The goals of this note are the following

- Give a quick summary of derived category / functors, explain & understand RHom, $\otimes^L$, to help me understand Künneth formula for varieties

- Explain derived category $D(S(X_{et}))$ and derived functors on it.

- Cartan Eilenberg resolution of a complex

- Proper base change / Smooth base change

Derived functors

· Triangulated Category

Triangulated Category is a replacement of Abelian category when we don't have a good Abelian category, it is
· An additive category $\mathcal{C}$
· With a transition functor $T: \mathcal{C} \to \mathcal{C}$
· Exact triangles replacement of exact sequence
this is simply a collection of objects & morphisms $(X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X))$ satisfying
TR1. $\forall\, X \xrightarrow{f} Y$ can be extended to an exact triangle

$$X \xrightarrow{f} Y \to Z \to T(X)$$

and $(X \xrightarrow{id_X} X \xrightarrow{0} 0 \xrightarrow{0} T(X))$ is always an exact triangle
and $\forall$ tuple isomorphic to an exact triangle is an exact triangle
TR2. (Rotation): $(X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X))$ is exact $\Leftrightarrow$ $(Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(Y))$ is exact
TR3. Given any two exact triangles $(X \to Y \to Z \to)$ & $(X' \to Y' \to Z' \to)$, and $f: X \to X'$, $g: Y \to Y'$, s.t.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r] & Y' \end{tikzcd}$$ commutes

then $\exists\, h: Z \to Z'$, s.t.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & T(X) \arrow[d, "T(f)"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X') \end{tikzcd}$$ commutes

TR4: complicated & not important for our purpose

There is also an concept of <u>Exact functor</u>: $F: C \to D$ is exact is it preserves exact triangles

Let's see one of the most important, and also the motivating example for triangulated category

· Homotopy Category

A is an additive category, $C(A)$ denote the category of complexes over A. then when A is abelian $\Leftrightarrow$ $C(A)$ abelian
But actually, $C(A)$ is not the most natural object in application. Instead, the following category is the most natural one!

$K(A)$: objects: complexes over A
morphisms: homotopy class of morphism of complexes

<u>Rmk</u>: when A is abelian, $K(A)$ is not Abelian. kernel, cokernel may not exists!

Next goal is to explain $K(\mathcal{A})$ is a triangulated category

- Transition functor: $T: K(\mathcal{A}) \to K(\mathcal{A})$
$$X \longmapsto T(X): T(X)^i = X^{i+1}$$, we simply denote $T(X) = X[1]$
- Exact triangle: For simplicity, we assum $\mathcal{A}$ is Abelian, then we consider the mapping cone of $f: X \to Y$,
$$C(f)^k := X^{k+1} \oplus Y^k, \quad d^k: C(f)^k := X^{k+1} \oplus Y^k \xrightarrow{\begin{pmatrix} -d_X^{k+1} & f^k \\ & d_Y^k \end{pmatrix}} C(f)^{k+1} := X^{k+2} \oplus Y^{k+1}$$
then exact triangles are those tuples isomorphic to $(X \xrightarrow{f} Y \to C(f) \to X[1])$

so we have from an exact triangle in $K(\mathcal{A})$ to a long exact sequence in Gps
$$\cdots \to H^i(X) \to H^i(Y) \to H^i(Z) \to H^{i+1}(X) \to \cdots$$

Claim: $K(\mathcal{A})$ is a triangulated category

But $K(\mathcal{A})$ is not good enough, one reason is that it contains too much "useless" morphisms. What the meaning of useless? Since we introduce complexes to compute cohomology, those complexes with the same cohomology are seen to be the "same". Let's introduce this concept

Def: A quasi-isomorphism $f: A \to B$ is a morphism of complexes, s.t. it induces isomorphisms.
$$f^k: H^k(A) \simeq H^k(B)$$

These "qis" are viewed as isomorphisms, to make the essense coming out, we use the technique of localization!
Our strategy is that: localize to make qis into isomorphism, therefore "should be isomorphic" is really "isomorphic"
For $\forall X, Y \in \mathrm{ob}\, K(\mathcal{A})$, we consider a "essential" morphism s.t
$$\varinjlim_{X' \overset{\sim}{\to} X} \mathrm{Hom}_{K(\mathcal{A})}(X', Y)$$
this morphism realize our hope: "quasi-isomorphism" becomes "isomorphism", but to guarantee the direct limit exists, we should check:
- f, g quasi-isomorphism $\Rightarrow fg$ quasi-isomorphism
- $\forall f: X' \xrightarrow{\sim} X,\ g: X'' \xrightarrow{\sim} X$, $\exists X'''$, and $X''' \to X'$, $X''' \to X''$ s.t. the following diagram commutes
$$\begin{array}{ccc} X''' & \xrightarrow{\sim} & X' \\ \downarrow{\scriptstyle s} & & {\scriptstyle f}\downarrow{\scriptstyle s} \\ X'' & \xrightarrow[g]{\sim} & X \end{array}$$

this two properties can be proved in a much general setting, and these weaker results can tell us more:
- $\forall$ diagram $\begin{array}{ccc} X & \to & Y \\ {\scriptstyle s}\downarrow{\scriptstyle s} & & \\ Z & & \end{array}$ $\left(\text{resp.} \begin{array}{ccc} & & Y \\ & & {\scriptstyle t}\downarrow{\scriptstyle s} \\ Z & \longrightarrow & W \end{array}\right)$ can be completed to a commutative diagram
$$\begin{array}{ccc} X & \to & Y \\ {\scriptstyle s}\downarrow{\scriptstyle s} & & {\scriptstyle t}\downarrow{\scriptstyle s} \\ Z & \to & W \end{array}$$
- $f, g: X \to Y$, then $\exists X' \xrightarrow[\sim]{s} X$, s.t. $fs = gs \iff \exists Y \xrightarrow{t} Y'$, s.t. $tf = tg$

With these results, we can even get better result $\varinjlim_{X' \overset{\sim}{\to} X} \mathrm{Hom}_{K(\mathcal{A})}(X', Y)$ exists and $\simeq \varinjlim_{Y \overset{\sim}{\to} Y'} \mathrm{Hom}_{K(\mathcal{A})}(X, Y')$

Now we come to the concept of derived category

Def: The derived category of an Abelian category $\mathcal{A}$ is $D(\mathcal{A})$

$D(\mathcal{A})$: objects : same as $K(\mathcal{A})$, same as $Com(\mathcal{A})$

morphisms: $\varinjlim_{X' \xrightarrow{\sim} X} Hom_{K(\mathcal{A})}(X', Y) \simeq \varinjlim_{Y \xrightarrow{\sim} Y'} Hom_{K(\mathcal{A})}(X, Y')$

Claim: $D(\mathcal{A})$ is a triangulated category

This category realizes our dreams: qis become isomorphism! morphism sets are much more essential!.
We also list the universal property of $D(\mathcal{A})$, recall we have natural $K(\mathcal{A}) \to D(\mathcal{A})$

- Universal property: For $\forall$ functor $K(\mathcal{A})$ (even $Com(\mathcal{A})$) $\xrightarrow{F} C$, which transforms qis to isomorphism, then F factors through $D(\mathcal{A})$, i.e.

$$K(\mathcal{A}) \; (Com(\mathcal{A})) \longrightarrow D(\mathcal{A}) \longrightarrow C$$

$D(\mathcal{A})$ is also better than $K(\mathcal{A})$ in that:

- Exactness property: For an exact sequence in $Com(\mathcal{A})$, i.e. $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ is exact, then $\exists\, C \to A[1]$ in $D(\mathcal{A})$, s.t.

may not exist in $K(A)$

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{+1} A[1]$$

is an exact triangle

· Our next goal is to define the all-important derived functors

General setting: $\mathcal{A}, \mathcal{B}$ are abelian categories, we consider a covariant functor $F: K(\mathcal{A}) \to K(\mathcal{B})$, then consider

$$\forall X \in \mathrm{ob}\, K(\mathcal{A}), \quad F'(X)(Y) := \varinjlim_{X \overset{\sim}{\to} X'} \mathrm{Hom}_{D(B)}(Y, F(X'))$$

Claim: $F'(X) \in \mathrm{Hom}(D(B)^{\circ}, \text{Sets})$

pf: For $\forall X_1 \to X_2 \in \mathrm{ob}\, K(A)$, and $\forall X_1 \overset{\sim}{\to} X_1'$, we can complete to get

$$\begin{array}{ccc} X_1 & \to & X_2 \\ \downarrow s & & \downarrow s \\ X_1' & \to & X_2' \end{array}$$

hence we get $\mathrm{Hom}_{D(B)}(Y, F(X_1')) \to \mathrm{Hom}_{D(B)}(Y, F(X_2'))$

taking limit, we can get $F'(X_1)(Y) \to F'(X_2)(Y)$

Now we obtain $K(\mathcal{A}) \xrightarrow{r} \mathrm{Hom}(D(B)^{\circ}, \text{Sets})$.

Claim: r factors through $D(A)$, hence we get $D(A) \xrightarrow{r_F} \mathrm{Hom}(D(B)^{\circ}, \text{Sets})$

pf: $\forall X_1 \overset{\sim}{\to} X_2$ qis, we know X_2^{+} naturally maps to X_1^{+}, and actually it is cofinal

Therefore our functor is $D(A) \longrightarrow \mathrm{Hom}(D(B)^{\circ}, \text{Sets})$, we denote it by RF

Remember Yoneda's lemma: $D(B) \hookrightarrow \mathrm{Hom}(D(B)^{\circ}, \text{Sets})$, we especially interested in those $X \in D(A)$, s.t. $RF(X) \in D(B)$, in this case, we say RF is defined at X, i.e.

$$\mathrm{Hom}_{D(B)}(Y, RF(X)) = \varinjlim_{X \to X'} \mathrm{Hom}_{D(B)}(Y, F(X'))$$

Rmk: Almost all books require $F: K(A) \to K(B)$ to be exact, I don't quite understand why. Maybe the following thm of Deligne tells the reason

Thm. Suppose F is exact, and $A \to B \to C \xrightarrow{+1}$ is an exact triangle, then

- If $RF(A)$, $RF(B)$ are defined, then $RF(C)$ is defined
- When they are all defined, then $RF(A) \to RF(B) \to RF(C) \xrightarrow{+1}$ is an exact triangle

It's not a difficult task to get an exact functor, actually we will almost always works with them:

- If $F: \mathcal{A} \to \mathcal{B}$ is additive, then $F: K(\mathcal{A}) \to K(\mathcal{B})$ is exact

Rmk: What's the idea behind derived functor RF?

Once we have a functor $F: K(A) \to K(B)$, we want to descend it to $D(A) \to D(B)$

But obviously it's not always possible, since most of the interesting F won't sends qis to qis

- Computation of RF

In this section, we explanian some situations where RF is defined, and see its connection with those cohomology functors we are familiar with. One of the most important objects in this section is injective object!

The most important property of injective (each component inj) complexes is the following cofinal lemma

<u>Lemma</u>: Suppose we have a qis $I^\cdot \overset{s}{\simeq} X^\cdot$, $I^\cdot$ is bounded below injective complex. then $\exists\, t: X^\cdot \to I^\cdot$, s.t. $ts = id_I$. in $K(A)$

pf: $I^\cdot \xrightarrow{s} X^\cdot$ is qis tells us that $\exists$ $C^\cdot \to I^\cdot \to X^\cdot \xrightarrow{+1}$, s.t. $C^\cdot$ is acyclic

then since $C^\cdot$ acyclic, $C^\cdot \to I^\cdot$ must be homotopic to 0, i.e. $C^\cdot \to 0 \to I^\cdot$, hence $\exists\, X^\cdot \to I^\cdot$, s.t.

$$\begin{tikzcd} C^\cdot \arrow[r] \arrow[d] & I^\cdot \arrow[r, "s"] \arrow[d] & X^\cdot \arrow[r, "+1"] \arrow[d, "t"] & {} \\ 0 \arrow[r] & I^\cdot \arrow[r] & I^\cdot \arrow[r, "+1"] & {} \end{tikzcd}$$

i.e. $ts = id_I$

especially, when $I^\cdot$, $J^\cdot$ are qis inj complexes, then $\exists$ inverse map to each other!

although qis is "isomorphism" but we not necessarily have an inverse! this lemma tells us it's possible for inj to construct a true inverse!

<u>Cor</u>: RF is defined at injective complexes, and $RF(I^\cdot) = F(I^\cdot)$

pf: For $\forall$ inj complex $I^\cdot$, $\varinjlim_{I^\cdot \simeq X^\cdot} Hom_{D(B)}(-, F(X^\cdot)) = Hom_{D(B)}(-, F(I^\cdot))$

This lemma & Cor is illuminating, now we consider a general $X^\cdot \in D(A)$, suppose we have a qis $X^\cdot \simeq I^\cdot$ then for $\forall\, X^\cdot \overset{\sim}{\to} X'^\cdot$, we can complete it to get a commutative diagram

$$\begin{tikzcd} X^\cdot \arrow[r, "\sim"] \arrow[d, "s"] & I^\cdot \arrow[d] \\ X'^\cdot \arrow[r] & Y^\cdot \end{tikzcd}$$

then by the lemma, $\exists\, Y^\cdot \to I^\cdot$, s.t. $I^\cdot \to Y^\cdot \to I^\cdot$ is homotopic to id_I. hence $I^\cdot$ is also a final object in X^+!, therefore

$$\varinjlim_{I^\cdot \simeq X^\cdot} Hom_{D(B)}(-, F(X^\cdot)) = Hom_{D(B)}(-, F(I^\cdot)), \quad \text{i.e.} \quad RF(X^\cdot) = F(I^\cdot)$$

Especially, we get for $\forall\, X \in A$, $R^iF(X) = H^i(RF(X[0])) \rightsquigarrow$ agree with our previous definition

<u>Thm</u>: If A has enough injectives, then

- For $\forall\, X^\cdot \in D^+(A)$, i.e. bounded below, $\exists$ qis $X^\cdot \to I^\cdot$ to an inj complex
- RF is defined everywhere in $D^+(A)$, and
$$RF(X^\cdot) = F(I^\cdot)$$
- When RF has finite cohomological dimension, i.e. for $\exists\, n>0$, s.t. $\forall\, X \in A$, $R^nF(X) = 0$, then RF is defined every in $D(A)$

Rmk: The proof of the last statement needs some work, we ponstpone it to th next section

Why inj resolutions are so good? We list the most important conditions. Let denote $\mathcal{L}$ by the subcategory of inj complexes

1. Every complex X admits a quasi-isomorphism to some complex in $\mathcal{L}$. i.e. $X \xrightarrow{\sim} I$, $I \in \mathcal{L}$

2. For quasi-iso between complexes in $\mathcal{L}$, i.e. $I_1 \xrightarrow[\sim]{qis} I_2$, $F(I_1) \to F(I_2)$ is also a qis!

With these two conditions, we see that the limit $\varinjlim_{X \xrightarrow{\sim} X'} \mathrm{Hom}_{D(\mathcal{B})}(Y, F(X'))$ is finally constant and $= \mathrm{Hom}_{D(\mathcal{B})}(Y, F(I))$

Sometimes we also have the last condition, which guarantees RF is exact!

3. For $\forall X_1 \to X_2$ with $X_1, X_2 \in \mathcal{L}$, there exists a distinguished triangle $X_1 \to X_2 \to X_3 \xrightarrow{+1}$, s.t. $X_3 \in \mathcal{L}$

With condition 3, RF is exact, because for $\forall X_1 \to X_2 \to X_3 \xrightarrow{+1}$, $\exists I_1, I_2, I_3 \in \mathcal{L}$, s.t.

$$\begin{array}{ccccccc} X_1 & \to & X_2 & \to & X_3 & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ I_1 & \to & I_2 & \to & I_3 & \xrightarrow{+1} & \end{array}$$

is an isomorphism between exact triangles

then $RF(X_1) \to RF(X_2) \to RF(X_3)$ is $F(I_1) \to F(I_2) \to F(I_3) \xrightarrow{+1}$ is exact since F is

· Two spectral sequence

The goal of this section is to prove the last part of the Thm on the previous page, and prove the following theorem which we have seen in de Rham theory. We will use the tool of Cartan-Eilenberg resolution

<u>Thm</u>: Suppose $F: \mathcal{A} \to \mathcal{B}$ is left exact functor, and $\mathcal{A}$ containes enough injectives. Then for $\forall K \in \mathrm{ob}\, D^+(\mathcal{A})$, i.e. a bounded below complex, we have two biregular spectral sequences

$$E_2^{pq} = R^pF(H^q(K^\cdot)) \Rightarrow R^{p+q}F(K^\cdot)$$

$$E_1^{pq} = R^qF(K^p) \Rightarrow R^{p+q}F(K^\cdot)$$

Suppose Furthermore that RF has finite cohomological dimension. Then the same result holds for $\forall K \in \mathrm{ob}\, D(\mathcal{A})$

<u>Blackbox</u>. Cartan-Eilenberg resolution

For a $\overset{\text{bounded below}}{\vee}$complex $X^\cdot$, we find an injective resolution $I^{p,\cdot}$ for each X^p, and chose $I^{p,\cdot} \to I^{p+1,\cdot}$ carefully to make sure $I^{\cdot,\cdot}$ becomes a double complex over $X^\cdot$, then the simple complex associated to $I^{\cdot,\cdot}$, denote by $I^\cdot$, is an injection resolution of $X^\cdot$. but this one is not good enough. because we don't have control over it's kernel, boundary and cohomology. so when we do composition functors, the situation is bad!

Claim: For any complex $X^\cdot$, $\exists$ double complex $I^{\cdot,\cdot}$ over $X^\cdot$, s.t.

- $I^{p,\cdot}$ is an injective resolution of X^p
- $Z^p(I^{\cdot,q}) \subset I^{p,q}$, $B^{p+1}(I^{\cdot,q}) \subset I^{p+1,q}$, $H^p(I^{\cdot,q}) = Z^p(I^{\cdot,q})/B^p(I^{\cdot,q})$ is injection resolutions of $Z^p(X^\cdot) \subset X^p$, $B^{p+1}(X^\cdot) \subset X^{p+1}$, $H^p(X^\cdot) = Z^p(X^\cdot)/B^p(X^\cdot)$ respectively

construction: Construct injective resolutions for $B^p(X^\cdot)$ & $H^p(X^\cdot)$ first, then direct sum to get inj res of $Z^p(X^\cdot)$ then by the same procedure to get inj for $X^\cdot$

pf of two spectral sequences: consider C-E resolution for $K^\cdot$, say $I^{\cdot,\cdot}$ Apply F, we get $F(I^{\cdot,\cdot})$, we denote this by E_0

$${}_{\uparrow}E_1^{p,q} = R^qF(K^p). \qquad {}_{\to}E_1^{p,q} = H^p(F(I^{\cdot,\cdot})), \text{ what is } {}_{\to}E_2^{p,q}\,?$$

Here comes <u>key ingredient</u> for C-E resolution, we know

$$0 \to Z^{p,\cdot} \to I^{p,\cdot} \to B^{p+1,\cdot} \to 0, \quad 0 \to B^{p,\cdot} \to Z^{p,\cdot} \to H^{p,\cdot} \to 0$$

since they are all injective objects, these exact sequence splits, hence

$$0 \to \underset{\substack{\| \\ Z^p(F(I^{\cdot,\cdot}))}}{FZ^{p,\cdot}} \to FI^{p,\cdot} \to FB^{p+1,\cdot} \to 0, \quad 0 \to FB^{p,\cdot} \to FZ^{p,\cdot} \to FH^{p,\cdot} \to 0$$

hence <u>$FZ^{p,\cdot} = Z^p(F(I^{\cdot,\cdot}))$, $FB^{p+1,\cdot} = B^{p+1}(F(I^{\cdot,\cdot}))$, $FH^{p,\cdot} = H^p(F(I^{\cdot,\cdot}))$</u>

hence ${}_{\to}E_2^{p,q} = H^q(FH^{p,\cdot}) = R^qF(H^p(K^\cdot))$

· Extend RF to $D(\mathcal{A})$ when RF has finite cohomological dimension

In this section, we give another way to compute derived functor: by F-acyclic resolution. We will show that by constructing acyclic resolution, we can extend RF to the whole $D(\mathcal{A})$

<u>Key point</u>: When RF has finite cohomological dimension, every complex admits F-acyclic resolution

· Left derived functor & Contravariance

Before we define the RF for a covariant exact functor $F: K(\mathcal{A}) \to K(\mathcal{B})$

Natural question 1: How to deal with contravariant functor?

Obviously, there are two ways:

· Follow the same definition as before, we define

$$F'(X)(Y) = \varinjlim_{X \xrightarrow{\sim} X'} \mathrm{Hom}_{D(\mathcal{B})}(Y, F(X'))$$

this time $F'(X) \in \mathrm{Hom}(D(\mathcal{B}), \mathrm{Sets})$, not $\mathrm{Hom}(D(\mathcal{B})^\circ, \mathrm{Sets})$!

· take the opposite category $K(\mathcal{B})^\circ = K(\mathcal{B}^\circ)$

Natural question 2: Why not we define (For covariant exact functor F)

$$F'(X)(Y) = \varinjlim_{X' \xrightarrow{\sim} X} \mathrm{Hom}_{D(\mathcal{B})}(F(X'), Y)$$

then also easy to check: $F'(X) \in \mathrm{Hom}(D(\mathcal{B}), \mathrm{Sets})$

Now for $\forall X_1 \to X_2$, $\forall X_2' \xrightarrow{\sim} X_2$, $\exists X_1' \to X_1$, s.t.

$$\begin{array}{ccc} X_1' & \to & X_2' \\ \downarrow & & \downarrow \\ X_1 & \to & X_2 \end{array}$$ commutes

we get:

$$\begin{array}{ccc} \varinjlim_{X_2' \to X_2} \mathrm{Hom}_{D(\mathcal{B})}(F(X_2'), Y) & \to & \varinjlim_{X_1' \to X_1} \mathrm{Hom}_{D(\mathcal{B})}(F(X_1'), Y) \\ \| & & \| \\ F'(X_2)(Y) & \to & F'(X_1)(Y) \end{array}$$

i.e. we get a contravariant functor $l: K(\mathcal{A}) \to \mathrm{Hom}(D(\mathcal{B}), \mathrm{Sets})$

easy to verify that l factors through $D(\mathcal{A})$, hence we get

$$LF: D(\mathcal{A}) \longrightarrow \mathrm{Hom}(D(\mathcal{B}), \mathrm{Sets}) \quad \text{contravariant!}$$

obviously, $D(\mathcal{B})^\circ \hookrightarrow \mathrm{Hom}(D(\mathcal{B}), \mathrm{Sets})$ by Yoneda lemma,

we say LF is defined at X, if $LF(X) \in D(\mathcal{B})^\circ$, if it is defined everywhere, we get

$$LF: D(\mathcal{A}) \longrightarrow D(\mathcal{B}) \quad \text{covariant!}$$

The dual version of "good" objects for left derived functors are:

1. Every complex X admits a quasi-isomorphism from some complex in $\mathcal{L}$. i.e. $I \xrightarrow{\sim} X$, $I \in \mathcal{L}$
2. For quasi-iso between complexes in $\mathcal{L}$, i.e. $I_1 \overset{qis}{\sim} I_2$, $F(I_1) \to F(I_2)$ is also a qis!

With these two conditions, we see that the limit $\varinjlim_{X' \xrightarrow{\sim} X} \mathrm{Hom}_{D(\mathcal{B})}(F(X), Y)$ is finally constant and $= \mathrm{Hom}_{D(\mathcal{B})}(F(I), Y)$

Sometimes we also have the last condition, which guarantees RF is exact!

3. For $\forall X_1 \to X_2$ with $X_1, X_2 \in \mathcal{L}$, there exists a distinguished triangle $X_1 \to X_2 \to X_3 \xrightarrow{+1}$, s.t. $X_3 \in \mathcal{L}$

$RHom$ & $-\otimes^L-$

We introduce and explain two naturally-arised functor $RHom$ & $-\otimes^L-$, and state the important projection formula, Künneth formula.

- $RHom$

For two complexes $X^\cdot$ & $Y^\cdot$, we want to introduce another Hom-complex, the point is: the grading on a complex is not canonical! Therefore to talk about the Hom-functor between them, we define the following:

$$Hom^\cdot(X^\cdot, Y^\cdot), \text{ the } n\text{-th component is } \prod_i Hom(X^i, Y^{n+i})$$

the differential operator $d_n: Hom^n(X^\cdot, Y^\cdot) \to Hom^{n+1}(X^\cdot, Y^\cdot)$ is

$$f \in \prod_i Hom(X^i, Y^{n+i}) \mapsto d_n f, \quad (d_n f)_i = d_Y^{i+n} \circ f_i - (-1)^n f_{i+1} \circ d_X^i$$

Then $\ker d^n = Hom_{Com}(X^\cdot, Y^\cdot[n])$, $\operatorname{im} d^{n-1}$: homotopic to 0-map $\Rightarrow H^n(Hom^\cdot(X^\cdot, Y^\cdot[n])) = Hom_{K(A)}(X^\cdot, Y^\cdot[n])$

Now we obtain a functor $Hom^\cdot: \quad Com(\mathcal{A}) \times Com(\mathcal{A}) \longrightarrow Com(Ab.gps)$

<u>Claim</u>: The $Hom^\cdot$-functor descent to $K(\mathcal{A}) \times K(\mathcal{A}) \to K(Ab.gps)$, and for fixed $X^\cdot$ or $Y^\cdot$

$\left.\begin{matrix} Hom^\cdot(X^\cdot, -) \\ Hom^\cdot(-, Y^\cdot) \end{matrix}\right. : K(A) \to K(Ab.gps)$ transforms exact triangles to exact triangles, i.e. exact

pf: For fixed $X^\cdot$, if $f: Y_1^\cdot \to Y_2^\cdot$ is homotopic to 0, then obviously $Hom^\cdot(f)$ homotopic to 0, hence the first claim

Now the claim that it is exact basically follows from Hom is additive

Hence we obtained $Hom^\cdot: K(\mathcal{A}) \times K(\mathcal{A}) \longrightarrow K(Ab.gps)$. especially, if we fix a $X^\cdot \in K(A)$, then we obtain an exact functor $Hom^\cdot(X^\cdot, -): K(\mathcal{A}) \to K(Ab.gps)$

Assume from now that $\mathcal{A}$ contains enough injectives, then we could define a derived functor

$$RHom: \ D^+(\mathcal{A}) \longrightarrow D(Ab.gps) \quad , \quad Ext^i := R^i Hom$$

the computation is, $\forall\, Y^\cdot$, take inj reso $Y^\cdot \xrightarrow[qis]{\sim} I^\cdot$, then $RHom(Y^\cdot) = Hom^\cdot(X^\cdot, I^\cdot)$

Hence we get: $RHom: K(\mathcal{A}) \times D^+(\mathcal{A}) \to D(Ab.gps)$, now we prove it can factor through $K(\mathcal{A})$

<u>Claim</u>: $RHom$ can factor as: $D(\mathcal{A}) \times D^+(\mathcal{A}) \longrightarrow D(Ab.gps)$

pf: We should show qis becomes iso, hence pick $X_1^\cdot \xrightarrow{\sim} X_2^\cdot$ is qis, then $X_1^\cdot \to X_2^\cdot \to C \xrightarrow{+1}$ exact. and C is acyclic

now $RHom(X_1, Y) = Hom^\cdot(X_1, I^\cdot)$, $RHom(X_2, Y) = Hom^\cdot(X_2, I^\cdot)$, then

$$Hom^\cdot(C, I^\cdot) \to Hom^\cdot(X_2, I^\cdot) \to Hom(X_1, I^\cdot) \xrightarrow{+1} \text{ is exact}$$

since C is acyclic, $I^\cdot$ inj, $\Rightarrow \forall f: C \to I^\cdot[n]$ is homotopic to $0 \Rightarrow Hom^\cdot(X_2, I^\cdot) \xrightarrow{\sim} Hom^\cdot(X_1^\cdot, I^\cdot)$

Therefore we obtain a functor $RHom: D(\mathcal{A}) \times D^+(\mathcal{A}) \longrightarrow D(Ab.gps)$

Rmk: Can we start from the first variable ?

Try: fix $Y^\bullet \in K(A)$, then $\mathrm{Hom}^\bullet : K(A) \to K(\text{Ab.gps})$ is contravariant, we consider

$$X \to \mathrm{Hom}^\bullet(X^\bullet, Y^\bullet)$$

$$F'(Y^\bullet)(T^\bullet) = \varinjlim_{Y'^\bullet \to Y^\bullet} \mathrm{Hom}_{D(Ab)}(T^\bullet, F(Y'^\bullet))$$

- $- \otimes_A^L -$

Now suppose we work in the category $\mathcal{S} = \mathcal{S}(X_G, A)$, where X_G is a G-topological space, A is a ring, $\mathcal{S}(X, A)$ is the category of sheaves of A-mods on X w.r.t. G

For $\forall$ Complexes $X^\cdot, Y^\cdot$, consider the following complex

$$X^\cdot \otimes_A Y^\cdot : \text{ n-th component is } \bigoplus_{i \in \mathbb{Z}} X^i \otimes_A Y^{n-i}$$

the differential operator $d : (X^\cdot \otimes_A Y^\cdot)^n \to (X^\cdot \otimes_A Y^\cdot)^{n+1}$ is

$$d_{X^i \otimes Y^{n-i}} = d_X^i \otimes 1_Y^{n-i} + (-1)^i 1_{X^i} \otimes d_Y^{n-i}$$

<u>Claim</u>: $- \otimes_A -$ defines a functor $K(X, A) \times K(X, A) \to K(X, A)$, which is exact in each variable

technical issue by the <u>Technical lemma 2</u>

Keep consistent with classical theory, fix $X^\cdot \in K^-(X, A)$, consider

$$X^\cdot \otimes_A - : K(X, A)^- \longrightarrow K(X, A)^-$$

We want the left derived functor of this, but before we do that, we need some technical lemmas

<u>Technical lemma 1</u>: For scheme X with étale topology, consider $\mathcal{L}$: bounded above complex with each component flat

- Every complex $X^\cdot \in K(X, A)^-$ admits a qis $F^\cdot \to X^\cdot$ with $F^\cdot \in \mathcal{L}$
- For $F_1^\cdot \xrightarrow{\simeq} F_2^\cdot$ qis, $X^\cdot \otimes_A F_1^\cdot \xrightarrow{\simeq} X^\cdot \otimes_A F_2^\cdot$ is a qis
- For $\forall$ map $F_1^\cdot \to F_2^\cdot$, $\exists$ exact triangle $F_1^\cdot \to F_2^\cdot \to F_3^\cdot \xrightarrow{+1}$ is exact (吗·写一下吧这是)

Therefore we get left derived functor

$$L_{\mathrm{II}}(X^\cdot \otimes_A -) : D(X, A)^- \to D(X, A)^-$$

Hence we get: $L_{\mathrm{II}}(- \otimes_A -) : K(X, A)^- \times D(X, A)^- \longrightarrow D(X, A)^-$, next lemma says that it descends to $D(X, A)^- \times D(X, A)^-$!

<u>Technical lemma 2</u>: Same notation as before, $X^\cdot, Y^\cdot \in K(X, A)$ (no bounded condition for now!). Suppose

- All components of $Y^\cdot$ are flat sheaves
- either $X^\cdot$ or $Y^\cdot$ is acyclic
- either $X^\cdot$ and $Y^\cdot$ are bounded below, or $Y^\cdot$ is bounded

Then $X^\cdot \otimes_A Y^\cdot$ is acyclic

<u>Rmk</u>: <u>Technical lemma 1</u> is also a consequence of <u>Technical lemma 2</u>, since

$F_1^\cdot \xrightarrow{\simeq} F_2^\cdot$ qis, then $F_1^\cdot \to F_2^\cdot \to F_3^\cdot \to$ exact with $F_3^\cdot \in \mathcal{L}$, and acyclic, then $X^\cdot \otimes_A F_3^\cdot$ is acyclic, hence $X^\cdot \otimes_A F_1^\cdot \xrightarrow{\simeq} X^\cdot \otimes_A F_2^\cdot$

<u>Technical lemma 2</u> implies descent to $D(X, A)^- \times D(X, A)^- \to D(X, A)^-$

take $X_1^\cdot \xrightarrow{\simeq} X_2^\cdot$, and $F^\cdot \to Y^\cdot$ qis, then $X_1^\cdot \otimes_A^L Y^\cdot = X_1^\cdot \otimes_A F^\cdot$, $X_2^\cdot \otimes_A^L Y^\cdot = X_2^\cdot \otimes_A F^\cdot$

now $X_1^\cdot \to X_2^\cdot \to X_3^\cdot \xrightarrow{+1}$, $X_3^\cdot$ acyclic, by 2, $X_3^\cdot \otimes_A F^\cdot$ is acyclic, hence the exact triangle

$$X_1^\cdot \otimes_A F^\cdot \to X_2^\cdot \otimes_A F^\cdot \longrightarrow X_3^\cdot \otimes_A F^\cdot \xrightarrow{+1}$$ tells us $X_1^\cdot \otimes_A^L Y^\cdot \cong X_1^\cdot \otimes_A F^\cdot \cong X_2^\cdot \otimes_A F^\cdot \cong X_2^\cdot \otimes_A^L Y^\cdot$