· Deformation of Abelian varieties

In this section, we aim to show that the Deformation functor for an Abelian variety is pro-representable by a power series ring:

<u>Thm</u> (A. Grothendieck)

Suppose X is an abelian variety of genus g over k, then the deformation functor Def_X^{AS} is pro-representable by: $W(k)[[t_1, t_2, \ldots, t_{g^2}]]$

Let's explain the deformation functor Def_X^{AS}, we not only want to deform the scheme itself, we also want to deform the group structure on the variety:

$$\mathrm{Def}_X^{AS} : \Lambda_W \longrightarrow \text{Sets}$$

~ Artinian local W-alg, with residue field $\simeq k$

$$A \longmapsto \{(X', \varphi) : X' \text{ AS}/A,\ X'_k \xrightarrow[\sim]{\varphi} X \text{ as AV}\} / \text{isomorphism}$$

Here, abelian scheme means proper smooth group scheme over a connected base scheme, with geometrically connected fibers. We also give the "usual" deformation functor

$$\mathrm{Def}_X : \Lambda_W \longrightarrow \text{Sets}$$

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here, since geometric fibers X_0 is automatically smooth, this condition can be replaced by smooth

$$A \longmapsto \{(X', \varphi) : X' \text{ flat}/A,\ X'_k \xrightarrow[\sim]{\varphi} X \text{ as } k\text{-variety}\} / \text{isomorphism}$$

To prove pro-representability, let's recall Schlessinger's criterion:

<u>Thm</u>: $F : \Lambda_W \to$ Sets is a covariant functor, consider the following condition, s.t. $F(W) = \{pt\}$, consider the following conditions on F:

$$F(A' \times_A A'') \xrightarrow{\alpha} F(A') \times_{F(A)} F(A'')$$

H_1) If $A' \twoheadrightarrow A$ is a small extension, then α is surj

H_2) If $A'' = k[\varepsilon]$, $A = k$, α is bijective

H_3) $\dim_k F(k[\varepsilon])$ is finite-dimensional

H_4) For $A'' \to A$ a small extension, then α is bij

Moreover, if H_4 & H_3 holds, and F is formally smooth, i.e. for $\forall R' \twoheadrightarrow R$, $F(R') \twoheadrightarrow F(R)$, then F is pro-representable by a power serise ring over W of $\dim_k F(k[\varepsilon])$ variables

So the main goal is to prove H_3) & H_4) for Def_X^{AS}, and we also should show the formally smooth property of Def_X^{AS}, then we compute the $\dim_k \mathrm{Def}_X^{AS}(k[\varepsilon])$ to get the full conclusion

We first prove a crucial lemma of rigidity, which will be used repeatedly to understand the geometry behind AS.

Geometry of AS

<u>Rigidity lemma</u>:

Proposition 6.1. (*Rigidity lemma.*) Given a diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "p"'] & Y \arrow[d, "q"] \\ & S \end{tikzcd}$$

suppose S is connected, p is flat and $H^0(X_s, \underline{o}_{X_s}) \cong \kappa(s)$, for all points $s \in S$ (X_s denoting the fibre of p over s). Assume that *one* of the following is true:

1) X has a section ε over S, and S consists of one point,
2) X has a section ε over S, and p is a closed map,
3) p is proper.

If, for one point $s \in S$, $f(X_s)$ is set-theoretically a single point, then there is a section $\eta: S \to Y$ of q such that $f = \eta \circ p$.

A corollary of this rigidity lemma is also important for our later use.

<u>Corollary</u>: Suppose X is an AS over S, G is a group scheme over S, if $f: X \to G$ is a morphism taking this identity section to identity section, then f is a homomorphism of group schemes over S.

pf: The idea is: if we want to show $f(x_1x_2) = f(x_1)\cdot f(x_2)$, then when x_2 fixed, $f(x_1x_2)\cdot f(x_1)^{-1}$ is constant

we consider the morphism: $f\circ\mu: X\times_S X \to X \to G$, and two others

$$\begin{tikzcd} X\times_S X \arrow[r] \arrow[dr] & G \arrow[d] \\ & S \end{tikzcd}$$

ψ_1: $(f\circ\mu, p_2): X\times_S X \longrightarrow G\times_S X$

$X(T)\times X(T) \longrightarrow G(T)\times X(T)$

$(x_1, x_2) \longmapsto (f(x_1x_2), x_2)$

ψ_2: $f\circ\mu\circ(1_X, \varepsilon\circ p)\times 1_X$: $X\times_S X \longrightarrow G\times_S X$

$X(T)\times X(T) \longrightarrow G(T)\times X(T)$

$(x_1, x_2) \longmapsto (f(x_1), x_2)$

$$\begin{tikzcd} X\times_S X \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr, "p_2"'] & & G_X \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

for $\forall s \in \varepsilon(S) \subset X$, $(\psi_1)_s = (\psi_2)_s$ $(x_2 = 1)$

then we get $\psi_2^{-1}\cdot\psi_1: X_X \to G_X$ satisfies the condition

so $\exists\, \eta: X \to G_X$, s.t. $\psi_2^{-1}\cdot\psi_1 = \eta\circ p_2 = (h, id)\circ p_2$, for $h: X\to G$

i.e. $\psi_1 = \psi_2\cdot(h, id)\circ p_2$

f: identity $\to$ identity

this implies: $f(x_1x_2) = f(x_1)\,h(x_2)$, take $x_1 = id \Rightarrow f = h: X \to G$

hence $f(x_1x_2) = f(x_1)\cdot f(x_2) \Rightarrow f$ is a homomorphism

$$\begin{tikzcd} & & X_s \arrow[r, "f_s"] \arrow[d] & G_s \arrow[d] \\ x_1 \arrow[r] \arrow[d] & f(x_1) \arrow[d] & X\times_S S \arrow[r, "f"] \arrow[d] & G\times_S S \arrow[d] \\ (x_1, e) \arrow[r, shift left] \arrow[r, shift right] & (f(x_1), e) & X\times_S X \arrow[r, shift left, "\psi_1"] \arrow[r, shift right, "\psi_2"'] & G\times_S X \end{tikzcd}$$

Rmk: this corollary alone has interesting consequence

- Abelian schemes are commutative

  pf: consider the inverse morphism $\iota: X \to X$ is a homomorphism $\Leftrightarrow X$ is commutative

- ϕ_1 & ϕ_2 are two homos of AS: $X \to Y$, then if $\exists s \in S$, s.t. $(\phi_1)_s = (\phi_2)_s$, then $\phi_1 = \phi_2$

  pf: rigidity lemma: $\phi_1 - \phi_2 = \eta\circ p: X \to S \xrightarrow{\eta} Y$, since $\varepsilon: S \to X$ is homo, we get:

  $$\varepsilon_Y = (\phi_1 - \phi_2)\circ\varepsilon_X = \eta\circ p\circ\varepsilon_X = \eta \Rightarrow \phi_1 = \phi_2$$

  $\eta_s = \varepsilon_s \rightsquigarrow \eta \sim \varepsilon$

  $$\begin{tikzcd} S \arrow[r] \arrow[d] & X \\ S \arrow[ur] & \end{tikzcd}$$

- $f: X \to Y$ is a morphism, then $\tilde{f} = f - f\circ\varepsilon_X\circ p_X$ is a homomorphism

  pf: $\tilde{f}\circ\varepsilon_X = f\circ\varepsilon_X - f\circ\varepsilon_X = \varepsilon_Y$

Let's make it more clearer what H_4 says for our Def_X^{AS}

H_4 for Def_X^{AS}:

For small extension $R'' \xrightarrow{\pi} R$, we should prove that there is a bijection

$$\mathrm{Def}_X^{AS}(R' \times_R R'') \xrightarrow{\sim} \mathrm{Def}_X^{AS}(R') \times_{\mathrm{Def}_X^{AS}(R)} \mathrm{Def}_X^{AS}(R'')$$

$$(\mathcal{X}, \varphi) \longrightarrow (\mathcal{X}_{R'}, \varphi) \times_{(\mathcal{X}_R, \varphi)} (\mathcal{X}_{R''}, \varphi)$$

$$\begin{tikzcd} R' \times_R R'' \arrow[r] \arrow[d, "\pi'"'] & R'' \arrow[d, "\pi"] \\ R' \arrow[r] & R \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Def}_X^{AS}(R' \times_R R'') \arrow[r] \arrow[d, "\pi'_{AS}"'] & \mathrm{Def}_X^{AS}(R'') \arrow[d, "\pi_{AS}"] \\ \mathrm{Def}_X^{AS}(R') \arrow[r] & \mathrm{Def}_X^{AS}(R) \\ (\mathcal{X}', \varphi') \arrow[r, mapsto] & (\mathcal{X}, \varphi) \end{tikzcd}$$

which is equivalent to the following bijection:

$$\pi'^{-1}_{AS}\big((\mathcal{X}', \varphi') \big) \xrightarrow{\sim} \pi^{-1}_{AS}\big((\mathcal{X}, \varphi) \big)$$

Next proposition identifies this two sets:

<u>Proposition</u>: $\pi^{-1}_{AS}\big((\mathcal{X}, \varphi) \big) \cong \left\{ (\mathcal{X}'', \varphi''_R) \mid \mathcal{X}'' \text{ flat}/R'',\ \mathcal{X}''_R \overset{\varphi''_R}{\xrightarrow{\sim}} \mathcal{X} \text{ as schemes over } R \right\} \big/ \mathrm{iso}_R \twoheadrightarrow \pi^{-1}\big((\mathcal{X}, \varphi) \big)$

pf: We first prove the last surjective, which is kind of "obvious"

$$\pi^{-1}\big((\mathcal{X}, \varphi) \big) = \left\{ (\mathcal{X}'', \varphi''_R) \mid \mathcal{X}'' \text{ flat}/R'',\ \mathcal{X}''_R \overset{\varphi''}{\xrightarrow{\sim}} X \text{ as schemes over } k \right\} \big/ \mathrm{iso}_k$$

here iso_R means: $(\mathcal{X}''_1, \varphi''_1) \sim (\mathcal{X}''_2, \varphi''_2)$ if $\exists\ \phi : \mathcal{X}''_1 \xrightarrow{\sim} \mathcal{X}''_2$ over R'', s.t.

$$\begin{tikzcd} \mathcal{X}''_{1R} \arrow[rr, "\phi_R"] \arrow[dr, "\varphi''_1"'] & & \mathcal{X}''_{2R} \arrow[dl, "\varphi''_2"] \\ & \mathcal{X} & \end{tikzcd}$$

commutes, i.e. $\varphi''_1 = \varphi''_2 \circ \phi_R$

iso_k means: $(\mathcal{X}''_1, \varphi''_1) \sim (\mathcal{X}''_2, \varphi''_2)$ if $\exists\ \phi : \mathcal{X}''_1 \xrightarrow{\sim} \mathcal{X}''_2$ over R'', s.t.

$$\begin{tikzcd} \mathcal{X}''_{1k} \arrow[rr, "\phi_k"] \arrow[dr, "(\varphi''_1)_k"'] & & \mathcal{X}''_{2k} \arrow[dl, "(\varphi''_2)_k"] \\ & \mathcal{X} & \end{tikzcd}$$

commutes, i.e. $(\varphi''_1)_k = (\varphi''_2)_k \circ \phi_k$

generally, iso_R implies iso_k, but iso_k not necessarily imply iso_R, but for abelian schemes, the situation is good, iso_k do imply iso_R.

Let's now show the first identification

$$\pi_{AS}^{-1}((\mathcal{X},\varphi)) \cong \{(\mathcal{X}'', \varphi''_R) \mid \mathcal{X}'' \text{ flat}/R'',\ \mathcal{X}''_R \xrightarrow[\sim]{\varphi''_R} \mathcal{X} \text{ as schemes over } R\}/_{iso_R}$$

this is the most striking part of the proposition, because on LHS we deform X as AS, but on RHS we deform X purely as schemes: it implicitly tells us that every deformation as a scheme will admit a "unique" AS structure

We first describe the map from LHS & RHS, by def

$$\pi_{AS}^{-1}((\mathcal{X},\varphi)) = \{(\mathcal{X}'', \varphi'') \mid \mathcal{X}'' \text{ AS}/R'',\ \mathcal{X}''_k \underset{\sim\,AV}{\xrightarrow{\varphi''}} X, \text{ and } \pi_{AS}((\mathcal{X}'', \varphi'')) = (\mathcal{X},\varphi)\}/_{iso_k}$$

$\forall\,(\mathcal{X}'', \varphi'')$, we know that $\exists\, \phi : \mathcal{X}''_R \overset{AS}{\xrightarrow{\sim}} \mathcal{X}$, s.t.

$$\begin{tikzcd} \mathcal{X}''_k \arrow[rr, "\phi_k"] \arrow[dr, "\varphi''"'] & & \mathcal{X}_k \arrow[dl, "\varphi"] \\ & X & \end{tikzcd}$$

commutes, i.e. $\varphi \circ \phi_k = \varphi''$

then the map is $(\mathcal{X}'', \varphi'') \longrightarrow (\mathcal{X}'', \phi)$, this is well-defined because:

1. ϕ is necessarily unique by rigidity lemma:
   because for another choice of ϕ', we get $\phi'_k = \phi_k$, then $\phi' - \phi = \eta \circ p''$ $\quad \phi' = \phi$
   now $\varepsilon_{\mathcal{X}''_R} = (\phi' - \phi)\circ \varepsilon_{\mathcal{X}} = \eta \circ p'' \circ \varepsilon_{\mathcal{X}} = \eta \Rightarrow \phi' = \phi$

2. if $(\mathcal{X}''_1, \varphi''_1) \overset{AS}{\sim} (\mathcal{X}''_2, \varphi''_2)$, i.e. $\exists\, \psi : \mathcal{X}''_1 \underset{AS}{\xrightarrow{\sim}} \mathcal{X}''_2$, s.t. $\varphi''_2 \circ \psi_k = \varphi''_1$

$$\begin{tikzcd} \mathcal{X}''_{1k} \arrow[r, "\psi_k"] \arrow[dr, "\varphi''_1"'] & \mathcal{X}''_{2k} \arrow[r, "(\phi_2)_k"] \arrow[d, "\varphi''_2"] & \mathcal{X}_k \arrow[dl, "\varphi"] \\ & X & \end{tikzcd}$$

$\Rightarrow (\phi_2)_k \circ \psi_k = (\phi_1)_k$

also by rigidity lemma, we get $\phi_2 \circ \psi_R = \phi_1 \Rightarrow (\mathcal{X}''_1, \phi_1) \sim_{iso_R} (\mathcal{X}''_2, \phi_2)$

now we should show that this map is both injective and surjective,

- Injective: Suppose $\left.\begin{array}{l} (\mathcal{X}''_1, \varphi''_1) \longrightarrow (\mathcal{X}''_1, \phi_1) \\ (\mathcal{X}''_2, \varphi''_2) \longrightarrow (\mathcal{X}''_2, \phi_2) \end{array}\right\}$ they are equivalent, i.e. $\exists\, \psi : \mathcal{X}''_1 \xrightarrow{\sim} \mathcal{X}''_2$, s.t. $\phi_2 \circ \psi_R = \phi_1$

  then obviously, $(\phi_2)_k \circ \psi_k = (\phi_1)_k$, but $(\phi_i)_k = \varphi^{-1} \circ \varphi''_i \Rightarrow \varphi''_2 \circ \psi_k = \varphi''_1$

  Can we conclude that $(\mathcal{X}''_1, \varphi''_1) \sim (\mathcal{X}''_2, \varphi''_2)$ from this? NO! because we don't know whether ψ is an iso as AS, by the corollary, we consider

  $$\psi' = \psi - \psi \circ \varepsilon''_1 \circ p''_1 \text{ is still an isomorphism } \mathcal{X}''_1 \xrightarrow{\sim} \mathcal{X}''_2$$

  then $\psi' \circ \varepsilon''_1 = 0$, i.e. $\psi' \circ \varepsilon''_1 = \varepsilon''_2$, $\Rightarrow \psi'$ is an isomorphism, moreover

  $$\psi'_R = \psi_R \text{ since } \psi_R \text{ is homomorphism, take identity to identity.}$$

  so we can replace ψ by ψ'

- Surjectivity: This is the most striking part of this identification, we should show any pair $(\mathcal{X}'', \varphi''_R)$ on RHS is equivalent to some $(\tilde{\mathcal{X}}, \tilde{\varphi})$, with $\tilde{\mathcal{X}}$ AS, $\tilde{\varphi}$ is an isomorphism of AS$/R$: $\tilde{\mathcal{X}}_R \xrightarrow{\tilde{\varphi}} \mathcal{X}$

Proposition 6.15. Let $S = \operatorname{Spec}(A)$, where A is an Artin local ring. Let $\mathfrak{m} \subset A$ be the maximal ideal, and let $I \subset A$ be an ideal such that $\mathfrak{m} \cdot I = (0)$. Let $\pi : X \to S$ be a smooth proper morphism, and let $\varepsilon : S \to X$ be a section. Let $S_0 = \operatorname{Spec}(A/I)$ and let $X_0 = X \times_S S_0$. Assume that X_0 is an abelian scheme over S_0 with identity $\varepsilon \mid S_0$. Then X is an abelian scheme over S with identity ε.

Thm : Obstruction of lifting a morphism

Suppose $X, Y/R \in \Lambda_k$ are smooth schemes. Now suppose $R' \overset{\pi}{\twoheadrightarrow} R$ is a small extension in Λ_k,

· $X' \in Def_X(R')$, $Y' \in Def_Y(R')$

Then for $\forall f: X \to Y/R$, there is a canonically associated class

$$o(f) \in H^1(X_k, f_k^* T_{Y_k/k}) \otimes_k \ker\pi$$

s.t. f can be deformed to a R'-morphism $X' \to Y' \iff o(f) = 0$

and the deformation of f is parametrized by $H^0(X_k, f_k^* T_{Y_k/k}) \otimes_k \ker\pi$

pf: the proof basically follows the same lines as classical deformation theory

we first consider affine case, where there should be no obstruction

$X = \operatorname{Spec} A \xrightarrow{f} Y = \operatorname{Spec} B$ over R, A, B are smooth R-algebras,

f is induced by $\varphi: B \to A$. Since X, Y are smooth, $X' \simeq \operatorname{Spec}(A \otimes_R R')$, $Y' \simeq \operatorname{Spec}(B \otimes_R R')$

then there exists a trivial deformation

$$\varphi_{R'}: B \otimes_R R' \longrightarrow A \otimes_R R' \qquad 0 \to \ker\pi \to R' \to R \to 0$$

Now we consider another deformation, φ', then $\varphi' - \varphi_{R'}: B \otimes_R R' \to \ker(A \otimes_R R' \to A) \simeq \ker\pi \otimes_R A \simeq \ker\pi \otimes_k A_0$

i.e. there difference induces $D: B \otimes_R R' \longrightarrow \ker\pi \otimes_k A_0$, which is R'-linear

$B \otimes_R R' \simeq B_0 \otimes_k R'$

then we consider $D_0 : B_0 \to B_0 \otimes_k R' \to \ker\pi \otimes_k A_0$, $(D_0)_{R'} = D$, and

$$D_0(ab) = D(ab) = \varphi'(ab) - \varphi_{R'}(ab) = \varphi'(a)\varphi'(b) - \varphi_{R'}(a)\varphi_{R'}(b)$$

$$= (\varphi_{R'}(a) + D_0(a))(\varphi_{R'}(b) + D_0(b)) - \varphi_{R'}(a)\varphi_{R'}(b)$$

$$= a \cdot D_0(b) + b \cdot D_0(a) \quad \left(\varphi_k(a) \cdot D_0(b) + \varphi_k(b) D_0(a)\right)$$

then $D_0 \in Der_k(B_0, A_0 \otimes_k \ker\pi) \cong Der_k(B_0, A_0) \otimes_k \ker\pi \simeq Hom_{B_0}(\Omega^1_{B_0/k}, A_0) \otimes_k \ker\pi = H^0(X_k, f^* T_{Y_k/k}) \otimes_k \ker\pi$

and easy to very, $\forall D_0$ belongs to this space corresponds to a deformation,

$$\varphi' = \varphi_{R'} + (D_0)_{R'}$$

φ' is also an isomorphism because of the following diagram

$$\begin{tikzcd} B_0 \otimes_k \ker\pi \arrow[r, "\simeq"] \arrow[d] & A_0 \otimes_k \ker\pi \arrow[d] \\ B_0 \otimes_k R' \arrow[r] \arrow[d] & A_0 \otimes_k R' \arrow[d] \\ B_0 \otimes_k R \arrow[r, "\simeq"] & A_0 \otimes_k R \end{tikzcd}$$

proof: How to show that there is an AS structure on X? Let's first consider How to define an "group scheme" structure, it is all contained in the following morphism

$$\mu : G \times_S G \longrightarrow G$$
$$(x, y) \longmapsto xy^{-1}$$

this μ will recover ε, m, ι, if we can show some commutative diagram (associativity, inverse, identity...)

so our case, we have a morphism

$$\mu_0 : X_0 \times_{S_0} X_0 \longrightarrow X_0$$

so our next job is to deform this morphism to $X \times_S X \to X$ and see among those deformations (if it has) which one has the properties we want

- Existence of the deformation

  By deformation theory for morphisms, there is a canonical class

$$o(\mu_0) \in H^1(\bar{X} \times_k \bar{X}, \bar{\mu}_0^*(T_{\bar{X}/k})) \otimes_k I$$

whose vanishing is equivalent to the existence of a deformation, our strategy is to use the functorial property to argue that $o(\mu_0) = 0$, consider following two morphisms:

$$g_1 : X_0 \xrightarrow{\Delta} X_0 \times_{S_0} X_0 \qquad \& \qquad g_2 : X_0 \xrightarrow{id \times \varepsilon p} X_0 \times_{S_0} X_0$$

then $\mu_0 \circ g_1 = \varepsilon_0 \circ \pi_0$, $\mu_0 \circ g_2 = 1_{X_0}$, $\varepsilon_0 \circ \pi_0$ & 1_{X_0} has obvious deformations, i.e.

$$\bar{g}_1^* o(\mu_0) = 0 \quad \& \quad \bar{g}_2^* o(\mu_0) = 0$$

Let's argue from this that $o(\mu_0) = 0$, consider

$$\bar{g}_1^* : H^1(\bar{X} \times_k \bar{X}, \bar{\mu}^*(T_{\bar{X}/k})) \longrightarrow H^1(\bar{X}, (\bar{\varepsilon} \circ \bar{\pi})^*(T_{\bar{X}/k}))$$

note, since $\bar{X}$ is AV, we have $T_{\bar{X}/k} \simeq \mathcal{O}_{\bar{X}} \otimes_k t$, $t = \mathrm{Lie}(\bar{X})$, hence

$$\begin{array}{ccc}
H^1(\bar{X} \times_k \bar{X}, \bar{\mu}^*(T_{\bar{X}/k})) & \longrightarrow & H^1(\bar{X}, (\bar{\varepsilon} \circ \bar{\pi})^*(T_{\bar{X}/k})) \\
\wr\!\!\mid & & \parallel \\
H^1(\bar{X} \times_k \bar{X}, \mathcal{O}_{\bar{X} \times_k \bar{X}}) \otimes_k t & \longrightarrow & H^1(\bar{X}, \mathcal{O}_{\bar{X}}) \otimes_k t \\
\wr\!\!\mid & & \\
\left(p_1^* H^1(\bar{X}, \mathcal{O}_{\bar{X}}) \oplus p_2^* H^1(\bar{X}, \mathcal{O}_{\bar{X}})\right) \otimes_k t & \longrightarrow & H^1(\bar{X}, \mathcal{O}_{\bar{X}}) \otimes_k t
\end{array}$$

$p_1 \circ g_1 = 1_{\bar{X}}$ $\qquad (x, y) \otimes v \longmapsto (x + y) \otimes v$ $\qquad$ hence $\bar{g}_1^* o(\mu_0) = 0$

$p_2 \circ g_1 = 1_{\bar{X}}$ $\qquad$ & $\bar{g}_2^* o(\mu_0) = 0$

same for g_2: $\qquad (x, y) \otimes v \longmapsto x \otimes v$ $\qquad \Rightarrow o(\mu_0) = 0$

$p_1 \circ g_2 = 1_{\bar{X}}$ $\qquad$ same for $\bar{g}_1^*, \bar{g}_2^* - \otimes_k I$

$p_2 \circ g_2 = 0$

· Find a good one

By previous computation, we know that μ_0 exists a deformation to X, all these deformations admits a free transitive action by

$$H^0(\bar{X}\times_k\bar{X},\ \bar{\mu}_0^*(T_{\bar{X}/k}))\otimes_k I \simeq t\otimes_k I$$

now we take an arbitrary extension μ, and consider $S\xrightarrow{(\varepsilon,\varepsilon)} X\times_S X\xrightarrow{\mu} X$, is deformation of $\varepsilon_0: S_0\to X_0$, hence the deformation of ε_0 is acted by:

$$H^0(\mathrm{Spec}\,k,\ \bar{\varepsilon}^* T_{\bar{X}/k})\otimes_k I \simeq t\otimes_k I$$

By the construction, we get a bijective, i.e. μ is totally detemined by $\mu\circ(\varepsilon,\varepsilon)$, since ε is a deformation of ε_0, hence we require our choice of μ should correspond to ε, i.e. $\mu\circ(\varepsilon,\varepsilon)=\varepsilon$

· (μ,ε) gives rise to an AS structure on X

we define inverse morphism $\iota=\mu\circ(\varepsilon\circ\pi, 1_X)$, $m=\mu\circ(1_X,\iota)$, so we should show associativity, etc.

all the 3 diagrams are of the following form, we should prove

$$\begin{tikzcd} X\times_S X\times_S\cdots\times_S X \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] \arrow[dr, "p"'] & X \arrow[d] \\ & S \end{tikzcd}$$

Now h_1 & h_2 are both deformations of $(h_1)_0=(h_2)_0$, they are parametrized by:

$$H^0(\bar{X}\times_k\bar{X}\times_k\cdots\times_k\bar{X},\ \bar{h}^* T_{\bar{X}/k})\simeq t\otimes_k I$$

if we restrict h_1 & h_2 to $S\xrightarrow{(\varepsilon,\varepsilon,\_,\varepsilon)} X\times_S X\times_S\cdots\times_S X$, we will get Claim $\underline{h_1\circ(\varepsilon,\_\varepsilon)=h_2\circ(\varepsilon,\_,\varepsilon)=\varepsilon}$

they are both deformations of some $\varepsilon_0: S_0\to X_0$, deformations of g are parametrized by

$$H^0(\mathrm{Spec}\,k(s),\ \bar{\varepsilon}^* T_{\bar{X}/k})\simeq t\otimes_k I$$

by the identification, we get $h_1=h_2$

Take m as an example:

$$\begin{tikzcd} X\times_S X\times_S X \arrow[r, shift left, "m\circ(m\times 1_X)"] \arrow[r, shift right, "m\circ(1_X\times m)"'] \arrow[dr] & X \arrow[d] \\ & S \end{tikzcd}$$

$$m\circ(m\times 1_X)\circ(\varepsilon,\varepsilon,\varepsilon)=m\circ(m\circ(\varepsilon,\varepsilon),\varepsilon)$$
$$m\circ(1_X\times m)\circ(\varepsilon,\varepsilon,\varepsilon)=m\circ(\varepsilon,m\circ(\varepsilon,\varepsilon))$$
$$\iota\circ\varepsilon=\mu\circ(\varepsilon\circ\pi,1_X)\circ\varepsilon=\mu\circ(\varepsilon,\varepsilon)=\varepsilon\Rightarrow m\circ(\varepsilon,\varepsilon)=\mu_0\circ(\varepsilon,\iota\circ\varepsilon)=\varepsilon$$

hence $m\circ(m\times 1_X)\circ(\varepsilon,\varepsilon,\varepsilon) = m\circ(1_X\times m)\circ(\varepsilon,\varepsilon,\varepsilon)$

Now back to the surjectivity, we know the obstruction of lifting $\varepsilon : \operatorname{Spec} R \to \mathcal{X}$ lies in $H^1(\operatorname{Spec} k, \varepsilon^* T_{\mathcal{X}/k}) \otimes_k \ker \pi = 0$

i.e. there always exists $\varepsilon'' : \operatorname{Spec} R'' \to \mathcal{X}''$, which is an R''-morphism, i.e. ε'' is a section. hence $\mathcal{X}''$ admits a unique structure of AS with ε'' the identity,

also consider $\varphi_R'' \circ \varepsilon'' : \operatorname{Spec} R'' \to \mathcal{X}_R'' \simeq \mathcal{X}$ also lifts to

$$c : \operatorname{Spec} R \longrightarrow \mathcal{X}''$$

then consider $\tilde{\varphi} : \mathcal{X}'' \xrightarrow{-c \circ p} \mathcal{X}''$. then

$$\mathcal{X}_R'' \xrightarrow[+c]{} \mathcal{X}_R'' \xrightarrow[\sim]{\varphi_R''} \mathcal{X}$$

$$\varphi_R'' = \phi + \varphi_R'' \circ \varepsilon'' p''$$

$$c : \operatorname{Spec} R \to \mathcal{X}_R''$$

$$c = -\phi^{-1} \circ \varphi_R'' \varepsilon_R'' p$$

$$\begin{aligned} g \mapsto g + c \to \varphi_R''(g+c) &= \phi(g+c) + \varphi_R''(e) \\ &= \phi(g) + \underbrace{\phi(c) + \varphi_R''(e)}_{\shortparallel \atop 0} \end{aligned}$$

· Smoothness

Our next goal is to prove the deformation functor Def_X^{AS} is formally smooth, i.e. for $\forall R' \overset{\pi}{\twoheadrightarrow} R$, we have
$$\mathrm{Def}_X^{AS}(R') \twoheadrightarrow \mathrm{Def}_X^{AS}(R)$$

Suppose $(\mathcal{X}, \varphi) \in \mathrm{Def}_X^{AS}(R)$, we only need to find a pre-image in the case of small extension $R' \overset{\pi}{\twoheadrightarrow} R$

By classical obstruction theory, we know that the obstruction of deforming $(\mathcal{X}, \varphi)$ is
$$o(\mathcal{X}, \varphi) \in H^2(X, T_{X/k}) \otimes_k \ker\pi$$

Short review of the construction:

- Since X is smooth, $\{U_i\}$ Zariski covering of X, $\mathcal{X}|_{U_i} \overset{\theta_i}{\underset{\sim}{\to}} U_i \times_k \mathrm{Spec} R$, define $\theta_{ij} = \theta_j \circ \theta_i^{-1}$ is automorphism of the trivial deformation $U_{ij} \times_k \mathrm{Spec} R$, we aims to lift θ_{ij} to $\tilde{\theta}_{ij}$ of automorphism of $U_{ij} \times_k \mathrm{Spec} R'$, but we should take care of the cocycle condition: $\tilde{\theta}_{ijk} = \tilde{\theta}_{ij} \circ \tilde{\theta}_{jk} \circ \tilde{\theta}_{ik}^{-1}$, which reduces to identity on $U_{ijk} \times_k \mathrm{Spec} R$, hence, essentially, $\tilde{\theta}_{ijk} = 1 + \varepsilon \tilde{d}_{ijk}$, $\tilde{d}_{ijk} \in \Gamma(U_{ijk}, T_{X/k})$

Note: $H^2(X, T_{X/k}) \simeq t \otimes_k H^2(X, \mathcal{O}_X) \simeq t \otimes_k (t^\vee \wedge t^\vee)$, $t^* = \mathrm{Hom}_k(t, k) \simeq H^1(X, \mathcal{O}_X) = t^\vee$
therefore if we consider the inverse $\iota: \mathcal{X} \to \mathcal{X}$, then ι_0^* induces -1 on t & t^*, hence by functorial property
$$o(\mathcal{X}, \varphi) = \iota_0^* o(\mathcal{X}, \varphi) = -o(\mathcal{X}, \varphi)$$

this concludes the proof if $\mathrm{char}\, k \neq 2$, because although here we only deform $\mathcal{X}$ as a scheme, by previous results
$$\pi_{AS}^{-1}((\mathcal{X}, \varphi)) \cong \{(\mathcal{X}'', \varphi_R'') \mid \mathcal{X}'' \text{ flat}/R'', \mathcal{X}_R'' \overset{\varphi_R''}{\underset{\sim}{\to}} \mathcal{X} \text{ as schemes over } R\} / \mathrm{iso}_R$$

we have shown RHS is nonempty, hence LHS is non-empty.
Now if $\mathrm{char}\, k = 2$, we can conclude by a slight different argument

· Dimension

By previous results, we know $\mathrm{Def}_X^{AS}(k[\varepsilon]) = \mathrm{Def}_X(k[\varepsilon])$
and the second one is parametrized by
$$H^1(X, T_{X/k}) \simeq t \otimes_k H^1(X, \mathcal{O}_X) \simeq t \otimes_k t^\vee$$

which is dimension g^2, hence we conclude the main theorem

· Deformation with polarizations

Our next task is to consider the deformation of an abelian variety with a polarization, but we should first figure out what is a polarization

Def: Dual AS

Suppose X/S is an AS, then we define the dual scheme X^t to be $\mathrm{Pic}^0_{X/S}$, i.e. the connected component of the Picard scheme of X. this X^t is also a proper smooth group scheme $/S$, i.e. X^t is an AS

Def: Polarization

Suppose X/S is an AS, a polarization of X is a morphism
$$\pi: X \longrightarrow X^t$$
s.t. for all geometric point $\bar{s}$ of S, $\pi_{\bar{s}}: X_{\bar{s}} \to X^t_{\bar{s}}$ is of the form $\Lambda(\bar{L})$, for some ample line bundle $\bar{L}$ of $X_{\bar{s}}$

Rmk: In the case that we are interested: $S = \mathrm{Spec}\, R$, for $R \in \Lambda_k$, then there is an equivalent definition
$$\pi: X \longrightarrow X^t$$

- π is a quasi-polarization:
  if $\exists$ some $\mathcal{L} \in \mathrm{Pic}(X)$, $\pi = \Lambda(\mathcal{L})$
- π is a polarization:
  if $\mathcal{L}$ is relatively ample w.r.t π

Rmk: What is $\Lambda(\mathcal{L})$?

$\Lambda(\mathcal{L})$ is a morphism: $X \to X^t = \mathrm{Pic}^0_{X/S} \hookrightarrow \mathrm{Pic}_{X/S}$, i.e. it is essentially an element in
$$\mathrm{Pic}_{X/S}(X) = \frac{\mathrm{Pic}(X \times_S X)}{p_2^* \mathrm{Pic}(X)}$$

what is this element? it is given by: $[m^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1}] = [m^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1}]$

and since $\varepsilon: S \longrightarrow X \longrightarrow \mathrm{Pic}_{X/S}$, gives rise to $X(S) \to \mathrm{Pic}_{X/S}(S) \simeq \mathrm{Pic}(X)/\varepsilon^*\mathrm{Pic}(S)$
$$[\varepsilon] \longmapsto (1_X \times \varepsilon)^*[m^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1}] = [\mathcal{L} \otimes \mathcal{L}^{-1}] = [\mathcal{O}_X]$$

i.e. $\Lambda(\mathcal{L}): X \to \mathrm{Pic}_{X/S}$ sends identity to identity, then this is a homomorphism

Since X is connected $\Rightarrow \Lambda(\mathcal{L})$ factors through $\mathrm{Pic}^0_{X/S}$, i.e. $\Lambda(\mathcal{L}): X \to X^t$

Example: $S = \mathrm{Spec}\, \mathbb{C}$

Suppose $X = V/\Lambda$, then X^t can be realized as: $V^\vee/\Lambda^\vee$, here
$$V^\vee = \mathrm{Hom}_{\bar{\mathbb{C}}}(V, \mathbb{C}) = \{ l: V \to \mathbb{C} \mid l(\lambda v) = \bar{\lambda}\, l(v) \}$$
$$\Lambda^\vee = \{ l \in V^\vee \mid \mathrm{Im}(l(\lambda)) \in \mathbb{Z}, \forall \lambda \in \Lambda \} \subset V^\vee$$

We first consider the problem of deforming a pair $(X, \mathcal{L})$, where X is a AV/k, $\mathcal{L} \in Pic(X)$, the classical result of deforming such a pair is the following:

- Deformation functor $Def_{(X,L)} : \Lambda_k \longrightarrow Sets$

$$A \longmapsto \{(\mathcal{X}, \mathcal{L}) \mid \mathcal{X} \text{ is a deformation of } X \text{ over } A,\ \mathcal{L}|_X \simeq L\} / iso_k$$

Note that we are deforming 2 objects: one is $\mathcal{X}$, one is $\mathcal{L}$, the obstruction tells us that the first one is related to T_X &. the second one is related to $End(\mathcal{L}) \simeq \mathcal{L}^\vee \otimes \mathcal{L} \simeq \mathcal{O}_X$ (at least in trivial det) the general theory is the following

Theorem 3.3.11 *Let (X, L) be a pair consisting of a nonsingular projective algebraic variety X and an invertible sheaf L on X. Then:*

(i) *The functor $\mathrm{Def}_{(X,L)}$ has a semiuniversal formal element.*

(ii) *there is a canonical isomorphism*

$$\mathrm{Def}_{(X,L)}(\mathbf{k}[\epsilon]) = \frac{\{\text{1-st order deformations of } (X,L)\}}{\text{isomorphism}} \cong H^1(X, \mathcal{E}_L)$$

and $H^2(X, \mathcal{E}_L)$ is an obstruction space for $\mathrm{Def}_{(X,L)}$.

(iii) *Given a first order deformation ξ of X, there is a first order deformation of L along ξ if and only if*

$$\kappa(\xi) \cdot c(L) = 0$$

where "·" denotes the composition:

$$H^1(X, T_X) \times H^1(X, \Omega^1_X) \xrightarrow{\cup} H^2(X, T_X \otimes \Omega^1_X) \to H^2(X, \mathcal{O}_X)$$

of the cup product of cohomology classes $\cup$ with the map induced by the duality pairing $T_X \otimes \Omega^1_X \to \mathcal{O}_X$ (therefore the left hand side is an element of $H^2(X, \mathcal{O}_X)$).

Sketch of the proof: Suppose $R' \to R \to 0$ is small, with ker one-dim'l.

given $(\mathcal{X}, \mathcal{L}) \in Def_{(X,L)}(R)$, we consider the following two questions

- Whether $\mathcal{X}$ admits a deformation to R' $\rightsquigarrow o(\mathcal{X}) \in H^2(X, T_X)$
- If $\mathcal{X}'$ is a deformation of $\mathcal{X}$. i.e. $o(\mathcal{X}) = 0$, then $\mathcal{X}' \in H^1(X, T_X)$

  How can we deform $\mathcal{L}$ along $\mathcal{X}'$?

$$0 \to B_i \otimes_k t \to B_i \otimes_k R' \to B_i \otimes_k R \to 0$$

Let's denote $[\xi]$ as $\mathcal{X}'$, on an open covering $\{U_\alpha\}_{\alpha \in I}$ of X, ξ is given by $\{\xi_\alpha\}$ here $\xi_\alpha \in \Gamma(U_\alpha, T_X) = Hom_{B_i}(\Omega^1_{B_i/k}, B_i) \simeq Der_k(B_i, B_i)$: $1 + t\xi_\alpha = \theta_\alpha$

we consider $\theta_{\alpha\beta}$ is a R'- isomorphism of $U_{\alpha\beta} \times_k Spec R'$, $\mathcal{X}'$ is glued together via $\theta_{\alpha\beta}$

Suppose $\mathcal{L}$ corresponds to $(f_{\alpha\beta}) \in H^1(X, \mathcal{O}_X^*)$

This theorem tells us that we only need to consider the Chern class $c(L)$ of a line bundle when we what to deform it to a first order deformation

Example of AV: Suppose $X = V/\Lambda$, $\mathcal{L}$ is an ample line bundle. we consider $\mathrm{Def}_{(X,\mathcal{L})}(\mathbb{C}[\varepsilon])$

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

gives rise to long exact sequence:

$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \simeq \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$$
$$\mathcal{L} \longmapsto c(\mathcal{L})$$

$$H^2(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^0(X, \Omega_X^2) \oplus H^1(X, \Omega_X^1) \oplus H^2(X, \mathcal{O}_X)$$

$$H^2(X, \mathbb{Z}) = \Lambda^2 H^1(X, \mathbb{Z}) = \Lambda^2(\Lambda^\vee)$$

i.e. $c(\mathcal{L}) : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ alternating form

more terms:
$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to \underset{\Lambda^\vee}{\underset{\cong}{H^1(X, \mathbb{Z})}} \longrightarrow \underset{V^\vee}{\underset{\cong}{H^1(X, \mathcal{O}_X)}}$$

V^* $\qquad \overline{V^*} \simeq \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C}) = V^\vee$

$$H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^0(X, \Omega_X^1) \oplus H^1(X, \mathcal{O}_X)$$

$$H^0(X, \Omega^2_{X/k}) = \Lambda^2 \overline{V^\vee}, \quad H^1(X, \Omega_X^1) = H^1(X, \mathcal{O}_X) \otimes_k \overline{V^\vee} \simeq V^\vee \otimes_k \overline{V^\vee}, \quad H^2(X, \mathcal{O}_X) = \Lambda^2 H^1(X, \mathcal{O}_X) = \Lambda^2 V^\vee$$

We also have an exact sequence: $0 \to X^t \to X \xrightarrow{\Lambda} \mathrm{Hom}(X, X^t)$

this tells us that is some sence, $C = \Lambda$, that's one of the reasons why we care deformation with polarization

Let's now consider the following map

$$\cdot c(\mathcal{L}) : \quad H^1(X, \mathcal{T}_X) \xrightarrow{\cup c(\mathcal{L})} H^2(X, \mathcal{T}_X \otimes \Omega_X^1) \longrightarrow H^2(X, \mathcal{O}_X)$$

we know, for a deformation $\xi \in H^1(X, \mathcal{T}_X)$, i.e. $\mathcal{X}/\mathbb{C}[\varepsilon]$ lifting $X/\mathbb{C}$, $\mathcal{L}$ admits a lifting iff

$$\xi \cdot c(\mathcal{L}) = 0$$

Suppose $c(\mathcal{L})$ corresponds to $h : V \times V \to \mathbb{C}$ is the corresponding positive-definite Hermitian form, then $\cdot c(\mathcal{L})$ can be described quite explicitly:

$$H^1(X, \mathcal{T}_X) \simeq V \otimes_{\mathbb{C}} V^\vee \longrightarrow H^2(X, \mathcal{T}_X \otimes \Omega_X^1) \simeq V \otimes \overline{V^\vee} \otimes \Lambda^2 V^\vee \longrightarrow H^2(X, \mathcal{O}_X) = \Lambda^2 V^\vee$$
$$v \otimes f \longmapsto \sum_i v \otimes f \otimes f_i \otimes g_i \mapsto \sum v \otimes f_i \otimes (f \wedge g_i) \longmapsto \sum f_i(v) f \wedge g_i = h(v, -) \wedge f$$

since h positive-definite gives $V \simeq V^\vee \Rightarrow$ this map is onto, hence

$$\dim_{\mathbb{C}} \ker = g^2 - \binom{g}{2} = \frac{g(g+1)}{2}$$

Next we show $\mathrm{Der}_{\mathbb{G}_m}(k[\varepsilon])$ is trivial, i.e. every Δ satisfying $(*)$ is of the form $f(T_1T_2)-f(T_1)-f(T_2)$

$f(T)=\sum a_i T^i \Rightarrow f(T_1T_2)-f(T_1)-f(T_2)$

$$= \sum a_i (T_1^i T_2^i - T_1^i - T_2^i)$$

$\Delta(T_1,T_2) = \sum_{i,j} \lambda_{ij} T_1^i T_2^j$

we can adjust Δ by some f, s.t. $\lambda_{i0}=0$, then $(*)$ becomes.

$$\sum_{i,j} \lambda_{ij} T_1^i T_2^j + \sum_{i,j} \lambda_{ij} T_1^i T_2^i T_3^j = \sum_{i,j} \lambda_{ij} T_1^i T_2^j T_3^j + \sum_{i,j} \lambda_{ij} T_2^i T_3^j$$

$$\underbrace{\sum_{i,j} \lambda_{ij} T_1^i T_2^j}_{\text{no } T_3} = \underbrace{\sum_{i,j} \lambda_{ij} T_1^i T_2^j T_3^j + \sum_{i,j} \lambda_{ij} T_2^i T_3^j - \sum_{i,j} \lambda_{ij} T_1^i T_2^i T_3^j}_{\text{all } T_3}$$

$\Rightarrow$ LHS $=0 \Rightarrow \lambda_{ij}=0$

· $\mathbb{G}_a$: We consider $\mathrm{Der}_{k[\varepsilon]}(\mathbb{G}_a)$

$$T \to T_1 + T_2 + \varepsilon\Delta(T_1, T_2)$$

associativity law: $\Delta(T_1, T_2) + \Delta(T_1+T_2, T_3) = \Delta(T_2, T_3) + \Delta(T_1, T_2+T_3)$ ①

trivial one: $\Delta(T_1, T_2) = f(T_1+T_2) - f(T_1) - f(T_2)$

$\frac{\partial}{\partial T_3}$: $\Delta_2(T_1+T_2, T_3) = \Delta_2(T_2, T_3) + \Delta_2(T_1, T_2+T_3)$ ②

$\frac{\partial}{\partial T_1}$: $\Delta_{12}(T_1+T_2, T_3) = \Delta_{12}(T_1, T_2+T_3)$

take $T_1 = 0 \Rightarrow \Delta_{12}(T_1, T_2) = f(T_1+T_2)$ for some $f \in k[T]$

then $\mathrm{char}\, k = 0 \Rightarrow \exists F$, s.t. $F' = f$, hence

$$\Delta_2(T_1, T_2) = F(T_1+T_2) + g(T_2)$$

use (2)

$$F(T_1+T_2+T_3) + g(T_3) = F(T_2+T_3) + g(T_3) + F(T_1+T_2+T_3) + g(T_2+T_3)$$

$$\Rightarrow g = -F$$

i.e. $\Delta_2(T_1, T_2) = F(T_1+T_2) - F(T_2)$, $\exists \lambda$, $\lambda' = F$, hence

$$\Delta(T_1, T_2) = \lambda(T_1+T_2) - \lambda(T_2) + h(T_1)$$

use ①: $\lambda(T_1+T_2) - \lambda(T_2) + h(T_1) + \lambda(T_1+T_2+T_3) - \lambda(T_3) + h(T_1+T_2)$

$$= \lambda(T_2+T_3) - \lambda(T_3) + h(T_2) + \lambda(T_1+T_2+T_3) - \lambda(T_2+T_3) + h(T_1)$$

$$\lambda(T_1+T_2) - \lambda(T_2) = h(T_2) - h(T_1+T_2), \quad h = -\lambda + a,$$

$\Rightarrow \Delta(T_1, T_2) = \lambda(T_1+T_2) - \lambda(T_2) - \lambda(T_1) + a$

$$= \left(\lambda(T_1+T_2) - a\right) - \left(\lambda(T_1) - a\right) - \left(\lambda(T_2) - a\right)$$

Deformed multiplication: $T \to T_1 + T_2 + \varepsilon \Delta(T_1, T_2)$

associativity law: $\Delta(T_1, T_2) + \Delta(T_1+T_2, T_3) = \Delta(T_2, T_3) + \Delta(T_1, T_2+T_3)$ ①

$\frac{\partial}{\partial T_3}$: $\Delta_2(T_1+T_2, T_3) = \Delta_2(T_2, T_3) + \Delta_2(T_1, T_2+T_3)$ ②

$\frac{\partial}{\partial T_1}$: $\Delta_{12}(T_1+T_2, T_3) = \Delta_{12}(T_1, T_2+T_3)$

take $T_1 = 0 \Rightarrow \Delta_{12}(T_1, T_2) = f(T_1+T_2)$ for some $f \in k[T]$

then $\Delta_2(T_1, T_2) = F(T_1+T_2) + g(T_1^p, T_2)$

existence of F: suppose $f = \sum a_n T^n + \sum_k b_k T^{pk-1}$

$\sum_k b_k (T_1+T_2)^{pk-1}$

$= \sum_k \sum_i b_k \binom{pk-1}{i} T_2^{pk-1-i} T_1^i$

$= \sum_i \left(\sum_k b_k \binom{pk-1}{i} T_2^{pk-1-i} \right) T_1^i$, $\quad \sum_k b_k \binom{pk-1}{i} = 0$

$\Rightarrow b_k = 0$

② $\Rightarrow$ $F(T_1+T_2+T_3) + g(T_1^p + T_2^p, T_3) = F(T_2+T_3) + g(T_2^p, T_3) + F(T_1+T_2+T_3) + g(T_1^p, T_2+T_3)$

$g(T_1^p + T_2^p, T_3) = F(T_2+T_3) + g(T_2^p, T_3) + g(T_1^p, T_2+T_3)$

$\xrightarrow{T_3=0}$ $g(T_1^p, T_2) = g(T_1^p+T_2^p, 0) - g(T_2^p, 0) - F(T_2)$

$\xrightarrow{T_2=0}$ $0 = F(T_3) + g(0, T_3) \Rightarrow F(T) = -g(0, T)$

$\longrightarrow$ $g(T_1^p, T_2) = g(T_1^p + T_2^p, 0) - g(T_2^p, 0) + g(0, T_2)$

$\Rightarrow \Delta_2(T_1, T_2) = F(T_1+T_2) + g(T_2)$

then $F(T_1+T_2+T_3) + g(T_3) = F(T_2+T_3) + g(T_3) + F(T_1+T_2+T_3) + g(T_2+T_3)$

$\Rightarrow g = -F$, i.e.

$\Delta_2(T_1, T_2) = F(T_1+T_2) - F(T_2)$

$\frac{\partial \Delta''}{\partial T_2}(T_1, T_2) = \sum_{\substack{n \not\equiv -1 \\ \bmod p}} a_n (T_1+T_2)^n - a_n T_2^n + \sum_k b_k \left((T_1+T_2)^{pk-1} - T_2^{pk-1} \right)$

$\binom{kp-1}{p-1}$

$\frac{(kp-1)(kp-2)\cdots(kp-(p-1))}{1 \quad 2 \quad \cdots \quad p-1} \neq 0$

$k=1$: no $p-1$

$k=2$: $b_2 \binom{2p-1}{p} T_1^p \quad T_2^{p-1}$

$\binom{2p-1}{p} \sim \frac{(2p-1)(2p-2)\cdots(2p-(p-1))}{}$

$k=3$: $b_3 \binom{3p-1}{2p} T_1^{2p} \cdot T_2^{p-1} + b_3 \binom{3p-1}{p} T_1^p \cdot T_2^{2p-1}$

$k=4$: $b_4 \binom{4p-1}{3p} T_1^{3p} \cdot T_2^{p-1} + b_5 \binom{4p-1}{2p} T_1^{2p} \cdot T_2^{2p-1} + b_6 \binom{4p-1}{p} T_1^p \cdot T_2^{3p-1}$

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3

$\Rightarrow b_k = 0$ for $k \geq 2$

$\Delta(T_1, T_2) = \tilde{F}(T_1+T_2) - \tilde{F}(T_2) - \tilde{F}(T_1) + b \cdot B_p(T_1, T_2) + h(T_1, T_2^p)$

$\Rightarrow \Delta_2(T_1, T_2) = \underbrace{F(T_1+T_2) - F(T_2)}_{\deg \text{ not } \equiv -1 \bmod p} + b\left((T_1+T_2)^{p-1} - T_2^{p-1} \right) \sim$

$\tilde{F}' = F$

$$\Delta(T_1,T_2) = \tilde{F}(T_1+T_2) - \tilde{F}(T_2) + b\cdot B_p(T_1,T_2) + h(T_1,T_2^p)$$ is a deformation

hence $h(T_1, T_2^p)$ is a deformation

$$h(T_1,T_2^p) + h(T_1+T_2, T_3^p) = h(T_2,T_3^p) + h(T_1, T_2^p+T_3^p)$$

$$\frac{\partial}{\partial T_2} \Rightarrow \quad h_1(T_1+T_2, T_3^p) = h_1(T_2, T_3^p) = h_1(0, T_3^p)$$

i.e. $\quad h_1(T_1, T_2^p) = f(T_2^p)$

$$\Rightarrow \quad h(T_1, T_2^p) = T_1 f(T_2^p) + h(T_1^p, T_2^p)$$

$$T_1 f(T_2^p) + h(T_1^p,T_2^p) + (T_1+T_2) f(T_3^p) + h(T_1^p+T_2^p, T_3^p) = T_2 f(T_3^p) + h(T_2^p,T_3^p) + T_1 f(T_2^p+T_3^p) + h(T_1^p, T_2^p+T_3^p)$$

$$T_1 f(T_2^p) + h(T_1^p,T_2^p) + T_1 f(T_3^p) + h(T_1^p+T_2^p, T_3^p) = h(T_2^p,T_3^p) + T_1 f(T_2^p+T_3^p) + h(T_1^p, T_2^p+T_3^p)$$

$$\xrightarrow{\frac{\partial}{\partial T_1}} \quad f(T_2^p) + f(T_3^p) = f(T_2^p+T_3^p) \Rightarrow f(T) = \sum_{n\geq 0} a_n T^{p^n} \sim f(T^p) = \sum_{n\geq 1} a_n T^{p^n}$$

$$h(T_1^p, T_2^p) + h(T_1^p+T_2^p, T_3^p) = h(T_2^p,T_3^p) + h(T_1^p, T_2^p+T_3^p)$$

<u>Claim</u>: every deformation is cohomological equivalent to

$\Delta \sim$ linear combination of

$$B_p(T_1,T_2)^{p^n}, \quad T_1^{p^n} T_2^{p^m}, \quad m \geq n+1$$

Up to now, we have seen that the deformation of $\mathbb{G}_m$ over char 0 field is trivial. what about char p ?
We have seen that $\mathbb{G}_a$ admits some non-trivial deformation over positive char, does this hold for $\mathbb{G}_m$?

Recall what we have done:

Suppose G is a smooth affine algebraic group $/k$. we consider the deformation of morphism:

$$m : G\times_k G \longrightarrow G$$

because Def_G is all contained in the deformation of this group law, suppose we picked up a deformation

$$m' : G_R \times_R G_R \longrightarrow G_R$$

we want this one satisfies the associativity law. so we consider the obstruction to be associative:

$$G_R \times_R G_R \times G_R \xrightarrow{m'(m'\times 1) - m'(1\times m')} G_R$$

$\rightsquigarrow$

$$\begin{tikzcd} & (A\otimes_k A\otimes_k A)\otimes_k m_R \arrow[d] \\ A\otimes_k R \arrow[r, "\varphi"] \arrow[d] & (A\otimes_k A\otimes_k A)\otimes_k R \arrow[d] \\ A \arrow[r] \arrow[u, bend left] & \mathrm{Spec}\,k \arrow[r] & A\otimes_k A\otimes_k A \end{tikzcd}$$

the trivial def

$\pi : \mathrm{Spec}\,k \hookrightarrow \mathrm{Spec}\,R$

$$\begin{tikzcd} \varphi : G_R\times_R G_R \times G_R \arrow[r, "{(m'(m'\times 1),\, m'(1\times m'))}"] \arrow[d] & G_R\times G_R \arrow[r, "m"] \arrow[d] & G_R \arrow[d] \\ \pi_* G \times \pi_* G\times \pi_* G \arrow[r] & \pi_* G\times \pi_* G \arrow[r] & \pi_* G \end{tikzcd}$$

$$\Rightarrow \quad G_R \times G_R \times G_R \longrightarrow \ker(G_R \to \pi_* G)$$
$$\simeq \mathrm{Der}(\varepsilon : \mathrm{Spec}\,R \to G)$$
$$\simeq \mathrm{Lie}(G_k)\otimes_k \ker\pi$$

S/R'

$$\begin{array}{ccc} G'(S) & \to & \pi_* G(S) \\ \| & & \| \\ \mathrm{Hom}_{Alg}(A\otimes_R R', S) & \to & G(S\otimes_{R'} R) \\ & & \| \\ & & \mathrm{Hom}_{Alg}(A, S\otimes_{R'} R) \end{array}$$

$$\begin{array}{cc} R' & G' \\ \downarrow \pi & \\ R & G \end{array}$$

$\mathrm{Spec}\,R \xrightarrow{\pi} \mathrm{Spec}\,R'$

$$G' \to \pi_* G$$
$$\wr$$
$$\pi^* G' \to G$$
$$\|$$
$$G$$

$$\begin{tikzcd} A\otimes_R R' \arrow[r, "\varphi"] \arrow[d] & S \arrow[d] \\ A \arrow[r, "\varphi_R"] \arrow[dr] & S\otimes_{R'} R \\ & R \arrow[u] \end{tikzcd}$$

$\mathrm{Lie}(G_k)\otimes_k \ker\pi$

$\otimes_{R'} S$

$$0 \to I \to R' \xrightarrow{\pi} R \to 0$$

$$\begin{tikzcd} A \otimes_R R' \arrow[r, "\varphi"] \arrow[d] & S \arrow[d] \\ A \arrow[r, "\varphi_R"] \arrow[dr] & S \otimes_{R'} R \\ & R \arrow[u] \end{tikzcd}$$

$$\mathrm{Der}(A \xrightarrow{\varepsilon} S \otimes_{R'} R) \qquad \mathrm{Tor}_{R'}(\to S \otimes_{R'} I \to S \to S \otimes_{R'} R \to$$

$$\varphi_1 - \varphi_2 : A \otimes_R R' \to I \cdot S \simeq I \otimes_k S_k \qquad S \otimes_{R'} R \simeq S \otimes_{R'} R'/I$$

$$\left(R'/\mathfrak{m}' e_1 + \cdots + R'/\mathfrak{m}' e_i \right) \cdot S' \qquad \simeq S/IS$$

$$S \to \mathrm{Def}_{R'/R}(A \xrightarrow{\varepsilon} R \to S \otimes_{R'} R) \qquad = \sum \bar{s}_i e_i = 0$$

$$\simeq \mathrm{Lie}(G_k) \otimes_{R'} S \qquad \sum s_i \otimes e_i$$

$$\ker(G' \to \pi_* G)(S) \simeq \ker(G'_S \to \pi_*(G_{S_R}))$$

$$\Rightarrow \quad G' \times G' \times G' \to \mathrm{Lie}(G_k) \otimes_k I \rightsquigarrow \mathrm{Lie}(G_k) \otimes (A_k \otimes_k A_k \otimes_k A_k) \otimes_k I \qquad 0$$

$$H^0(G_k \times_k G_k, m^* \mathrm{Lie}_{G_k}) \otimes_k I$$

$$\simeq \mathrm{Lie}(G_k) \otimes (A_k \otimes_k A_k) \otimes_k I \qquad m$$

$$(g, h) \quad g^{-1} h^{-1}$$

$$\begin{tikzcd} G' \times G' \arrow[r, "m"] \arrow[d] & G' \arrow[d] \\ \pi_* G \times \pi_* G \arrow[r, "m"] & \pi_* G \end{tikzcd}$$

Thm: G affine smooth algebraic group$/k$, then first order deformation

$$\mathrm{Def}_G(k[I]) \simeq H^2(G, \mathrm{Lie}(G)) \otimes_k I \qquad k\text{-alg}$$

pf: $C_n(G) = \mathbb{Z}$-combination of natural transformation

$$G^n \to \mathrm{Lie}(G)$$

$$d : C_n(G) \to C_{n+1}(G) \qquad dc(g_0, \dots, g_n) = g_0 c(g_1, \dots, g_n) + \sum_{i=1}^{n} (-1)^i c(g_0, \dots, g_{i-1} g_i, \dots, g_n) + (-1)^{n+1} c(g_0, \dots, g_{n-1})$$

$$G^n \to \mathrm{Lie}(G) \sim \mathrm{Lie}(G) \otimes_k \underbrace{A \otimes_k \cdots \otimes_k A}_{n}$$

G_a : Z^2: Δ , B^2 : f

G_m : Z^2: Δ B^2: f

Theorem 2.4. *(cf. [5]).*

(1) $H^*(\mathbb{G}_a, k) = \Lambda^*(y_1, y_2, \dots) \otimes k[x_1, x_2 \dots], \quad p \neq 2.$

$$H^*(\mathbb{G}_a, k) = k[y_1, y_2, \dots], \quad p = 2$$

where each $y_i \in H^1(\mathbb{G}_a, k), x_i \in H^2(\mathbb{G}_a, k)$.

(2) *Let* $F : \mathbb{G}_a \to \mathbb{G}_a$ *be the (geometric) Frobenius endomorphism. Then*

$$F^*(x_i) = x_{i+1}, \quad F^*(y_i) = y_{i+1}.$$

(3) *The weight of* x_i *is* $-p^i$ *and of* y_i *is* $-p^{i-1}$.

(4) *If* $(\alpha \cdot -) : \mathbb{G}_a \to \mathbb{G}_a$ *denotes multiplication by* $\alpha \in k$, *then*

$$(\alpha \cdot -)^*(x_i) = \alpha^{p^i} x_i, \quad (\alpha \cdot -)^*(y_i) = \alpha^{p^{i-1}} y_i,$$

(5) $H^*(\mathbb{G}_{a(r)}, k) = \Lambda^*(y_1, \dots, y_r) \otimes k[x_1, \dots x_r], \quad p \neq 2.$

$$H^*(\mathbb{G}_{a(r)}, k) = k[y_1, \dots, y_r], \quad p = 2.$$

For reductive groups, $G_R \in \mathrm{Def}_G(R)$, trivial one, then

$m' : G_{R'} \times G_{R'} \to G_{R'}$ is

a deformation of

$m : G_R \times G_R \to G_R$

$\sim H^0(G \times_k G, \mathcal{O}_{G \times_k G}) \otimes_k \mathrm{Lie}(G) \simeq A \otimes_k A \otimes_k \mathrm{Lie}(G)$

$m'(m' \times 1) - m(1 \times m')$ is deformation of $G_R \times G_R \times G_R \to \mathrm{Spec}\, R \to G_R$ $\quad d$

$G_{R'} \times G_{R'} \times G_{R'} \to G_{R'}$

$H^0(G \times_k G \times_k G, \mathcal{O}) \otimes_k \mathrm{Lie}(G) \simeq A \otimes_k A \otimes_k A \otimes_k \mathrm{Lie}(G)$

m' associative $\Leftrightarrow d([m']) = 0$

m' trivial $\Leftrightarrow [m'] \in \mathrm{Im}$

$\Rightarrow \quad \underline{H^2(G, \mathrm{Lie}(G))}$

right derived functor for $V \to V^G$