Iwasawa Main Conjecture over $\mathbb{Q}$

Selmer groups (Motivation?)

$\mathbb{Q}_\infty = \mathbb{Q}(\mu_{p^\infty})^{\mu_{p-1}}$ is cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Z}_p$, $F/\mathbb{Q}$ finite, denote $F_\infty = F\cdot\mathbb{Q}_\infty$, cyclotomic $\mathbb{Z}_p$-ext of F

F_n = the fixed field corresponding to $p^n\mathbb{Z}_p$, $\mathrm{Gal}(F_n/F) \simeq \mathbb{Z}/p^n\mathbb{Z}$

$A_n :=$ class group of F_n, there is a natural action $\mathrm{Gal}(F_n/F) \curvearrowright A_n$

we have maps $N_n : A_{n+1} \to A_n$ which is compatible with Galois action, then let

$$X_\infty = \varprojlim_n A_n, \quad X_\infty \text{ is } \Gamma := \mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_p\text{-module}$$

<u>Note</u>: X_∞ is a f.g. torsion $\Lambda = \mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T]]$ - module

<u>Recall</u>: structure theorem of f.g. torsion Λ-module

$$M \sim \bigoplus_{i=1}^{s} \Lambda/f_i(T)^{k_i}\Lambda \oplus \bigoplus_{j=1}^{t} \Lambda/p^{l_j}\Lambda, \quad k_i, l_j \geq 1, \ f_i \text{ distinguished}$$

$$\lambda(M) = \sum_{i=1}^{s} s_i \deg f_i, \quad \mu(M) = \sum_{j=1}^{t} l_j \quad \text{are invariants of } M$$

$$\mathrm{Ch}_\Lambda(M) := \left(p^{\mu(M)} \cdot \prod_{i=1}^{s} f_i(T)^{k_i}\right)\Lambda \subset \Lambda$$

relating to Selmer groups:

pontryagin dual

$$X_\infty^* \simeq \ker\left(H^1(G_{F_\infty,\Sigma}, \mathbb{Q}_p/\mathbb{Z}_p) \longrightarrow \bigoplus_{v\in\Sigma} H^1(I_v, \mathbb{Q}_p/\mathbb{Z}_p)\right)$$

Σ : primes lying over p

CONJECTURE 6.1.1 (The Iwasawa Main Conjecture). *Let p be an odd prime. Let χ be a nontrivial, even finite order p-adic character of $G_{\mathbb{Q}}$ of conductor not divisible by p^2, and let F be the the fixed field of the kernel of χ. For the cyclotomic $\mathbb{Z}_p$-extension F_∞ of F, we have*

$$\mathrm{char}_{\Lambda_\chi} X_\infty^{(\omega\chi^{-1})} = (f_\chi),$$

where $f_\chi \in \Lambda_\chi$ satisfies

$$f_\chi((1+p)^s - 1) = L_p(\chi, s)$$

for all $s \in \mathbb{Z}_p$.

Generalization

$\chi : Gal(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \simeq \mathbb{Z}_p^\times$, $\Delta = Gal(\mathbb{Q}(\mu_p)/\mathbb{Q}) \simeq \mu_{p-1}$, $\Gamma = Gal(\mathbb{Q}_\infty/\mathbb{Q})$, $\gamma \in \Gamma$ generator (topologically)

we consider the following character. $k \geq 0$, ζ is a p-th power root of unity

$$\phi_{k,\zeta} : \begin{array}{ccc} \Gamma & \longrightarrow & \mathbb{Z}_p[\zeta]^\times \\ \gamma & \longmapsto & \zeta \cdot \chi^k(\gamma) \end{array}$$

it extends to homo: $\Lambda \xrightarrow{\phi_{k,\zeta}} \mathbb{Z}_p[\zeta]$

we also consider $\Psi : G_\mathbb{Q} \longrightarrow \Lambda^\times$ be

$$G_\mathbb{Q} \to Gal(\mathbb{Q}_\infty/\mathbb{Q}) = \Gamma \hookrightarrow \Lambda^\times$$

define $\Psi_{k,\zeta} = \phi_{k,\zeta} \circ \Psi : G_\mathbb{Q} \longrightarrow \mathbb{Z}_p[\zeta]^\times$, then $\boxed{\Psi_{k,\zeta} = \psi_\zeta \cdot \omega^{-k} \cdot \chi^k}$

where $\omega : G_\mathbb{Q} \to \mu_{p-1} \subset \mathbb{Z}_p^\times$ is the Teichmüller char

$$\begin{tikzcd} G_\mathbb{Q} \arrow[r, "\psi_\zeta"] \arrow[d] & \mathbb{Z}_p[\zeta]^\times \\ \Gamma \arrow[ur] & \end{tikzcd} \qquad \gamma \mapsto \zeta$$

because $\Psi_{k,\zeta}(g) = \zeta^s \cdot \chi(\gamma)^{ks}$

$g|_{\mathbb{Q}(\mu_{p^\infty})} = \omega(g) \cdot \gamma^s$, $s \in 1 + p\mathbb{Z}_p$

$\Rightarrow g|_{\mathbb{Q}_\infty} = \gamma^s$

and $\psi_\zeta \cdot \omega^{-k} \cdot \chi^k(g) = \zeta^s \cdot \omega^{-k}(g) \cdot \omega^k(g) \cdot \chi(\gamma)^{ks} = \Psi_{k,\zeta}(g)$

ψ : primitive odd Dirichlet character, $\mathcal{O}_\psi = \mathbb{Z}_p[\psi]$, $F_\psi = \mathbb{Q}_p[\psi]$, $\Lambda_\psi = \Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_\psi$

$G_\mathbb{Q} \longrightarrow \mathbb{C}^\times$

$$\psi\Psi : G_\mathbb{Q} \longrightarrow \Lambda_\psi^\times$$

if Σ is a finite set of primes not including p, we define

$$Sel^\Sigma(\psi\chi^{-k}) = \ker\left(H^1(G_\mathbb{Q}, F_\psi/\mathcal{O}_\psi(\psi\chi^{-k})) \longrightarrow \bigoplus_{\ell \notin \Sigma} H^1(I_\ell, F_\psi/\mathcal{O}_\psi(\psi\chi^{-k})) \right)$$

and also

$$Sel_\infty^\Sigma(\psi) = \ker\left(H^1(G_\mathbb{Q}, \Lambda_\psi^*(\psi\Psi^{-1})) \longrightarrow \bigoplus_{\ell \notin \Sigma} H^1(I_\ell, \Lambda_\psi^*(\psi\Psi^{-1})) \right)$$

also

$$X^\Sigma(\psi\chi^{-k}) = Sel^\Sigma(\psi\chi^{-k})^*, \qquad X_\infty^\Sigma(\psi) = Sel_\infty(\psi)^*$$

the relation between these two groups are

<u>Prop</u>: Let $k \geq 0$, suppose $\forall\, \ell \neq p,\ \ell \notin \Sigma,\ p^2 \nmid$ order of $\psi|_{I_\ell}$, then

$$Sel^{\Sigma}(\psi\psi_{\zeta}^{-1}\omega^k\chi^{-k}) \xrightarrow{\sim} \left(Sel_{\infty}^{\Sigma}(\psi) \otimes_{\mathcal{O}_\psi} \mathcal{O}_\psi[\zeta]\right)[\gamma - \zeta\chi^k(\gamma)]$$

and

$$\left(X_{\infty}^{\Sigma}(\psi) \otimes_{\mathcal{O}_\psi} \mathcal{O}_\psi[\zeta]\right) \Big/ (\gamma - \zeta\chi^k(\gamma)) \xrightarrow{\sim} X^{\Sigma}(\psi\psi_{\Sigma}^{-1}\omega^k\chi^{-k})$$

Let $L = F_\psi(\mu_p)$, $L_\infty = F_\psi(\mu_{p^\infty})$, M_∞^Σ = maximal abelian pro-p extension unramified outside Σ

We have a natural action of $Gal(L/\mathbb{Q})$ on $Gal(M_\infty^\Sigma/L_\infty)$ given by the following:

$$Gal(L/\mathbb{Q}) \overset{\text{lifts}}{\rightsquigarrow} Gal(L_\infty/\mathbb{Q}_\infty), \text{ since } L_\infty = L\cdot\mathbb{Q}_\infty$$

$$\overset{\text{lifts}}{\rightsquigarrow} Aut(M_\infty^\Sigma/\mathbb{Q}_\infty)$$

$$\begin{tikzcd} & L_\infty \arrow[dl, no head] \arrow[dr, no head] & \\ L \arrow[dr, no head] & & \mathbb{Q}_\infty \arrow[dl, no head] \\ & \mathbb{Q} & \end{tikzcd}$$

$$Gal(L_\infty/\mathbb{Q}_\infty) \xrightarrow{\sim} Gal(L/\mathbb{Q}) \quad \left(F_\psi \cap \mathbb{Q}_\infty = \mathbb{Q}\right)$$

$\sigma \rightsquigarrow \tilde{\sigma} \in Aut(M_\infty^\Sigma/\mathbb{Q}_\infty)$, define $\sigma\cdot g = \tilde{\sigma}\cdot g\cdot\tilde{\sigma}^{-1} \in Gal(M_\infty^\Sigma/L_\infty)$ since g acts trivially on L_∞

$g \in Gal(M_\infty^\Sigma/L_\infty)$

independence of lifting $\tilde{\sigma}$: another lift $\tilde{\sigma}'$ will differ with $\tilde{\sigma}$ by $\tilde{\sigma}' = \tilde{\sigma}\cdot\tau$, $\tau \in Gal(M_\infty^\Sigma/L_\infty)$ then $\tau g\tau^{-1} = g$ since $Gal(M_\infty^\Sigma/L_\infty)$ is abelian, the

<u>Prop</u>: if ψ is odd & $F_\psi \cap \mathbb{Q}_\infty = \mathbb{Q}$, then

$$X_\infty^\Sigma(\psi) \simeq Gal(M_\infty^\Sigma/L_\infty)^{\psi^{-1}} \quad \text{as } \Lambda\text{-mod}$$

<u>Prop</u>: $X_\infty^\Sigma(\psi)$ is a finite torsion Λ_ψ-mod, $Ch_\infty^\Sigma(\psi)$ be its characteristic ideal

Review: p-adic L-functions

For any odd Dirichlet character ψ, we know

$$L(s,\psi) = \prod_{\ell} (1-\psi(\mathrm{Frob}_\ell)\ell^{-s})^{-1}$$ has nice analytic properties
well-defined except possibly pole at $s=1$
when ψ is trivial

$$L^{\{p\}}(s,\psi) = \prod_{\ell \neq p} (1-\psi(\mathrm{Frob}_\ell)\ell^{-s})^{-1} \rightsquigarrow$$ remove the p-part

we associate to $\psi \Psi_{k,\zeta}$ the following L-function

$$L(s, \psi\Psi_{k,\zeta}) = L(s-k, \underbrace{\psi\psi_\zeta}_{} \omega^{-k})$$

$\psi\Psi_{k,\zeta}$ " $\psi\psi_\zeta\omega^{-k}\chi^k$; Dirichlet char (finite order) ; $(\omega^{-1}\chi)^{n_\chi} \cdot \psi_\zeta$

$$L^{\{p\}}(s, \psi\Psi_{k,s}) = L^{\{p\}}(s-k, \psi\psi_\zeta\omega^{-k})$$

ψ finite $\downarrow$, Ψ $\downarrow$: $\chi_\Lambda = \chi \cdot \varepsilon_\Lambda$

$\exists\, g_\psi \in \Lambda_\psi$ such that

$$\mathcal{L}_\psi = \frac{g_\psi}{h_\psi} \quad \text{with} \quad h_\psi = \begin{cases} \zeta\chi(\gamma)\gamma - 1, & \text{if } \psi = \omega^{-1}\psi_\zeta^{-1} \\ 1, & \text{otherwise} \end{cases}$$

then

$$\phi_{k,\zeta}(\mathcal{L}_\psi) := \frac{\phi_{k,\zeta}(g_\psi)}{\phi_{k,\zeta}(h_\psi)} = L^{\{p\}}(0, \psi\Psi_{k,\zeta}) \quad \text{for } k>0$$

we also consider the following

Σ : finite set of primes doesn't contain p. modified L-function is

$$\mathcal{L}_\psi^\Sigma = \frac{g_\psi^\Sigma}{h_\psi}, \quad g_\psi^\Sigma = g_\psi \times \prod_{\ell\in\Sigma} (1-\ell^{-1}\psi^{-1}\Psi^{-1}(\mathrm{Frob}_\ell))$$

then $$\phi_{k,\zeta}(\mathcal{L}_\psi^\Sigma) = L^{\{p\}}(-k, \psi\psi_\zeta\omega^{-k}) \prod_{\ell\in\Sigma} (1-\psi^{-1}\psi_\zeta^{-1}\omega^k(\ell)\ell^{-1-k})$$

- **Iwasawa's Main Conjecture for Q:** *Let ψ be an odd primitive Dirichlet character. Then*

$$Ch_\infty(\psi^{-1}) = (g_\psi) \subseteq \Lambda_\psi \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

and even in Λ_ψ if p does not divide the order of ψ.

This can be modified to include a finite set Σ of primes different from p:

Iwasawa's Main Conjecture for Q (general form): *Let ψ be an odd primitive Dirichlet character. Then*

$$Ch_\infty^\Sigma(\psi^{-1}) = (g_\psi^\Sigma) subseteq \Lambda_\psi \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

and even in Λ_ψ if p does not divide the order of ψ.

<u>Prop.</u>: $X_\infty^\Sigma(\psi)$ is a f.g. Λ_ψ-module

pf. Let $\Sigma' = \Sigma \cup \{p\}$, $G_{\mathbb{Q},\Sigma'}$: Galois group $/\mathbb{Q}$ of maximal extension of $\mathbb{Q}$ unramified outside Σ'

$\lambda :=$ uniformizer of $\mathcal{O}_\psi$, $\mathfrak{m} = (\lambda, \gamma - 1) \subset \Lambda_\psi$ is the maximal ideal

Let $M = \Lambda_\psi^*(\psi \Psi^{-1})$, then we have the following exact sequences

$$0 \to M[\gamma-1] \to M \xrightarrow{\times(\gamma-1)} M \to 0$$

$$0 \to M[\mathfrak{m}] \to M[\gamma-1] \xrightarrow{\times\lambda} M[\gamma-1] \to 0 \quad \rightsquigarrow$$

$T \infty =$

$$\begin{tikzcd}
0 \arrow[r] & M[\lambda] \arrow[r] \arrow[d, "\gamma-1"] & M \arrow[r, "\lambda"] \arrow[d, "\gamma-1"] & M \arrow[r] \arrow[d, "\gamma-1"] & 0 \\
0 \arrow[r] & M[\lambda] \arrow[r] & M \arrow[r, "\lambda"] & M \arrow[r] & 0
\end{tikzcd}$$

$M[\lambda]/_{\gamma-1} = 0$

<u>Question</u>: Why is $M \xrightarrow{\gamma-1} M$ surjective?

follows by: $0 \to \Lambda_\psi(\psi^{-1}\Psi) \xrightarrow{\times(\gamma-1)} \Lambda_\psi(\psi^{-1}\Psi) \to \Lambda_\psi(\psi^{-1}\Psi)/(\gamma-1) \to 0$

taking pontryagin dual, we get exact sequence

$$0 \to M[\gamma-1] \to M \xrightarrow{\gamma-1} M \to 0$$

<u>Question</u>: Is M still f.g. Λ_ψ-mod?

consider $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p) = \varinjlim_n \mathrm{Hom}(\mathbb{Z}_p, p^{-n}\mathbb{Z}_p/\mathbb{Z}_p) \simeq \varinjlim_n p^{-n}\mathbb{Z}_p/\mathbb{Z}_p \simeq \mathbb{Q}_p/\mathbb{Z}_p$

is no longer a f.g. $\mathbb{Z}_p$-mod!

We get exact sequences by taking Galois cohomology

$$0 \to M^{G_{\mathbb{Q},\Sigma'}}/_{\gamma-1} \to H^1(G_{\mathbb{Q},\Sigma'}, M[\gamma-1]) \to H^1(G_{\mathbb{Q},\Sigma'}, M)[\gamma-1] \to 0$$

$$0 \to M[\gamma-1]^{G_{\mathbb{Q},\Sigma'}}/_\lambda \to H^1(G_{\mathbb{Q},\Sigma'}, M[\mathfrak{m}]) \to H^1(G_{\mathbb{Q},\Sigma'}, M[\gamma-1])[\lambda] \to 0$$

Note that

$$\begin{tikzcd}
0 \arrow[r] & M^{G_{\mathbb{Q},\Sigma'}}/_{\gamma-1}[\lambda] \arrow[r] \arrow[d] & H^1(G_{\mathbb{Q},\Sigma'}, M[\gamma-1])[\lambda] \arrow[r] \arrow[d] & H^1(G_{\mathbb{Q},\Sigma'}, M)[\mathfrak{m}] \arrow[d] & \\
0 \arrow[r] & M^{G_{\mathbb{Q},\Sigma'}}/_{\gamma-1} \arrow[r] \arrow[d, "\times\lambda"] & H^1(G_{\mathbb{Q},\Sigma'}, M[\gamma-1]) \arrow[r] \arrow[d, "\times\lambda"] & H^1(G_{\mathbb{Q},\Sigma'}, M)[\gamma-1] \arrow[r] \arrow[d, "\times\lambda"] & 0 \\
0 \arrow[r] & M^{G_{\mathbb{Q},\Sigma'}}/_{\gamma-1} \arrow[r] \arrow[d] & H^1(G_{\mathbb{Q},\Sigma'}, M[\gamma-1]) \arrow[r] & H^1(G_{\mathbb{Q},\Sigma'}, M)[\gamma-1] \arrow[r] & 0 \\
 & M^{G_{\mathbb{Q},\Sigma'}}/_{\mathfrak{m}} \arrow[r] & & &
\end{tikzcd}$$

we obtain morphism: $H^1(G_{\mathbb{Q},\Sigma'}, M[\mathfrak{m}]) \to H^1(G_{\mathbb{Q},\Sigma'}, M)[\mathfrak{m}]$

kernel of $H^1(G_{\mathbb{Q},\Sigma'}, M[m]) \to H^1(G_{\mathbb{Q},\Sigma'}, M)[m]$ is controlled by

$$\left.\begin{array}{l} M[\gamma-1]^{G_{\mathbb{Q},\Sigma'}}/\lambda \quad \& \quad M^{G_{\mathbb{Q},\Sigma'}}/_{\gamma-1}[\lambda] \\ \text{cokernel is controlled by } M^{G_{\mathbb{Q},\Sigma'}}/m \end{array}\right\} \Rightarrow \text{all are finite order } \Lambda_\psi\text{-modules}$$

why??

<u>Question</u>: $\mathrm{Hom}_{cont}(\mathbb{Z}_p[[T]], \mathbb{Q}_p/\mathbb{Z}_p)$?

$$\mathrm{Hom}_{cont}(\mathbb{Z}_p[[T]], \mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Hom}_{cont}\left(\prod \mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p\right)$$

$M[m]$ finite order $\Rightarrow H^1(G_{\mathbb{Q},\Sigma'}, M[m])$ has finite order

$\Rightarrow H^1(G_{\mathbb{Q},\Sigma'}, M)[m]$ has finite order

since $Sel_\infty^\Sigma(\psi)[m]$ is sub-module of $H^1(G_{\mathbb{Q},\Sigma'}, M)[m] \Rightarrow Sel_\infty^\Sigma(\psi)[m]$ has finite order

hence $X_\infty^\Sigma(\psi)/m X_\infty^\Sigma(\psi)$ is the dual of $Sel_\infty^\Sigma(\psi)[m] \Rightarrow X_\infty^\Sigma(\psi)/m X_\infty^\Sigma(\psi)$ has finite order

therefore by Nakayama, $X_\infty^\Sigma(\psi)$ is f.g. Λ_ψ-module

Prop: $X_\infty^\Sigma(\psi)$ is a torsion Λ_ψ-module

pf: 'Structure theorem' of f.g. Λ_ψ-mod gives the following map

$$X_\infty^\Sigma(\psi) \longrightarrow \prod_{i=1}^{n^\Sigma(\psi)} \Lambda_\psi / (f_{i,\psi}^\Sigma) \quad \text{with finite kernel \& cokernel}$$

Note: $f_{i,\psi}^\Sigma$ & $n^\Sigma(\psi)$ are not uniquely determined. $f_{i,\psi}$ may be 0, but

$$f_\psi^\Sigma = \prod_{i=1}^{n^\Sigma(\psi)} f_{i,\psi}^\Sigma \quad \text{is uniquely determined up to some units}$$

$$Ch_\infty^\Sigma(\psi) := (f_\psi^\Sigma) \subset \Lambda_\psi$$

$X_\infty^\Sigma(\psi)$ is torsion $\Longleftrightarrow f_\psi^\Sigma \neq 0$

Next we prove $f_\psi^\Sigma \neq 0$ by comparing it with another polynomial

· Let $K = \mathbb{Q}(\mu_N)$ containing F_ψ, or $\psi \in \widehat{Gal(K/\mathbb{Q})}$, $\mathcal{O}$ be an integer ring contained $\mathbb{Z}_p[\psi]$

Shapiro's lemma $\Rightarrow$

$$H_\Sigma^1(K, \Lambda_\mathcal{O}^*(\Psi^{-1})) \cong H_\Sigma^1(\mathbb{Q}, \mathrm{Hom}_\mathcal{O}(\mathcal{O}[Gal(K/\mathbb{Q})], \Lambda_\mathcal{O}^*(\Psi^{-1})))$$

$\wr$ ⟶ has finite-order ker & Coker annilated by a power of p; trivial if $p \nmid \varphi(N)$

$$\prod_{\psi \in \widehat{Gal(K/\mathbb{Q})}} H_\Sigma^1(\mathbb{Q}, \Lambda_\mathcal{O}^*(\psi^{-1}\Psi^{-1}))$$

⟵ $Gal(K/K^+)$-action

then

$$\prod_{\substack{\psi \in \widehat{Gal(K/\mathbb{Q})} \\ \psi \text{ odd}}} H_\Sigma^1(\mathbb{Q}, \Lambda_\mathcal{O}^*(\psi^{-1}\Psi^{-1})) \sim H_\Sigma^1(K, \Lambda_\mathcal{O}^*(\Psi^{-1}))^-$$

$$\| $$

$$\prod_{\substack{\psi \in \widehat{Gal(K/\mathbb{Q})} \\ \psi \text{ odd}}} Sel_\infty^\Sigma(\psi) \otimes_{\mathcal{O}_\psi} \mathcal{O}$$

taking Pontryagin dual, we get:

$$\underbrace{X_\infty^\Sigma(K)^-}_{\text{Pontryagin dual of the RHS}} \longrightarrow \prod_{\substack{\psi \in \widehat{Gal(K/\mathbb{Q})} \\ \psi \text{ odd}}} X_\infty^\Sigma(\psi) \otimes_{\mathcal{O}_\psi} \mathcal{O}$$

ker & coker finite annilated by a power of p then we get

$$\mathrm{Ch}_\infty^\Sigma(K)^- = \prod_{\substack{\psi \in \widehat{\mathrm{Gal}(K/\mathbb{Q})} \\ \psi \text{ odd}}} (f_\psi^\Sigma) \subset \Lambda_\mathcal{O} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

inclusion holds in $\Lambda_\mathcal{O}$ is $p \nmid \varphi(N)$ (because both sides up to p^m for some m)

we will show, $\mathrm{Ch}_\infty^\Sigma(K)^- \neq 0$

By definition

$$H^1_\Sigma(K, \Lambda_\mathcal{O}^*(\Psi^{-1})) = \ker\Big(H^1(G_K, \Lambda_\mathcal{O}^*(\Psi^{-1})) \to \bigoplus_{l \notin \Sigma} H^1(I_l, \Lambda_\mathcal{O}^*(\Psi^{-1})) \Big)$$

$$\simeq \mathrm{Hom}_\Gamma (X_\infty^\Sigma, \Lambda_\mathcal{O}^*(\Psi^{-1}))$$

Γ: $\mathrm{Gal}(K_\infty/K)$

X_∞^Σ: $\mathrm{Gal}(M_\infty^\Sigma/K_\infty)$, M_∞^Σ: maximal abelian pro-p extension of K_∞ unramified outside Σ

taking Pontiyagin dual, we get

$$X_\infty^{\Sigma,-,\iota} \otimes_{\mathbb{Z}_p[[\Gamma]]} \Lambda_\mathcal{O} \simeq X_\infty^\Sigma(K)^-$$

Iwasawa: let $n_0 > 0$ be such that $\gamma^{p^{n_0}} \in \Gamma_K$ & a $\mathbb{Z}_p[[\Gamma_K]]$-submodule $Y \subset X_\infty^{\Sigma,-,\iota}$ such that if $n \geq n_0$, then

$$X_\infty^{\Sigma,-,\iota} / \nu_n Y \xrightarrow{\sim} \mathrm{Gal}(M_n^\Sigma/K_n)^-, \qquad \nu_n = \frac{\gamma^{p^n}-1}{\gamma^{p^{n_0}}-1}$$

M_n^Σ: maximal abelian pro-p extension of K_n unramified outside Σ

$\Rightarrow$ Y has finite index in $X_\infty^{\Sigma,-}$, hence characteristic ideal of $Y \otimes_{\mathbb{Z}_p[[\Gamma]]} \Lambda_\mathcal{O}$ is $\mathrm{Ch}_\infty^\Sigma(K)^-$

then
$$\# \Lambda_\mathcal{O} / (\mathrm{Ch}_\infty^\Sigma(K)^-, \nu_n) = \# \, Y \otimes_{\mathbb{Z}_p[[\Gamma]]} \Lambda_\mathcal{O} / \nu_n (Y \otimes_{\mathbb{Z}_p[[\Gamma]]} \Lambda_\mathcal{O})$$
$$= \left(\# \, Y/\nu_n Y \right)^{[K':\mathbb{Q}][\mathcal{O}:\mathbb{Z}_p]}$$
$$= \# \left(X_\infty^{\Sigma,-}/\nu_n Y \right)^{[K':\mathbb{Q}][\mathcal{O}:\mathbb{Z}_p]} \Big/ \# \left(X_\infty^{\Sigma,-}/Y \right)^{[K':\mathbb{Q}][\mathcal{O}:\mathbb{Z}_p]}$$
$$= \left(h_n^\Sigma / h_{n_0}^\Sigma \right)^{[K':\mathbb{Q}][\mathcal{O}:\mathbb{Z}_p]}$$
is finite! $\Rightarrow$ $\mathrm{Ch}_\infty^\Sigma(K)^- \neq 0$

$K' = K_\infty \cap \mathbb{Q}$

$$[K':\mathbb{Q}] = \frac{[K:\mathbb{Q}]}{[K_\infty:\mathbb{Q}_\infty]}$$

$$\begin{tikzcd}
 & K_\infty \arrow[dl, dash] \arrow[dr, dash] & \\
K \arrow[r, dash] & K' \arrow[d, dash] \arrow[r, dash] & \mathbb{Q}_\infty \\
 & \mathbb{Q} \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

Explain .

- first equality

$$\# \Lambda_{\mathcal{O}}\big/\left(Ch^{\Sigma}_{\infty}(K)^-, \nu_n\right) \leq \# \, Y \otimes_{\mathbb{Z}_p[[\Gamma]]} \Lambda_{\mathcal{O}} \big/ \nu_n\left(Y \otimes_{\mathbb{Z}_p[[\Gamma]]} \Lambda_{\mathcal{O}}\right)$$

We have the following more general result

Lemma: Suppose M is a f.g. Λ-module, with $Ch_\infty(M) \subset \Lambda$, such that there is no non-zero finite order submodule of M

let $0 \neq r \in \Lambda$, then the follow holds if any side is finite

$$\# M/rM = \# \Lambda\big/(Ch_\infty(M), r)\Lambda \, ,$$ in this case, r & $Ch_\infty(M)$ are coprime to each other

pf: we have the following homo with finite ker & Coker

$$\Lambda/(r_1 t, r_2 t) \twoheadrightarrow \underset{\substack{\uparrow \\ \text{infinite}}}{\Lambda/(t)}$$

$$M \xrightarrow{\varphi} \prod_{i=1}^{n} \Lambda/f_i$$

then we get

$$0 \to M \to \prod_{i=1}^{n} \Lambda/f_i \to \mathrm{Coker} \to 0$$

$$\begin{array}{ccccccccc}
0 & \to & M & \to & \prod_{i=1}^{n} \Lambda/f_i & \to & \mathrm{Coker} & \to & 0 \\
 & & \downarrow \times r & & \downarrow \times r & & \downarrow \times r & & \\
0 & \to & M & \to & \prod_{i=1}^{n} \Lambda/f_i & \to & \mathrm{Coker} & \to & 0
\end{array}$$

$$\cdots \to \mathrm{Coker}[r] \to M/rM \longrightarrow \prod_{i=1}^{n} \Lambda/(r, f_i) \to \mathrm{Coker}/r\,\mathrm{Coker} \to 0$$

then since both $\mathrm{Coker}[r]$ & $\mathrm{Coker}/r \cdot \mathrm{Coker}$ are finite, then M/rM finite $\Leftrightarrow \prod_{i=1}^{n} \Lambda/(r, f_i)$ finite $\underbrace{\Leftrightarrow \Lambda/(r, Ch_\infty(M))}_{\text{is finite}}$

$$\# M/rM \leq \# \prod_{i=1}^{n} \Lambda/(r, f_i) = \# \Lambda/(r, Ch_\infty(M))$$

this is actually an equality since r & $Ch_\infty(M)$ are coprime to each other $\square$

Rmk: Iwasawa shows $X^{\Sigma,-}_\infty$ has no non-zero finite order submodules

$\Rightarrow X^{\Sigma}_\infty(K)^-$ has no non-zero finite order submodules

if $p \nmid \varphi(N) \Rightarrow X^{\Sigma}_\infty(\psi) \otimes_{\mathcal{O}_\psi} \mathcal{O}$ has no non-zero finite order submodules

· By $Ch_\infty^\Sigma(K)^- \neq 0$, we obtain the following

Theorem 4.9 (Iwasawa). *Suppose $K = \mathbf{Q}(\mu_N)$. Let $\mathcal{O}$ be the integer ring of any finite extension of $\mathbf{Q}_p$ containing $\mathbf{Z}_p[\psi]$ for all odd characters $\psi \in \widehat{\mathrm{Gal}(K/\mathbf{Q})}$. For any finite set Σ of primes different from p:*

$$\sum_{\substack{\psi \in \widehat{\mathrm{Gal}(K/\mathbf{Q})} \\ \psi \ odd}} \lambda(f_\psi^\Sigma) = \sum_{\substack{\psi \in \widehat{\mathrm{Gal}(K/\mathbf{Q})} \\ \psi \ odd}} \lambda(g_\psi^\Sigma),$$

and if $p \nmid [K : \mathbf{Q}]$:

$$\sum_{\substack{\psi \in \widehat{\mathrm{Gal}(K/\mathbf{Q})} \\ \psi \ odd}} \mu(f_\psi^\Sigma) = \sum_{\substack{\psi \in \widehat{\mathrm{Gal}(K/\mathbf{Q})} \\ \psi \ odd}} \mu(g_\psi^\Sigma)$$

pf: Multiplying the short exact sequence

$$0 \to \Lambda_{\mathcal{O}[\mu_{p^n}]}/(\nu_n) \to \prod_{\zeta^{p^n}=1,\ \zeta^{p^{n_0}} \neq 1} \Lambda_{\mathcal{O}[\mu_{p^n}]}/(\gamma - \zeta) \to K_n \to 0$$

by $g_K^\Sigma := \prod_{\substack{\psi \in \widehat{\mathrm{Gal}(K/\mathbf{Q})} \\ \psi \ \mathrm{odd}}} g_\psi^\Sigma$ and appealing to the snake lemma (noting that K_n has finite order) yields the first equality in the following chain:

$$\begin{aligned}
\#\Lambda_{\mathcal{O}[\mu_{p^n}]}/(g_K^\Sigma, \nu_n) &= \prod_{\zeta^{p^n}=1,\ \zeta^{p^{n_0}} \neq 1} \#\Lambda_{\mathcal{O}[\mu_{p^n}]}/(g_K^\Sigma, \gamma - \zeta) \\
&= \prod_{\zeta^{p^n}=1,\ \zeta^{p^{n_0}} \neq 1} \#\mathcal{O}[\mu_{p^n}]/(\phi_{0,\zeta}(g_K^\Sigma)) \\
&= \#\mathcal{O}[\mu_{p^n}]/(\prod_{\zeta^{p^n}=1,\ \zeta^{p^{n_0}} \neq 1} \phi_{0,\zeta}(g_K^\Sigma)) \\
&= \#\mathcal{O}[\mu_{p^n}]/((h_n^\Sigma/h_{n_0}^\Sigma)^{[K':\mathbf{Q}]}).
\end{aligned}$$

$$\begin{aligned}
\phi_{0,\zeta}(g_K^\Sigma) = \prod_{\substack{\psi \in \widehat{\mathrm{Gal}(K/\mathbb{Q})} \\ \psi \ odd}} \phi_{0,\zeta}(g_\psi^\Sigma) &= \prod_{\substack{\psi \in \widehat{\mathrm{Gal}(K/\mathbb{Q})} \\ \psi \ odd}} L^{\{p\}}(0, \psi \Psi_{0,\zeta}) \\
&= \prod_{\substack{\psi \in \widehat{\mathrm{Gal}(K/\mathbb{Q})} \\ \psi \ odd}} L^{\{p\}}(0, \psi \psi_\zeta)
\end{aligned}$$

then

Let f and e be the residue class degree and ramification degree of $\mathcal{O}$ over $\mathbf{Z}_p$. Let $f_n, g_n \in \mathbf{Z}$ be defined by $p^{f_n} = \#\Lambda_{\mathcal{O}}/(Ch_\infty^\Sigma(K)^-, \nu_n)$ and $p^{g_n} = \#\Lambda_{\mathcal{O}}/(\prod_{\substack{\psi\in\widehat{\mathrm{Gal}(K/\mathbf{Q})}\\ \psi \text{ odd}}} g_\psi^\Sigma, \nu_n)$. A calculation similar to those done to prove (4.6) and (4.7) shows that for $n \gg n_0$

$$f_n = \mu(Ch_\infty^\Sigma(K)^-)fp^{n-n_0} + \lambda(Ch_\infty^\Sigma(K)^-)ef(n-n_0) + e_n$$

and

$$g_n = \mu(\prod_{\substack{\psi\in\widehat{\mathrm{Gal}(K/\mathbf{Q})}\\ \psi \text{ odd}}} g_\psi^\Sigma)fp^{n-n_0} + \lambda(\prod_{\substack{\psi\in\widehat{\mathrm{Gal}(K/\mathbf{Q})}\\ \psi \text{ odd}}} g_\psi^\Sigma)ef(n-n_0) + e'_n,$$

with e_n and e'_n bounded independently of n. From this together with (4.6) and (4.7) (which imply that $f_n = g_n$ for $n \gg n_0$) it is easy to deduce that

$$\mu(Ch_\infty^\Sigma(K)^-) = \mu(\prod_{\substack{\psi\in\widehat{\mathrm{Gal}(K/\mathbf{Q})}\\ \psi \text{ odd}}} g_\psi^\Sigma) \ textand \lambda(Ch_\infty^\Sigma(K)^-) = \lambda(\prod_{\substack{\psi\in\widehat{\mathrm{Gal}(K/\mathbf{Q})}\\ \psi \text{ odd}}} g_\psi^\Sigma).$$

As $\lambda(Ch_\infty^\Sigma(K)^-) = \lambda(\prod_{\substack{\psi\in\widehat{\mathrm{Gal}(K/\mathbf{Q})}\\ \psi \text{ odd}}} f_\psi^\Sigma)$ by (4.4) and even $\mu(Ch_\infty^\Sigma(K)^-) = \mu(\prod_{\substack{\psi\in\widehat{\mathrm{Gal}(K/\mathbf{Q})}\\ \psi \text{ odd}}} f_\psi^\Sigma)$ if $p \nmid [K:\mathbf{Q}]$, this proves Theorem 4.9.

Statement of IMC

Iwasawa's Main Conjecture for Q: *Let ψ be an odd primitive Dirichlet character. Then*

$$Ch_\infty(\psi^{-1}) = (g_\psi) \subseteq \Lambda_\psi \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

and even in Λ_ψ if p does not divide the order of ψ.

This can be modified to include a finite set Σ of primes different from p:

Iwasawa's Main Conjecture for Q (general form): *Let ψ be an odd primitive Dirichlet character. Then*

$$Ch_\infty^\Sigma(\psi^{-1}) = (g_\psi^\Sigma) subseteq \Lambda_\psi \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

and even in Λ_ψ if p does not divide the order of ψ.

Sketch of the proof: By the previous results, we know that we only need to show

$$g_\psi^\Sigma \mid Ch_\infty^\Sigma(\psi^{-1})$$

(same λ & μ-invariants of sum of both sides)

we will introduce an intermediate ideal Eis_ψ, and then prove: $g_\psi^\Sigma \mid Eis_\psi$, $Eis_\psi \mid Ch_\infty^\Sigma(\psi^{-1})$

Λ-adic forms

$M_k(N, \psi, \mathcal{O}_\psi)$ = (q-expansion of) p-ordinary modular forms of wt k, level $\Gamma_1(N)$, nebentype $\psi \hookrightarrow Q_\psi[[q]]$

for any subring A of $\mathbb{C}_p/\mathbb{C}$ containing $\mathcal{O}_\psi$, define $M_k(N,\psi,A) = M_k(N,\psi,\mathcal{O}_\psi) \otimes_{\mathcal{O}_\psi} A$

recall $\psi\Psi_{k,\zeta} : G_{\mathbb{Q}} \longrightarrow \Gamma \longrightarrow \Lambda^\times \longrightarrow \mathbb{C}_p^\times$

$\gamma \longrightarrow [\gamma] \longmapsto \zeta \cdot \chi^k(\gamma)\cdot\psi(\gamma)$

for a character $\epsilon : Gal(\mathbb{Q}(\mu_{p^n})/\mathbb{Q}) \simeq Gal(\mathbb{Q}(\mu_p)/\mathbb{Q}) \times Gal(\mathbb{Q}_n/\mathbb{Q}) \longrightarrow \mu_{p^\infty}$ of the form $\Psi_{k,\zeta}$

$$\mathcal{M}_\Lambda(N,\psi) = \left\{ F \in \Lambda_\psi[[q]] \;\middle|\; \psi\Psi_{k,\zeta}(F) \in \bigoplus_{i=0}^{p-1} M_{k+1}^{ord}(Np^n, \psi\psi_\zeta\omega^{i-k}, \mathcal{O}_\psi[\zeta]) \text{ for all but finitely many } k \geq 2,\ \zeta \in \mu_{p^n} \right\}$$

ordinary Λ-adic form of level N, character ψ (odd primitive)

we can similarly define $S_k(N,\psi,A)$ & $S_\Lambda(N,\psi)$ of ordinary cusp (Λ-adic) forms

Example: Eisenstein family: $\mathcal{E}_\psi = g_\psi + h_\psi \sum_{n=1}^{\infty} \left(\sum_{\substack{d \mid N \\ (\ell, pN_\psi)=1}} \prod_{\ell^e \| d} \psi\Psi(Frob_\ell^e) \right) q^n$

$\Rightarrow \psi\Psi_{k,\zeta}(\mathcal{E}_\psi) = \phi_{k,\zeta}(h_\psi) \cdot E^{ord}_{k+1,\psi\psi_\zeta\omega^{-k}}$

· <u>Fundamental exact sequence</u>

$$0 \longrightarrow S_\Lambda(N,\psi) \longrightarrow \mathcal{M}_\Lambda(N,\psi) \xrightarrow{\Phi_\Lambda} C_\Lambda(N) \longrightarrow 0$$

$$C_\Lambda(N) = \bigoplus_{\substack{g \in \tilde{C}_N \\ \underset{\iota}{\cong}}} \Lambda_\psi[g_a]$$

$$C_N : \quad B(\mathbb{Q}) \backslash SL_2(\mathbb{A}_f) / \widehat{\Gamma_0(N)}$$

satisfying the following commutative diagram

$$\begin{tikzcd}
\mathcal{M}_\Lambda(N,\psi) \arrow[r, "\Phi_\Lambda"] \arrow[d, "\psi\bar{\Psi}_{k,\zeta}"] & C_\Lambda(N) \arrow[d, "\psi\bar{\Psi}_{k,\zeta}"] \\
\bigoplus_{i=0}^{p-1} M_{k+1}^{\mathrm{ord}}(Np^n, \psi\psi_\zeta \cdot \omega^{i-k}, \mathcal{O}_\psi[\zeta]) \arrow[r] & \bigoplus_{i=0}^{p-1} \bigoplus_{g_\alpha \in \tilde{C}_N} \mathcal{O}_\psi[\zeta][g_\alpha]_{Np^n}
\end{tikzcd}$$

usual map: the constant term at the cusp $[g_\alpha]_{Np^n}$

<u>Eisenstein congruence</u>

$S_\Lambda(N,\psi)$ & $\mathcal{M}_\Lambda(N,\psi)$ are equipped with actions of Hecke operators T_ℓ, $\ell \nmid Np$, U_ℓ for $\ell \mid Np$

$\mathfrak{h}_\Lambda(N,\psi)$:= Hecke algebra generated by T_ℓ ($\ell \nmid Np$) & U_ℓ $\subset \mathrm{End}_{\Lambda_\psi} S_\Lambda(N,\psi)$

I_ψ = the ideal generated by $T_\ell - (1 + \psi\bar{\Psi}(\mathrm{Frob}_\ell))$, $\ell \nmid Np$, $U_\ell - 1$, $\ell \mid Np$

$i : \Lambda_\psi \longrightarrow \mathfrak{h}_\Lambda(N,\psi)$ be the structure map, we define

$$\mathrm{Eis}_\psi := i^{-1}(I_\psi)$$

Proof of $g_\psi \mid \mathrm{Eis}_\psi$

Prop: $\exists$ a formal power series $\mathcal{E}_\psi \in \Lambda_\psi[[q]]$, s.t.

① $\mathcal{E}_\psi \in \mathcal{M}_\Lambda(N, \psi)$ & $T_\ell \mathcal{E}_\psi = (1 + \psi\Psi(\mathrm{Frob}_\ell))\mathcal{E}_\psi$

② $g_\psi \mid \Phi_\Lambda(\mathcal{E}_\psi)$ (constant term)

③ $a(1, \mathcal{E}_\psi) \in \Lambda_\psi^\times$

the fundamental exact sequence tells us, $\exists\, \mathcal{E}' \in \mathcal{M}_\Lambda(N, \psi)$, s.t.

$$F := \mathcal{E}_\psi - g_\psi \cdot \mathcal{E}' \in S_\Lambda(N, \psi)$$

$$T_\ell F = T_\ell \mathcal{E}_\psi - g_\psi \cdot T_\ell \mathcal{E}' \Rightarrow T_\ell F = (1+\psi\Psi(\mathrm{Frob}_\ell)) F \mod g_\psi$$

if $g_\psi \in \Lambda_\psi^\times$, nothing to be proved

so we assume $g_\psi \in \mathfrak{m}_{\Lambda_\psi}$, then $a(1, F) = a(1, \mathcal{E}_\psi) - g_\psi \cdot a(1, \mathcal{E}') \in \Lambda_\psi^\times$

then we get a Λ_ψ-alg homo

$$\lambda_F : h_\Lambda(N, \psi) \longrightarrow \Lambda/(g_\psi)$$

$$T_\ell \longmapsto \frac{a(1, T_\ell F)}{a(1, F)}$$

$$\ker(\lambda_F) \supset I_\psi \Rightarrow \begin{array}{c} h_\Lambda(N,\psi)/I_\psi \longrightarrow \Lambda/g_\psi \\ \uparrow \\ \Lambda/\mathrm{Eis}_\psi \end{array} \Rightarrow g_\psi \mid \mathrm{Eis}_\psi$$