p} x_2y_j\cdot\omega_{j2}\right) \end{aligned}$$

then the first term in (48) is

$$(a+\frac{1}{4\pi})\left(\Omega\wedge d\varphi_V^+\right)\cdot g = -(1+4\pi a)\varphi_V^+\cdot g\cdot\left(x_1\sum_{i,j} y_j\omega_{i1}\wedge\omega_{i2}\wedge\omega_{j1} + x_2\sum_{i,j} y_j\omega_{i1}\wedge\omega_{i2}\wedge\omega_{j2}\right)$$

the second term in (48) is

$$\begin{aligned} \left(\phi + a\Omega\cdot\varphi_V^+\right)\cdot dg &= a\Omega\cdot\varphi_V^+\cdot g'\cdot\left(\sum_{j=1}^{p} x_1y_j\cdot\omega_{j1} + \sum_{j=1}^{p} x_2y_j\cdot\omega_{j2}\right) \\ &= a\varphi_V^+\cdot g'\cdot\left(x_1\sum_{i,j} y_j\omega_{i1}\wedge\omega_{i2}\wedge\omega_{j1} + x_2\sum_{i,j} y_j\omega_{i1}\wedge\omega_{i2}\wedge\omega_{j2}\right) \end{aligned}$$

Note that

$$x_1\sum_{i,j} y_j\omega_{i1}\wedge\omega_{i2}\wedge\omega_{j1}+x_2\sum_{i,j} y_j\omega_{i1}\wedge\omega_{i2}\wedge\omega_{j2}$$
$$=\sum_{i<j} x_1(y_j-y_i)\omega_{i1}\wedge\omega_{i2}\wedge\omega_{j1}+\sum_{i<j} x_2(y_j-y_i)\omega_{i1}\wedge\omega_{i2}\wedge\omega_{j2}$$

is non-zero, then we necessarily have

$$\big(ag'-(1+4\pi a)g\big)\cdot\varphi_V^+=0 \tag{49}$$

If g is not a constant, then $deg(g')<deg(g)$, we must have $1+4\pi a=0$, but then $ag'=0$, since g is not a constant, $a=0$, which is a contradiction. Therefore we must have g is a constant, then $g'=0$, this implies $1+4\pi a=0$, i.e. $a=-\frac{1}{4\pi}$. This finishes the proof the theorem.

Lemma 5.1.1. *If f and g'_{jl} are both polynomial functions, then the only solutions (up to nonzero constant) to (45) are given by*

$$f(T)=T^2+a,\ g'_{jl}(T_1,T_2,\cdots,T_n)=T_jT_l$$

for some $a\in\mathbb{C}$

Proof: We make a particular choice of $A\in SO(V_+)$

$$Aw_i=cos(\theta)w_i+sin(\theta)w_j$$
$$Aw_i=-sin(\theta)w_i+cos(\theta)w_j$$

then (45) becomes,

$$f(cos(\theta)y_i+sin(\theta)y_j)=cos(\theta)^2f(y_i)+sin(\theta)^2f(y_j)+cos(\theta)sin(\theta)(g'_{ij}(y)+g'_{ji}(y))$$

suppose $f(T)=\sum_{m\geqslant 0}c_mT^m$, then set $y_j=\sqrt{-1}y_i$, and suppose $(g'_{ij}+g'_{ji})(y_i,\sqrt{-1}y_i)=\sum_{m\geqslant 0}a_my_i^m$, we get

$$\sum_{m\geqslant 0}c_me^{im\theta}y_i^m=\sum_{m\geqslant 0}\big((cos(\theta)^2+(-i)^msin(\theta)^2)c_m+cos(\theta)sin(\theta)a_m\big)\,y_i^m$$

therefore by comparing coefficients,

$$c_me^{im\theta}=(cos(\theta)^2+(-i)^msin(\theta)^2)c_m+cos(\theta)sin(\theta)a_m,\ \forall m\geqslant 0,\ \forall\theta\in\mathbb{R} \tag{50}$$

For $m\neq 0,2$, the only possibility for (50) to hold is $c_m=a_m=0$ (view θ as a complex variable and compare degrees). This concludes the proof.

5.2. **Interaction with Weil representation.**

Appendix A. Spin-Oscillator representation

A.1. **Oscialltor representation and inifinitesimal Fock model.** In this appendix, we construct the infinitesimal representation associated to the Weil representation at the Archimedean place. Suppose $(W,<,>)$ is a symplectic space of dimension $2n$ over $\mathbb{R}$, we fix a standard basis $\{e_1,e_2,\cdots,e_n,f_1,f_2,\cdots,f_n\}$ for W, i.e.

$$<e_i,e_j>=<f_i,f_j>=0,<e_i,f_j>=\delta_{ij} \tag{51}$$

There is a complex structure on W given by

$$J:W\longrightarrow W$$
$$e_i\longmapsto f_i,f_i\longmapsto -e_i$$

We then have the following Hodge decomposition,

$$W\otimes_{\mathbb{R}}\mathbb{C}=W_1+W_2$$

where

$$W_1 = span_{\mathbb{C}}\{w_{1j} = e_j - if_j\}_{j=1}^n$$
$$W_2 = span_{\mathbb{C}}\{w_{2j} = e_j + if_j\}_{j=1}^n$$

It's easy to check W_1 and W_2 are dually paired Lagrangians of the symplectic space $W \otimes_{\mathbb{R}} \mathbb{C}$. We use z_j to denote the following linear functional on W_1,

$$z_j(w) = < w, w_{2j} >$$

then z_j spans the space W_1^*.

Lemma A.1.1. *There is a isomorphism between $Sym^2(W)$ and $\mathfrak{sp}(W)$ given by*

$$Sym^2(W) \xrightarrow{\varphi} \mathfrak{sp}(W)$$
$$x \circ y \longmapsto \varphi(x \circ y)$$

where $x \circ y = x \otimes y + y \otimes x$, and

$$\varphi(x \circ y)(z) = < x, z > y + < y, z > x$$

Remark A.1.1. *Under the symplectic basis we are using, we identify $\mathfrak{sp}(n)$ with the following*

$$\mathfrak{sp}(n) = \{\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} | B, C \in Sym_n(\mathbb{R})\}$$

under the isomorphism φ,

$$e_i \circ e_j \longmapsto \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix}$$
$$f_i \circ f_j \longmapsto \begin{pmatrix} 0 & 0 \\ -E_{ij} - E_{ji} & 0 \end{pmatrix}$$
$$e_i \circ f_j \longmapsto \begin{pmatrix} -E_{ij} & 0 \\ 0 & E_{ji} \end{pmatrix}$$

Definition A.1.1. *For $\lambda \in \mathbb{C}^*$, the Weyl algebra $\mathbf{W}_\lambda$ associated to $(W, < \, , \, >)$ is*

$$\mathbf{W}_\lambda = \frac{T(W \otimes_{\mathbb{R}} \mathbb{C})}{ideal \ generated \ by \ x \otimes y - y \otimes x - \lambda < x, y >}$$

Remark A.1.2. *There is a natural filtration $F^p\mathbf{W}_\lambda$ on $\mathbf{W}_\lambda$ inherited by the degree filtration on the tensor algebra $T(W)$, and it's easy to check that*

$$[F^p\mathbf{W}_\lambda, F^q\mathbf{W}_\lambda] \subset F^{p+q-2}\mathbf{W}_\lambda$$

where $[\, , \,]$ is the Lie bracket on the associated algebra $\mathbf{W}_\lambda$. Therefore $F^2\mathbf{W}_\lambda$ forms a complex Lie algebra. There is natural isomorphisms,

$$F^2\mathbf{W}_\lambda / F^1\mathbf{W}_\lambda \simeq S^2(W)_{\mathbb{C}} \simeq sp(W)_{\mathbb{C}}$$

We can even define a spliting homomorphism between Lie algebras,

$$j : \mathfrak{sp}(W)_{\mathbb{C}} \simeq Sym^2(W)_{\mathbb{C}} \longrightarrow F^2\mathbf{W}_\lambda$$
$$x \circ y \longmapsto -\frac{1}{2\lambda}(xy + yx)$$

Now we consider the left ideal $\mathcal{I}$ of the Weyl algebra $\mathbf{W}_\lambda$ generated by W_1, then it's easy to see that,

$$\mathbf{W}_\lambda / \mathcal{I} \simeq Sym(W_2) \simeq Sym(W_1^*) \simeq \mathbb{C}[z_1, z_2, \cdots, z_n]$$

there is a natural action ρ_λ of $\mathbf{W}_\lambda$ on $\mathbf{W}_\lambda/\mathcal{I}$, hence on $\mathbb{C}[z_1, z_2, \cdots, z_n]$, by left multiplication. It is given explicitly by

$$\rho_\lambda(w_{1j}) = 2i\lambda\frac{\partial}{\partial z_j}$$
$$\rho_\lambda(w_{2j}) = z_j$$

then we can construct a representation of Lie algebra $sp(W)_\mathbb{C}$ via j, i.e. $\sigma_\lambda = \rho_\lambda \circ j$. Note that $Sym^2(W \otimes_\mathbb{R} \mathbb{C}) \simeq Sym^2(W_1) \oplus (W_1 \otimes_\mathbb{C} W_2) \oplus Sym^2(W_2)$. σ_λ can be given explicitly by

$$\sigma_\lambda(w_{1j} \circ w_{1k}) = 4\lambda\frac{\partial^2}{\partial z_j \partial z_k}$$
$$\sigma_\lambda(w_{2j} \circ w_{2k}) = -\frac{1}{\lambda} z_j z_k$$
$$\sigma_\lambda(w_{1j} \circ w_{2k}) = -i(z_k\frac{\partial}{\partial z_j} + z_j\frac{\partial}{\partial z_k})$$

It's natural to ask whether this Lie algebra representation gives rise to a Lie group representation, this is not a trivial question because the representation space $\mathbb{C}[z_1, z_2, \cdots, z_n]$ is infinite dimensional. It turns out we should consider a larger space, before we state the result, let's first consider the relation between σ_λ and the Weil representation of $Sp(W)$ on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. The Heisenberg group $H(W) = W \times \mathbb{R}$ acts on $\mathcal{S}(\mathbb{R}^n)$ with central character

$$e_\infty : \mathbb{R} \longrightarrow \mathbb{C}^*$$
$$t \longmapsto e^{2\pi i t}$$

it is an irreducible representation of the Heisenberg group, and the associated infinitesimal representation is given by

$$\omega(e_j) = \frac{\partial}{\partial x_j}$$
$$\omega(f_j) = 2\pi i x_j$$

It's easy to verify that,

$$\omega(e_i) \circ \omega(f_j) - \omega(f_j) \circ \omega(e_i) = 2\pi i < e_i, f_j >$$

therefore ω extends to a representation of $\mathbf{W}_{2\pi i}$. Now we compare ω with $\sigma_{2\pi i}$

Theorem A.1.1. *There exists an injective $\mathbf{W}_{2\pi i}$−intertwining operator*

$$\iota : \mathbb{C}[z_1, z_2, \cdots, z_n] \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

s.t. 1 *is mapped to the Gaussian φ_0 on $\mathbb{R}^n$, the image is a dense subspace of $\mathcal{S}(\mathbb{R}^n)$*

Proof: We consider the subrepresentation $\mathcal{S}'$ of ω generated by the Gaussian φ_0. It's trivial to see that W_1 annihilates φ_0, hence

$$\mathcal{S}' = \mathbf{W}_{2\pi i} \cdot \varphi_0 \simeq \mathbf{W}_{2\pi i}/\mathcal{I} \simeq \mathbb{C}[z_1, z_2, \cdots, z_n]$$

Remark A.1.3. *It's easy to check that $\mathcal{S}'$ is exactly the space of Hermite functions, i.e.*

$$\mathcal{S}' = \{P(x) \cdot \varphi_0 | \, P \textit{ is a polynomial}\}$$

it is said in [3] that this subspace is exactly the $U(n)$−finite functions of $\mathcal{S}(\mathbb{R}^n)$.

Now we find the right space to extend the Lie algebra representation to Lie group representation,

Proposition A.1.1. *$\sigma_{2\pi i}$ gives rise to a representation on the Schwatz space $\mathcal{S}(\mathbb{R}^n)$ of the complex symplectic group $Sp(W)(\mathbb{C})$, also a representation of real group $Mp(W)$, which is a double cover of $Sp(V)$.*

A.2. **Spin representation.** Most of the results in this section is parallel to the previous one. Suppose V is a finite dimensional vector space over $\mathbb{R}$, equipped with a nondegenerate symmetric bilinear form $(\ ,\ )$.

Lemma A.2.1. *The following map is an isomorphism*

$$\wedge^2 V \xrightarrow{\phi} \mathfrak{o}(V)$$
$$x \wedge y \longmapsto \phi(x \wedge y)$$

where

$$\phi(x \wedge y)(z) = (x, z)y - (y, z)x$$

Definition A.2.1. *For $\lambda \in \mathbb{C}^*$, the complex Clifford algebra $\mathbf{V}_\lambda$ associated to $(V, (\ ,\ ))$ is*

$$\mathbf{V}_\lambda = \frac{T(V \otimes_{\mathbb{R}} \mathbb{C})}{\textit{ideal generated by } x \otimes y + y \otimes x - \lambda(x, y)}$$

Remark A.2.1. *Contrary to the Weyl algebra associated to a symplectic space, Clifford algebra is of finite dimensional $2^{dim_{\mathbb{R}} V}$ over $\mathbb{C}$.*

Remark A.2.2. *There is a natural filtration $F^p\mathbf{V}_\lambda$ on $\mathbf{V}_\lambda$ inherited by the degree filtration on the tensor algebra $T(V)$, and it's easy to check that*

$$[F^p\mathbf{V}_\lambda, F^q\mathbf{V}_\lambda] \subset F^{p+q-2}\mathbf{V}_\lambda$$

where $[\ ,\ ]$ is the Lie bracket on the associated algebra $\mathbf{V}_\lambda$. Therefore $F^2\mathbf{V}_\lambda$ forms a complex Lie algebra. There is natural isomorphisms,

$$F^2\mathbf{V}_\lambda / F^1\mathbf{V}_\lambda \simeq \wedge^2 V_{\mathbb{C}} \simeq \mathfrak{o}(V)_{\mathbb{C}}$$

We can even define a splitting homomorphism between Lie algebras,

$$j : \mathfrak{o}(V)_{\mathbb{C}} \simeq \wedge^2(V)_{\mathbb{C}} \longrightarrow F^2\mathbf{V}_\lambda$$
$$x \circ y \longmapsto -\frac{1}{2\lambda}(xy - yx)$$

Suppose there exists a basis of V, $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ s.t.

$$(u_i, u_j) = (v_i, v_j) = 0, (u_i, v_j) = (v_j, u_i) = \delta_{ij} \tag{52}$$

define $V_1 = span_{\mathbb{R}} u_i{}_{i=1}^n$, $V_2 = span_{\mathbb{R}} v_i{}_{i=1}^n$. We use α_i (resp. β_i) to denote the coordinate function on V_1 (resp. V_2) of u_i (resp. v_i). Now we consider the left ideal $\mathcal{J}$ of the Clifford algebra $\mathbf{V}_\lambda$ generated by V_1, then it's easy to see that,

$$\mathbf{V}_\lambda / \mathcal{J} \simeq \wedge(V_2)_{\mathbb{C}} \simeq \wedge(V_1^*)_{\mathbb{C}}$$

this gives a natural action ρ_λ of $\mathbf{V}_\lambda$ on the wedge algebra $\wedge(V_1^*)_{\mathbb{C}}$, it is given explicitly by

$$\rho_\lambda(u_j) = \lambda I_{u_j} \tag{53}$$
$$\rho_\lambda(v_j) = \alpha_i \wedge \tag{54}$$

here I_{u_j} is the interior multiplication on the wedge algebra

$$(I_{u_j} f)(X) = f(u_j, X)$$

Therefore by the splitting map j, we get a representation of $\mathfrak{o}(V)$ on the wedge algebra $\wedge(V_1^*)_{\mathbb{C}}$, i.e. $\sigma_\lambda = \rho_\lambda \circ j$, and it is given explicitly by

$$\sigma_\lambda(u_j \circ u_k) = -\frac{\lambda}{2}(I_{u_j} I_{u_k} - I_{u_k} I_{u_j})$$
$$\sigma_\lambda(u_j \circ v_k) = \alpha_k \wedge I_{u_j} - \frac{1}{2}\delta_{jk}$$
$$\sigma_\lambda(v_j \circ v_k) = \frac{1}{2\lambda}(\alpha_k \wedge \alpha_j \wedge -\alpha_j \wedge \alpha_k \wedge)$$

from these formulas, we see that $u_j \circ u_k$ lower the degree by 2, $u_j \circ v_k$ keeps the degree unchanged, $v_j \circ v_k$ raise the degree by 2. Since the wedge algebra $\wedge(V_1^*)_{\mathbb{C}}$ is of finite dimensional, we have

Proposition A.2.1. *σ_λ gives rise to a finite dimensional representation of the complex orthogonal group $SO(V)(\mathbb{C})$, also a finite dimensional representation of real group $Spin(V)$, which is a double cover of $SO(V)$.*

A.3. **Ortho-symplectic algebra.** In this section, we combine the construction in the previous two sections together and construct a representation which is important in finding the K_∞−invariant.

Suppose $(W, < \, , \, >)$ is a symplectic space, and $(V, (\, , \,))$ is a symmetric space, we use w (resp. v) to denote the vector in W (resp. V).

Definition A.3.1. *For $\lambda \in \mathbb{C}^*$, the ortho-symplectic algebra associated to $(W, < \, , \, >)$ and $(V, (\, , \,))$ is*

$$\mathcal{T}_\lambda(W,V) = \frac{T((W \oplus V)\otimes_{\mathbb{R}} \mathbb{C})}{\{v\otimes v' + v'\otimes v - \lambda(v,v'), w\otimes w' - w'\otimes w - \lambda < w, w' >, w\otimes v - v\otimes w\}}$$

Then we know from the previous two sections that when V is split orthogonal, i.e. admits a basis of the form in (52), $\mathcal{T}_\lambda(W,V)$ admits an action on the following algebra

$$Sym(W_1^*) \bigotimes \wedge {V_1^*}_{\mathbb{C}} \tag{55}$$

i.e. we have a homomorphism of associated algebra,

$$\mathcal{T}_\lambda(W,V) \longrightarrow End_{\mathbb{C}}(Sym(W_1^*) \bigotimes \wedge {V_1^*}_{\mathbb{C}})$$

we use $\tilde{\mathcal{T}}_\lambda(W,V)$ to denote the image of this homomorphism.

$\mathcal{T}_\lambda(W,V)$ also admits a degree filtration inherited from the tensor algebra, heuristics from previous two sections motivate us to consider the degree 2 part, the natural bracket on the associated algebra gives,

$$[F^p\mathcal{T}_\lambda(W,V), F^q\mathcal{T}_\lambda(W,V)] \subset F^{p+q-2}\mathcal{T}_\lambda(W,V)$$

therefore $F^2\mathcal{T}_\lambda(W,V)$ is a Lie algebra. We have the following isomorphism

$$F^2\mathcal{T}_\lambda(W,V)/F^1\mathcal{T}_\lambda(W,V) \simeq Sym^2 W_{\mathbb{C}} \oplus \wedge^2 V_{\mathbb{C}} \oplus (W\otimes V)_{\mathbb{C}} \tag{56}$$

and the splitting is given by

$$\begin{aligned} j : Sym^2 W_{\mathbb{C}} \oplus \wedge^2 V_{\mathbb{C}} \oplus (W\otimes V)_{\mathbb{C}} &\longrightarrow F^2\mathcal{T}_\lambda(W,V) \\ w\circ w' &\longmapsto -\frac{1}{2\lambda}(ww' + w'w) \\ v\circ v' &\longmapsto -\frac{1}{2\lambda}(vv' - v'v) \\ v\circ w &\longmapsto -\frac{1}{2\lambda}(vw + vw) \end{aligned}$$

By the identification

$$\mathfrak{sp}(W) \simeq Sym^2 W, \ \mathfrak{o}(V) \simeq \wedge^2 V$$

we therefore obtain an Lie algebra representation of $\mathfrak{sp}(W)\oplus\mathfrak{o}(V)$ on (55) when $\lambda = 2\pi i$. Actually we can get more, with the Lie bracket given by the embedding j, the space (56) admits a structure of Lie superalgebra, we denote it by $\tilde{\mathfrak{g}}$, the grading is given by

$$\tilde{\mathfrak{g}}^0 = \mathfrak{sp}(W)\oplus\mathfrak{o}(V), \ \tilde{\mathfrak{g}}^1 = W\otimes V$$

What we actually get is a representation of the Lie superalgebra $\tilde{\mathfrak{g}}$.

We consider the following special choice of W and V. We choose X to be an arbitrary finite dimensional vector space over $\mathbb{R}$ equipped with a non-degenerate symmetric form $(\, , \,)_X$, we use W and V_0 to denote the standard symplectic space and split orthogonal space, then we define

$$\mathsf{W} = W \otimes X, \ \mathsf{V} = V_0 \otimes X$$

it's easy to see that W and V are also symplectic and split orthogonal space if we equip W with $(\, , \,)_X \otimes < \, , \, >$ and $(\, , \,)_X \otimes (\, , \,)$ respectively. Note that $\mathfrak{o}(X)$ naturally injects into $\mathfrak{sp}(\mathsf{W})$ and $\mathfrak{o}(\mathsf{V})$.

Theorem A.3.1. *The centralizer of $\mathfrak{o}(X)$ in $\tilde{\mathfrak{g}}$ is also a Lie superalgebra, we denote it by Γ, then*

$$\Gamma^0 = \mathfrak{sp}(W)_{\mathbb{C}} \oplus \mathfrak{o}(V_0)_{\mathbb{C}} \subset \tilde{\mathfrak{g}}^0, \ \Gamma^1 = (W \otimes V_0)_{\mathbb{C}} \subset \tilde{\mathfrak{g}}^1$$

and moreover,

$$\tilde{\mathcal{T}}_{2\pi i}(\mathsf{W}, \mathsf{V})^{SO(V)^+} = \textit{the subalgebra generated by } \Gamma \textit{ inside } \tilde{\mathcal{T}}_{2\pi i}(\mathsf{W}, \mathsf{V})$$

Remark A.3.1. *Here the second inclusion*

$$\Gamma^1 = (W \otimes V_0)_{\mathbb{C}} \subset \tilde{\mathfrak{g}}^1$$

is realized as

$$(W \otimes V_0)_{\mathbb{C}} \simeq Hom_{\mathbb{C}}(W_{\mathbb{C}}, V_{0\mathbb{C}}) \hookrightarrow Hom_{\mathbb{C}}(W_{\mathbb{C}} \otimes X_{\mathbb{C}}, V_{0\mathbb{C}} \otimes X_{\mathbb{C}}) \simeq Hom_{\mathbb{C}}(\mathsf{W}_{\mathbb{C}}, \mathsf{V}_{\mathbb{C}}) \simeq (\mathsf{W} \otimes \mathsf{V})_{\mathbb{C}}$$

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