Th 6/20

**Definite integral:**

$$\int_a^b f(x)\,dx = \lim_{|P| \to 0} \sum f(x_i)\,\Delta x_i$$

$\Rightarrow$ area under a function!

**The fundamental theorem of Calculus.**

1. Let $f$ be a continuous function on $[a, b]$, then

$$g(x) = \int_a^x f(t)\,dt \text{ is an anti-derivative of } f$$

i.e. $g'(x) = f(x)$

2. Let $F$ be an antiderivative of $f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

or,

$$\int_a^b \underbrace{F'(x)}_{\substack{\text{instantaneous} \\ \text{rate of change}}} \underbrace{dx}_{\substack{\text{instantaneous} \\ \text{change in } x}} = \underbrace{F(b) - F(a)}_{\text{total change}}$$

Recall: Anti-derivative

$F$, s.t. $F' = f$ is called an anti-derivative of $f$

anti-derivatives are not unique:

$F + 1$, $F + 100$, ... $F + C$ will all be anti-derivatives of $f$

we use the symbol

$\int f(x)\,dx$ to represent the family of anti-derivatives of $f$

We need a convenient notation for antiderivatives that makes them easy to work with. Because of the relation between antiderivatives and integrals given by the Fundamental Theorem, the notation $\int f(x)\,dx$ is traditionally used for an antiderivative of $f$ and is called an **indefinite integral**. Thus

$$\boxed{\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)}$$

For example, we can write

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \text{because} \qquad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

Distinition with definite integral.

We don't have integral region, s.t. [a, b]

**1 Table of Indefinite Integrals**

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

**EXAMPLE 4** Find $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx$ and interpret the result in terms of areas.

**SOLUTION** The Fundamental Theorem gives

$$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx = 2\,\frac{x^4}{4} - 6\,\frac{x^2}{2} + 3\tan^{-1}x\Big]_0^2$$

$$= \tfrac{1}{2}x^4 - 3x^2 + 3\tan^{-1}x\Big]_0^2$$

$$= \tfrac{1}{2}(2^4) - 3(2^2) + 3\tan^{-1}2 - 0$$

$$= -4 + 3\tan^{-1}2$$

**EXAMPLE 2** Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$.

**SOLUTION** This indefinite integral isn't immediately apparent from Table 1, so we use trigonometric identities to rewrite the function before integrating:

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right) d\theta$$

$$= \int \csc\theta\,\cot\theta\,d\theta = -\csc\theta + C$$

■

# Substitution Rule

In general, finding anti-derivatives for an arbitrary function is very difficult!
although finding derivatives is very simple (limit formula)

Recall:

given pts $x_1, \ldots, x_r$, $m_1, \ldots, m_r$ $\rightsquigarrow$ construct $P(x) = \underbrace{(x-x_1)^{m_1} \cdots (x-x_r)^{m_r}}_{\text{has zero } x_1, \ldots, x_r}$

but the reverse direction is basically impossible!
Finding zeroes of functions

Finding indefinite integral $\rightsquigarrow$ Finding solutions of equations

but aside from examples in the table, there are other tricks!

**4 The Substitution Rule** If $u = g(x)$ is a differentiable function whose range is an interval $I$ and $f$ is continuous on $I$, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

pf: Suppose $F(x) + C = \int f(g(x))\, g'(x)\, dx$

$G(u) + D = \int f(u)\, du$

$\Rightarrow G'(u) = f(u)$

$\big(G(g(x))\big)' = f'(g(x)) \cdot g'(x) = F'(x)$

$\Rightarrow G(u) \rightsquigarrow F(x)$

Another viewpoint

$$\int f'(g(x))\, g'(x)\, dx \overset{u = g(x)}{=\!=} \int f'(u)\, du$$

$$u = g(x) \Rightarrow u' = \underbrace{g'(x)}_{\uparrow} \;\text{:::}\; \frac{du}{dx} \begin{array}{l} \Leftarrow \text{ instantaneous chg in } u \\ \Leftarrow \text{ in --- chg in } x \end{array}$$

instantaneous rate of
chge of $u$, w.r.t. $x$

**EXAMPLE 1** Find $\int x^3 \cos(x^4 + 2)\, dx$.

**SOLUTION** We make the substitution $u = x^4 + 2$ because its differential is $du = 4x^3\, dx$, which, apart from the constant factor 4, occurs in the integral. Thus, using $x^3\, dx = \frac{1}{4}\, du$ and the Substitution Rule, we have

$$\begin{aligned}
\int x^3 \cos(x^4 + 2)\, dx &= \int \cos u \cdot \tfrac{1}{4}\, du = \tfrac{1}{4} \int \cos u\, du \\
&= \tfrac{1}{4} \sin u + C \\
&= \tfrac{1}{4} \sin(x^4 + 2) + C
\end{aligned}$$

Notice that at the final stage we had to return to the original variable $x$. ■

**EXAMPLE 2** Evaluate $\int \sqrt{2x + 1}\, dx$.

**SOLUTION 1** Let $u = 2x + 1$. Then $du = 2\, dx$, so $dx = \frac{1}{2}\, du$. Thus the Substitution Rule gives

$$\begin{aligned}
\int \sqrt{2x + 1}\, dx &= \int \sqrt{u} \cdot \tfrac{1}{2}\, du = \tfrac{1}{2} \int u^{1/2}\, du \\
&= \frac{1}{2} \cdot \frac{u^{3/2}}{3/2} + C = \tfrac{1}{3} u^{3/2} + C \\
&= \tfrac{1}{3}(2x + 1)^{3/2} + C
\end{aligned}$$

**SOLUTION 2** Another possible substitution is $u = \sqrt{2x + 1}$. Then

$$du = \frac{dx}{\sqrt{2x + 1}} \qquad \text{so} \qquad dx = \sqrt{2x + 1}\, du = u\, du$$

(Or observe that $u^2 = 2x + 1$, so $2u\, du = 2\, dx$.) Therefore

$$\begin{aligned}
\int \sqrt{2x + 1}\, dx &= \int u \cdot u\, du = \int u^2\, du \\
&= \frac{u^3}{3} + C = \tfrac{1}{3}(2x + 1)^{3/2} + C
\end{aligned}$$ ■

**EXAMPLE 2** Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$.

**SOLUTION** This indefinite integral isn't immediately apparent from Table 1, so we use trigonometric identities to rewrite the function before integrating:

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right) d\theta$$

$$= \int \csc\theta\,\cot\theta\,d\theta = -\csc\theta + C \qquad \blacksquare$$

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \frac{1}{\sin^2\theta}\,d(\sin\theta)$$

$$\underset{u=\sin\theta}{=\!=} \int \frac{1}{u^2}\,du = -\frac{1}{u} + C = -\frac{1}{\sin\theta} + C$$

$$= -\csc\theta + C$$

**EXAMPLE 3** Find $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

**SOLUTION** Let $u = 1 - 4x^2$. Then $du = -8x\,dx$, so $x\,dx = -\frac{1}{8}\,du$ and

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = -\frac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{8}\int u^{-1/2}\,du$$

$$= -\frac{1}{8}(2\sqrt{u}) + C = -\frac{1}{4}\sqrt{1-4x^2} + C \qquad \blacksquare$$

**EXAMPLE 4** Evaluate $\int e^{5x}\,dx$.

**SOLUTION** If we let $u = 5x$, then $du = 5\,dx$, so $dx = \frac{1}{5}\,du$. Therefore

$$\int e^{5x}\,dx = \frac{1}{5}\int e^u\,du = \frac{1}{5}e^u + C = \frac{1}{5}e^{5x} + C \qquad \blacksquare$$

**EXAMPLE 5** Find $\int \sqrt{1 + x^2}\, x^5\, dx$.

**SOLUTION** An appropriate substitution becomes more apparent if we factor $x^5$ as $x^4 \cdot x$. Let $u = 1 + x^2$. Then $du = 2x\, dx$, so $x\, dx = \frac{1}{2}\, du$. Also $x^2 = u - 1$, so $x^4 = (u - 1)^2$:

$$\begin{aligned} \int \sqrt{1 + x^2}\, x^5\, dx &= \int \sqrt{1 + x^2}\, x^4 \cdot x\, dx \\ &= \int \sqrt{u}\, (u - 1)^2 \cdot \tfrac{1}{2}\, du = \tfrac{1}{2} \int \sqrt{u}\, (u^2 - 2u + 1)\, du \\ &= \tfrac{1}{2} \int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du \\ &= \tfrac{1}{2}\left(\tfrac{2}{7}u^{7/2} - 2 \cdot \tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right) + C \\ &= \tfrac{1}{7}(1 + x^2)^{7/2} - \tfrac{2}{5}(1 + x^2)^{5/2} + \tfrac{1}{3}(1 + x^2)^{3/2} + C \end{aligned}$$

**EXAMPLE 6** Evaluate $\int \tan x\, dx$.

**SOLUTION** First we write tangent in terms of sine and cosine:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

This suggests that we should substitute $u = \cos x$, since then $du = -\sin x\, dx$ and so $\sin x\, dx = -du$:

$$\begin{aligned} \int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx = -\int \frac{1}{u}\, du \\ &= -\ln|u| + C = -\ln|\cos x| + C \end{aligned}$$





Notice that $-\ln|\cos x| = \ln(|\cos x|^{-1}) = \ln(1/|\cos x|) = \ln|\sec x|$, so the result of Example 6 can also be written as

**5** $$\boxed{\int \tan x\, dx = \ln|\sec x| + C}$$

Substitution Rule for definite integrals

When evaluating a *definite* integral by substitution, two methods are possible. One method is to evaluate the indefinite integral first and then use the Fundamental Theorem. For instance, using the result of Example 2, we have

$$\int_0^4 \sqrt{2x+1}\,dx = \int \sqrt{2x+1}\,dx\Big]_0^4$$
$$= \tfrac{1}{3}(2x+1)^{3/2}\Big]_0^4 = \tfrac{1}{3}(9)^{3/2} - \tfrac{1}{3}(1)^{3/2}$$
$$= \tfrac{1}{3}(27-1) = \tfrac{26}{3}$$

Another method, which is usually preferable, is to change the limits of integration when the variable is changed.

**6 The Substitution Rule for Definite Integrals** If $g'$ is continuous on $[a, b]$ and $f$ is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

**PROOF** Let $F$ be an antiderivative of $f$. Then, by (3), $F(g(x))$ is an antiderivative of $f(g(x))\,g'(x)$, so by Part 2 of the Fundamental Theorem, we have

$$\int_a^b f(g(x))\,g'(x)\,dx = F(g(x))\Big]_a^b = F(g(b)) - F(g(a))$$

But, applying FTC2 a second time, we also have

$$\int_{g(a)}^{g(b)} f(u)\,du = F(u)\Big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$ ■

**EXAMPLE 7** Evaluate $\int_0^4 \sqrt{2x+1}\,dx$ using (6).

**SOLUTION** Using the substitution from Solution 1 of Example 2, we have $u = 2x + 1$ and $dx = \frac{1}{2}\,du$. To find the new limits of integration we note that

$$\text{when } x = 0,\ u = 2(0) + 1 = 1 \qquad \text{and} \qquad \text{when } x = 4,\ u = 2(4) + 1 = 9$$

Therefore

$$\begin{aligned} \int_0^4 \sqrt{2x+1}\,dx &= \int_1^9 \tfrac{1}{2}\sqrt{u}\,du \\ &= \tfrac{1}{2} \cdot \tfrac{2}{3} u^{3/2} \Big]_1^9 \\ &= \tfrac{1}{3}(9^{3/2} - 1^{3/2}) = \tfrac{26}{3} \end{aligned}$$

**EXAMPLE 8** Evaluate $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$.

**SOLUTION** Let $u = 3 - 5x$. Then $du = -5\,dx$, so $dx = -\frac{1}{5}\,du$. When $x = 1$, $u = -2$ and when $x = 2$, $u = -7$. Thus

$$\begin{aligned} \int_1^2 \frac{dx}{(3-5x)^2} &= -\frac{1}{5}\int_{-2}^{-7} \frac{du}{u^2} = -\frac{1}{5}\left[-\frac{1}{u}\right]_{-2}^{-7} = \frac{1}{5u}\Bigg]_{-2}^{-7} \\ &= \frac{1}{5}\left(-\frac{1}{7} + \frac{1}{2}\right) = \frac{1}{14} \end{aligned}$$

**EXAMPLE 9** Evaluate $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$.

**SOLUTION** We let $u = \ln x$ because its differential $du = (1/x)\,dx$ occurs in the integral. When $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$. Thus

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \frac{u^2}{2}\Bigg]_0^1 = \frac{1}{2}$$

Symmetry in definite integral

## Symmetry

The next theorem uses the Substitution Rule for Definite Integrals (6) to simplify the calculation of integrals of functions that possess symmetry properties.

**7 Integrals of Symmetric Functions** Suppose $f$ is continuous on $[-a, a]$.

(a) If $f$ is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If $f$ is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

**PROOF** We split the integral in two:

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \tag{8}$$

In the first integral on the far right side we make the substitution $u = -x$. Then $du = -dx$ and when $x = -a$, $u = a$. Therefore

$$-\int_{0}^{-a} f(x)\,dx = -\int_{0}^{a} f(-u)\,(-du) = \int_{0}^{a} f(-u)\,du$$

and so Equation 8 becomes

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx \tag{9}$$

(a) If $f$ is even, then $f(-u) = f(u)$ so Equation 9 gives

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

**EXAMPLE 10** Because $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even and so

$$\int_{-2}^{2} (x^6 + 1)\,dx = 2\int_{0}^{2} (x^6 + 1)\,dx$$

$$= 2\Big[\tfrac{1}{7}x^7 + x\Big]_0^2 = 2\big(\tfrac{128}{7} + 2\big) = \tfrac{284}{7} \qquad \blacksquare$$

**EXAMPLE 11** Because $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\,dx = 0 \qquad \blacksquare$$

Applications of definite integral

Area between curves

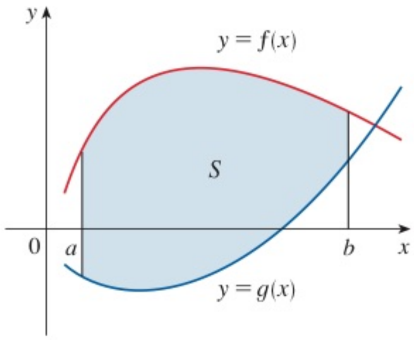



**FIGURE 1**
$S = \{(x, y) \mid a \leqslant x \leqslant b,$
$g(x) \leqslant y \leqslant f(x)\}$

How do we find the area?

## Area Between Curves: Integrating With Respect to $x$

Consider the region $S$ shown in Figure 1 that lies between two curves $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b$, where $f$ and $g$ are continuous functions and $f(x) \geqslant g(x)$ for all $x$ in $[a, b]$.

Just as we did for areas under curves in Section 5.1, we divide $S$ into $n$ strips of equal width and then we approximate the $i$th strip by a rectangle with base $\Delta x$ and height $f(x_i^*) - g(x_i^*)$. (See Figure 2. If we like, we could take all of the sample points to be right endpoints, in which case $x_i^* = x_i$.) The Riemann sum

$$\sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\, \Delta x$$

is therefore an approximation to what we intuitively think of as the area of $S$.

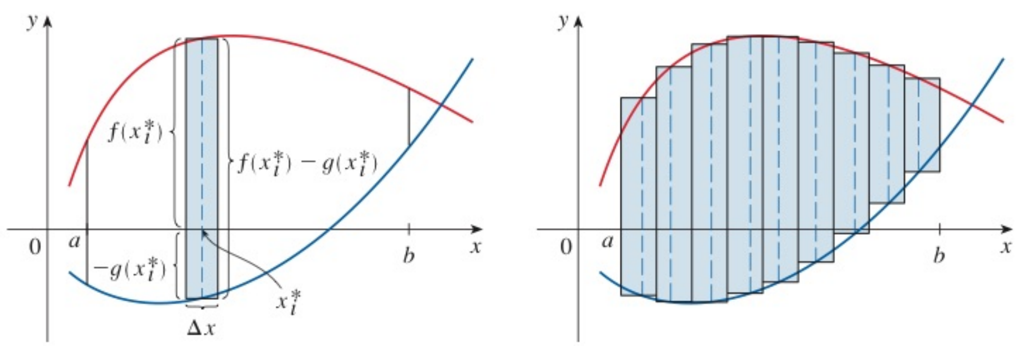



**FIGURE 2** (a) Typical rectangle (b) Approximating rectangles

This approximation appears to become better and better as $n \to \infty$. Therefore we define the **area** $A$ of the region $S$ as the limiting value of the sum of the areas of these approximating rectangles.

**1**
$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\, \Delta x$$

We recognize the limit in (1) as the definite integral of $f - g$. Therefore we have the following formula for area.

**2** The area $A$ of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where $f$ and $g$ are continuous and $f(x) \geqslant g(x)$ for all $x$ in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\, dx$$

**EXAMPLE 1** Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

**SOLUTION** The region is shown in Figure 4. The upper boundary curve is $y = e^x$ and the lower boundary curve is $y = x$. So we use the area formula (2) with $f(x) = e^x$, $g(x) = x$, $a = 0$, and $b = 1$:

$$A = \int_0^1 (e^x - x)\,dx = e^x - \tfrac{1}{2}x^2\Big]_0^1$$

$$= e - \tfrac{1}{2} - 1 = e - 1.5$$

**FIGURE 5**

**EXAMPLE 2** Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

**SOLUTION** We first find the points of intersection of the parabolas by solving their equations simultaneously. This gives $x^2 = 2x - x^2$, or $2x^2 - 2x = 0$. Thus $2x(x - 1) = 0$, so $x = 0$ or $1$. The points of intersection are $(0, 0)$ and $(1, 1)$.

We see from Figure 6 that the top and bottom boundaries are

$$y_T = 2x - x^2 \qquad \text{and} \qquad y_B = x^2$$

The area of a typical rectangle is

$$(y_T - y_B)\,\Delta x = (2x - x^2 - x^2)\,\Delta x = (2x - 2x^2)\,\Delta x$$

and the region lies between $x = 0$ and $x = 1$. So the total area is

$$A = \int_0^1 (2x - 2x^2)\,dx = 2\int_0^1 (x - x^2)\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

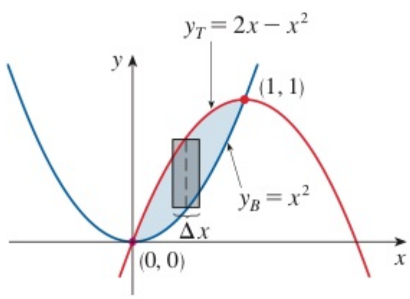



**FIGURE 6**

General steps:

- Find the integration area!

  solving equations

- Calculating the definite integral

  $\int_a^b (f(x) - g(x))\,dx$

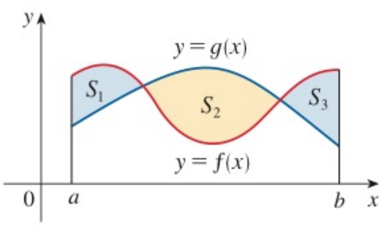



**FIGURE 8**

If we are asked to find the area between the curves $y = f(x)$ and $y = g(x)$ where $f(x) \geqslant g(x)$ for some values of $x$ but $g(x) \geqslant f(x)$ for other values of $x$, then we split the given region $S$ into several regions $S_1, S_2, \ldots$ with areas $A_1, A_2, \ldots$ as shown in Figure 8. We then define the area of the region $S$ to be the sum of the areas of the smaller regions $S_1, S_2, \ldots$, that is, $A = A_1 + A_2 + \cdots$. Since

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{when } f(x) \geqslant g(x) \\ g(x) - f(x) & \text{when } g(x) \geqslant f(x) \end{cases}$$

we have the following expression for $A$.

> **3** The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is
>
> $$A = \int_a^b |f(x) - g(x)|\, dx$$

**EXAMPLE 4** Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

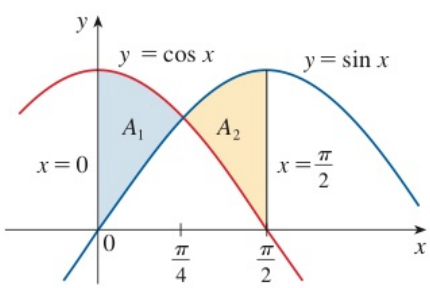



**FIGURE 9**

**SOLUTION** The points of intersection occur when $\sin x = \cos x$, that is, when $x = \pi/4$ (since $0 \leqslant x \leqslant \pi/2$). The region is sketched in Figure 9.

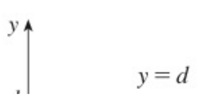
Observe that $\cos x \geqslant \sin x$ when $0 \leqslant x \leqslant \pi/4$ but $\sin x \geqslant \cos x$ when $\pi/4 \leqslant x \leqslant \pi/2$. Therefore the required area is

$$\begin{aligned} A &= \int_0^{\pi/2} |\cos x - \sin x|\, dx = A_1 + A_2 \\ &= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx \\ &= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2} \\ &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \\ &= 2\sqrt{2} - 2 \end{aligned}$$

In this particular example we could have saved some work by noticing that the region is symmetric about $x = \pi/4$ and so

$$A = 2A_1 = 2\int_0^{\pi/4} (\cos x - \sin x)\, dx$$

■



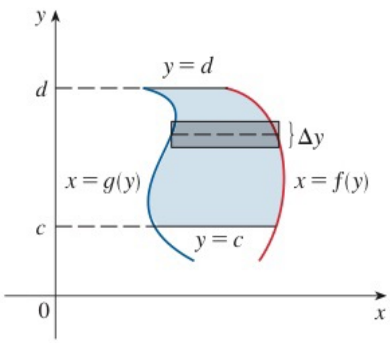


FIGURE 10

$$A = 2A_1 = 2\int_0^{\pi/4} (\cos x - \sin x)\,dx$$

## Area Between Curves: Integrating With Respect to $y$

Some regions are best treated by regarding $x$ as a function of $y$. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where $f$ and $g$ are continuous and $f(y) \geqslant g(y)$ for $c \leqslant y \leqslant d$ (see Figure 10), then its area is

$$A = \int_c^d [f(y) - g(y)]\,dy$$

**EXAMPLE 5** Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

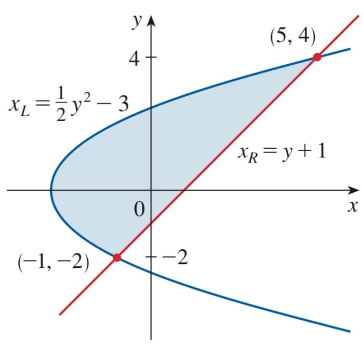


FIGURE 12

Can't be represent in x-coordinate

**SOLUTION** By solving the two equations simultaneously we find that the points of intersection are $(-1, -2)$ and $(5, 4)$. We solve the equation of the parabola for $x$ and





notice from Figure 12 that the left and right boundary curves are

$$x_L = \tfrac{1}{2}y^2 - 3 \qquad \text{and} \qquad x_R = y + 1$$

We must integrate between the appropriate $y$-values, $y = -2$ and $y = 4$. Thus

$$A = \int_{-2}^{4} (x_R - x_L)\,dy = \int_{-2}^{4} \left[(y + 1) - \left(\tfrac{1}{2}y^2 - 3\right)\right] dy$$

$$= \int_{-2}^{4} \left(-\tfrac{1}{2}y^2 + y + 4\right) dy$$

$$= -\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y \Bigg]_{-2}^{4}$$

$$= -\tfrac{1}{6}(64) + 8 + 16 - \left(\tfrac{4}{3} + 2 - 8\right) = 18$$

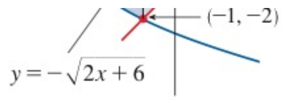



**FIGURE 13**

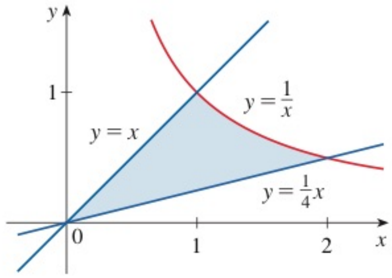



**FIGURE 14**

**EXAMPLE 6** Find the area of the region enclosed by the curves $y = 1/x$, $y = x$, and $y = \frac{1}{4}x$, using (a) $x$ as the variable of integration and (b) $y$ as the variable of integration.

**SOLUTION** The region is graphed in Figure 14.

(a) If we integrate with respect to $x$, we must split the region into two parts because the top boundary consists of two separate curves, as shown in Figure 15(a). We compute the area as

$$A = A_1 + A_2 = \int_0^1 \left(x - \tfrac{1}{4}x\right) dx + \int_1^2 \left(\frac{1}{x} - \frac{1}{4}x\right) dx$$

$$= \left[\tfrac{3}{8}x^2\right]_0^1 + \left[\ln x - \tfrac{1}{8}x^2\right]_1^2 = \ln 2$$

(b) If we integrate with respect to $y$, we also need to divide the region into two parts because the right boundary consists of two separate curves, as shown in Figure 15(b). We compute the area as

$$A = A_1 + A_2 = \int_0^{1/2} (4y - y)\, dy + \int_{1/2}^1 \left(\frac{1}{y} - y\right) dy$$

$$= \left[\tfrac{3}{2}y^2\right]_0^{1/2} + \left[\ln y - \tfrac{1}{2}y^2\right]_{1/2}^1 = \ln 2$$

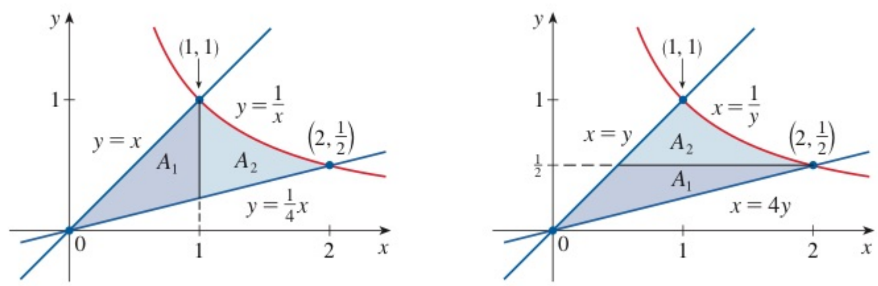



**FIGURE 15** (a) (b)

■

HW.

**51.** $\int_0^{1/\sqrt{3}} \frac{t^2 - 1}{t^4 - 1}\, dt$

**52.** $\int_0^{2} |2x - 1|\, dx$

$$\int_0^{1/\sqrt{3}} \frac{1}{t^2+1}\, dt$$

$$\int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^{2}$$

**29.** $\int \left(x - \frac{1}{x^2}\right)\left(x^2 + \frac{2}{x}\right)^5 dx$

$$u = \frac{1}{2}\left(x^2 + \frac{2}{x}\right)$$

$$du = x - \frac{1}{x^2}$$

**51.** $\int \frac{1 + x}{1 + x^2} dx$

$$\int \frac{1}{1+x^2} dx + \int \frac{x}{1+x^2} dx$$

**78.** Evaluate

$$\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right]$$

**19.** Evaluate

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\,\sqrt{n+1}} + \frac{1}{\sqrt{n}\,\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\,\sqrt{n+n}}\right)$$

**19.** $y = 12 - x^2, \quad y = x^2 - 6$

**28.** $y = \cos x, \quad y = 1 - \cos x, \quad 0 \leqslant x \leqslant \pi$