Tu 6/18

Area Problem

Silly questions: 1. What is the area of a rectangular?

2. ... area of a triangle?

3 ... area of geometric objects with stright sides
$\Rightarrow$ divide into triangles

properties of $\underline{\underline{A}}$ as a function: planar bounded object $\to$ a number

- rectangular $a, b \Rightarrow A(\square) = ab$
- $A(S_1 \sqcup S_2) = A(S_1) + A(S_2)$
- $S' \subset S, \Rightarrow A(S') \leq A(S)$

Approximation: · If S_1 & S_2 are close enough, the $A(S_1) \approx A(S_2)$

4. What is the area of a circle

answer $= \pi r^2$ (r = radius of the circle)

i.e. we give a definition:

$$A(\odot) = \pi r^2$$

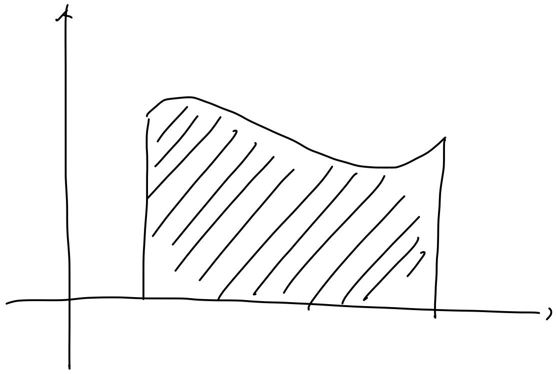
but is it compatible with the orignal function?

Goal: define the area function rigorously!

then use it to compute Area of a circle

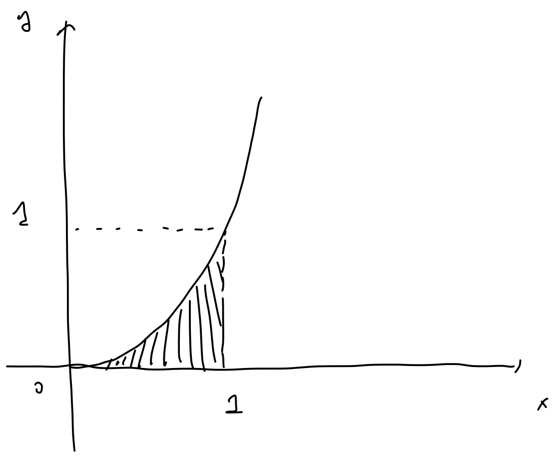
Area under a function

$f(x)$: continuous on $[a, b]$ $\Rightarrow$ continuous curve



Area of the shadowed region ?

e.g. $f(x) = x^2$, $x \in [0, 1]$



Method 1:

Possibility test: randomly choose a point in $[0,1] \times [0,1]$

$$A(S) = \text{probability of the point lies in } S$$

$$\approx \frac{S_N}{N}$$

$S_N \leftarrow$ choose n points independently
the time that point $\in S_N$

Our intuition: $A(S) = \lim_{N \to \infty} \frac{S_N}{N}$

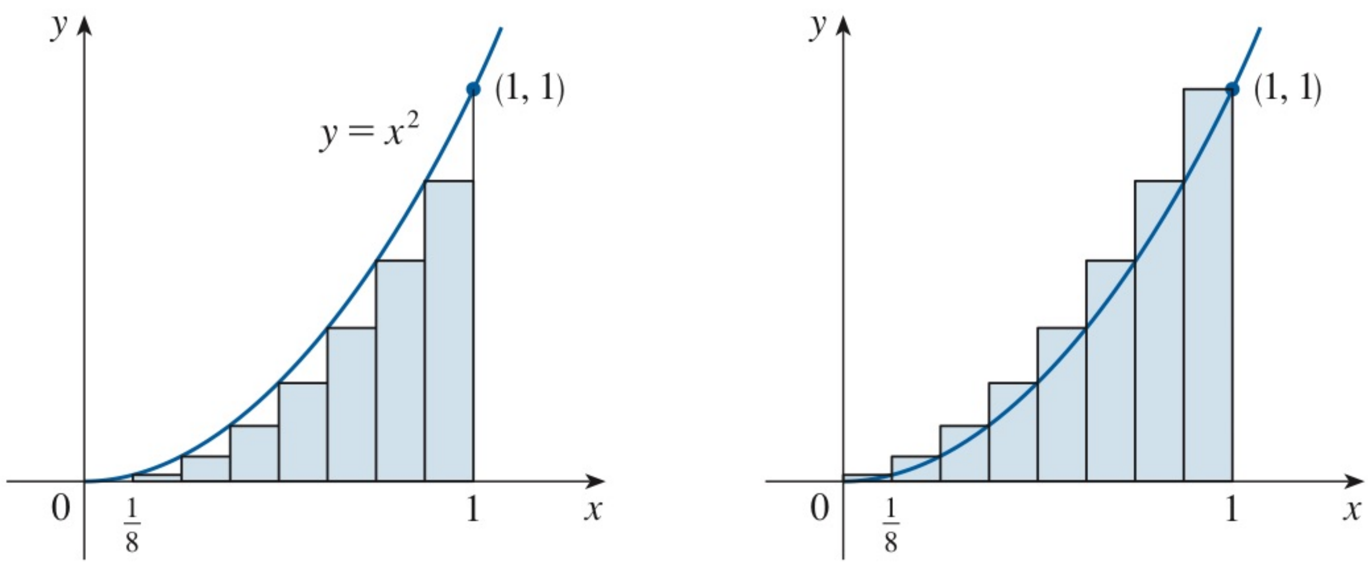
· In real life, we can only do finitely many tests, taking $N \to +\infty$ means we have to do infinitely many times! which is impossible

know all the values S_N

· But by doing test, we can only get a sequence of rational numbers

different test $\Rightarrow$ different sequence, how do we argue they have same limit?

Method 2: Approximation by rectangular



(a) Using left endpoints (b) Using right endpoints

Cut $[0,1]$ into N parts

$$[0, \tfrac{1}{N}], \quad [\tfrac{1}{N}, \tfrac{2}{N}], \quad \dots, \quad [\tfrac{N-1}{N}, 1]$$

(region using left endpoints) $\subset S \subset$ (region using right endpoints)

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$S_N^{\ell} \qquad\qquad\qquad\qquad S_N^{r}$$

$$A(S_N^{\ell}) = 0 \times \frac{1}{N} + \frac{1}{N^2} \times \frac{1}{N} + \frac{4}{N^2} \times \frac{1}{N} + \cdots + \frac{(N-1)^2}{N^2} \times \frac{1}{N}$$

$$= \sum_{i=1}^{N-1} \frac{i^2}{N^2} \times \frac{1}{N} = \frac{1}{N^3} \sum_{i=1}^{N-1} i^2$$

$$= \frac{1}{N^3} \times \frac{(N-1)N(2N-1)}{6}$$

$$= \frac{1}{3} - \frac{1}{2N} + \frac{1}{N^2}$$

$$A(S_N^r) = \frac{1}{N} \times \frac{1}{N} + \left(\frac{2}{N}\right)^2 \times \frac{1}{N} + \cdots + \left(\frac{N}{N}\right)^2 \times \frac{1}{N}$$

$$= \sum_{i=1}^{N} \frac{i^2}{N^2} \times \frac{1}{N} = \frac{1}{N^3} \times \frac{N(N+1)(2N+1)}{6}$$

$$= \frac{1}{3} + \frac{1}{2N} + \frac{1}{N^2}$$

$$A(S_N^{\ell}) \leq A(S) \leq A(S_N^r)$$

$$\frac{1}{3} - \frac{1}{2N} + \frac{1}{N^2} \leq A(S) \leq \frac{1}{3} + \frac{1}{2N} + \frac{1}{N^2}$$

for all $N \geq 1$, taking $N \to +\infty$

$$\frac{1}{3} \leq A(S) \leq \frac{1}{3} \Rightarrow A(S) = \frac{1}{3}$$

EXAMPLE 3 Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.

e.g.

(a) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.
(b) Estimate the area by taking the sample points to be midpoints and using four subintervals and then ten subintervals.

SOLUTION

(a) Since $a = 0$ and $b = 2$, the width of a subinterval is

$$\Delta x = \frac{2 - 0}{n} = \frac{2}{n}$$

So $x_1 = 2/n$, $x_2 = 4/n$, $x_3 = 6/n$, $x_i = 2i/n$, and $x_n = 2n/n$. The sum of the areas of the approximating rectangles is

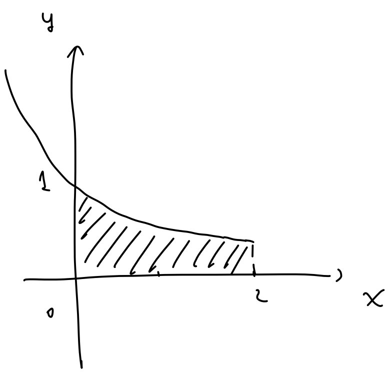
$$\begin{aligned} R_n &= f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x \\ &= e^{-x_1}\,\Delta x + e^{-x_2}\,\Delta x + \cdots + e^{-x_n}\,\Delta x \\ &= e^{-2/n}\left(\frac{2}{n}\right) + e^{-4/n}\left(\frac{2}{n}\right) + \cdots + e^{-2n/n}\left(\frac{2}{n}\right) \end{aligned}$$

According to Definition 2, the area is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2}{n}\left(e^{-2/n} + e^{-4/n} + e^{-6/n} + \cdots + e^{-2n/n}\right)$$

Using sigma notation we could write

$$A = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} e^{-2i/n}$$



$$\begin{aligned} A(S^{l}_{N}) &= \sum_{i=1}^{N} \frac{2}{N} f\left(\frac{2(i-1)}{N}\right) \\ &= \sum_{i=1}^{N} \frac{2}{N} e^{\frac{2(1-i)}{N}} \\ &= \frac{2}{N} e^{\frac{2}{N}} \cdot \sum_{i=1}^{N} e^{-\frac{2i}{N}} \\ &= \frac{2}{N} e^{\frac{2}{N}} \cdot e^{-\frac{2}{N}} \cdot \frac{e^{-2}-1}{e^{-\frac{2}{N}}-1} \\ &= 2(1-e^{-2}) \cdot \frac{1}{N(1-e^{-\frac{2}{N}})} \end{aligned}$$

$$A(S^{r}_{N}) = 2(1-e^{-2}) \cdot e^{-\frac{2}{N}} \cdot \frac{1}{N(1-e^{-\frac{2}{N}})}$$

$$\lim_{N \to +\infty} \frac{1}{N\left(1-e^{-\frac{2}{N}}\right)}$$

Calculating: $\lim_{x \to +\infty} \frac{1}{x\left(1-e^{-\frac{2}{x}}\right)}$

$$\overset{t=\frac{1}{x}}{=\!=} \lim_{t \to 0^+} \frac{t}{1-e^{-2t}}$$

$$\overset{\text{L'Hospital}}{=} \lim_{t \to 0^+} \frac{1}{2e^{-2t}} = \frac{1}{2}$$

$$2\left(1-e^{-2}\right) \cdot e^{-\frac{2}{N}} \cdot \frac{1}{N\left(1-e^{-\frac{2}{N}}\right)} \leq A(S) \leq 2\left(1-e^{-2}\right) \cdot \frac{1}{N\left(1-e^{-\frac{2}{N}}\right)}$$

$$1-e^{-2} \leq A(S) \leq 1-e^{-2}$$

$$\Rightarrow A(S) = 1-e^{-2}$$

For more general regions, we can also do this

Cut $[a, b]$ into N parts:

$$[a, a+\frac{b-a}{N}], \quad \ldots, \quad [a+\frac{(N-1)(b-a)}{N}, b]$$

$$I_1 \qquad\qquad I_N$$

then choose a sample point $x_i^* \in I_i$ $\Rightarrow$ region "near" S

$$\sum_{i=1}^{N} \frac{b-a}{N} f(x_i^*) \rightsquigarrow \text{an approximation for } A(S)$$

S_N^{lower} = take x_i^* to be the absolute minimum pt in I_i

S_N^{upper} = take x_i^* to be the absolute maximum pt in I_i

$$\Rightarrow \quad S_N^{lower} \subset S \subset S_N^{upper}$$

$$\Rightarrow \quad A(S_N^{lower}) \leq A(S) \leq A(S_N^{upper})$$

<u>Thm</u>: $\lim_{N\to+\infty} A(S_N^{lower}) = \lim_{N\to+\infty} A(S_N^{upper})$

if f is continuous

2 Definition of a Definite Integral If f is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let x_0 $(= a)$, $x_1, x_2, \ldots, x_n$ $(= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of *f* from *a* to *b*** is

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

NOTE 1 The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and was chosen because an integral is a limit of sums. In the notation $\int_a^b f(x)\,dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**. For now, the symbol dx has no meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol. The dx simply indicates that the independent variable is x. The procedure of calculating an integral is called **integration**.

NOTE 2 The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

NOTE 3 The sum

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

that occurs in Definition 2 is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826–1866). So Definition 2 says that the definite integral of an integrable function can be approximated to within any desired degree of accuracy by a Riemann sum.

We can form the Riemann sum for any functions
but limit may not exist !!

We have defined the definite integral for an integrable function, but not all functions are integrable (see Exercises 81–82). The following theorem shows that the most commonly occurring functions are in fact integrable. The theorem is proved in more advanced courses.

3 Theorem If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

Moreover, if f is integrable over $[a,b]$,

we can divide $[a,b]$ aribitrarily, as long as

subintervals $I_i \longrightarrow 0$,

the pt x_i^* can also be chosen aribitrarily!

If f takes on both positive and negative values, as in Figure 3, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the *negatives* of the areas of the rectangles that lie below the x-axis (the areas of the blue rectangles *minus* the areas of the gold rectangles). When we take the limit of such Riemann sums, we get the situation illustrated in Figure 4. A definite integral can be interpreted as a **net area**, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.

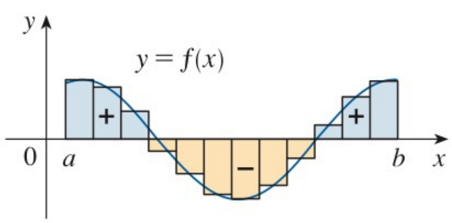


FIGURE 3
$\sum f(x_i^*)\,\Delta x$ is an approximation to the net area.

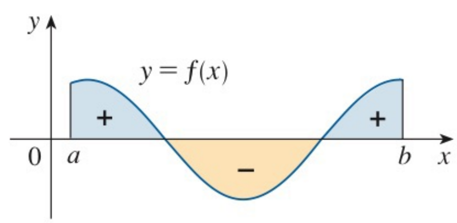


FIGURE 4
$\int_a^b f(x)\,dx$ is the net area.

e.g. $$\int_0^1 x^2\,dx = \frac{1}{3}$$

$$\int_0^2 e^{-x}\,dx = 1 - e^{-2}$$

Basic properties

When we defined the definite integral $\int_a^b f(x)\,dx$, we implicitly assumed that $a < b$. But the definition as a limit of Riemann sums makes sense even if $a > b$. Notice that if we interchange a and b, then Δx changes from $(b - a)/n$ to $(a - b)/n$. Therefore

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

If $a = b$, then $\Delta x = 0$ and so

$$\int_a^a f(x)\,dx = 0$$

Properties of the Integral

1. $\displaystyle\int_a^b c\,dx = c(b - a)$, where c is any constant

2. $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant

4. $\displaystyle\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

5. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

Comparison Properties of the Integral

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\displaystyle\int_a^b f(x)\,dx \geqslant 0$.

7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\displaystyle\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$

Back to the area problem

If we have a rectangular $a \times b$

consider $f(x) = b$, $0 \leq x \leq a$

$$A(\square) = \int_0^a f(x)\,dx = \int_0^a b\,dx = b \cdot \int_0^a 1\,dx$$

$$= ab$$

EXAMPLE 7 Use the properties of integrals to evaluate $\int_0^1 (4 + 3x^2)\,dx$.

SOLUTION Using Properties 2 and 3 of integrals, we have

$$\int_0^1 (4 + 3x^2)\,dx = \int_0^1 4\,dx + \int_0^1 3x^2\,dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$$

We know from Property 1 that

$$\int_0^1 4\,dx = 4(1 - 0) = 4$$

and we found in Example 5.1.2 that $\int_0^1 x^2\,dx = \frac{1}{3}$. So

$$\int_0^1 (4 + 3x^2)\,dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$$

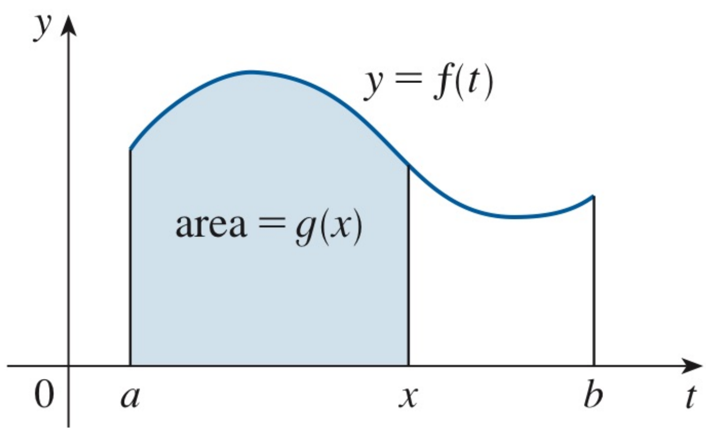
$$= 4 + 3 \cdot \tfrac{1}{3} = 5$$

The fundamental theorem of Calculus

· The "area so far" function

Give a continuous function f on $[a,b]$, define

$$g(x) = \int_a^x f(t)\,dt$$



e.g. $f(t) = t$,

$$g(x) = \int_0^x t\,dt$$

$\frac{N}{2}$

$$= \lim_{N\to+\infty} \sum_{i=1}^{N} f\left(\frac{ix}{N}\right)\cdot\frac{x}{N}$$

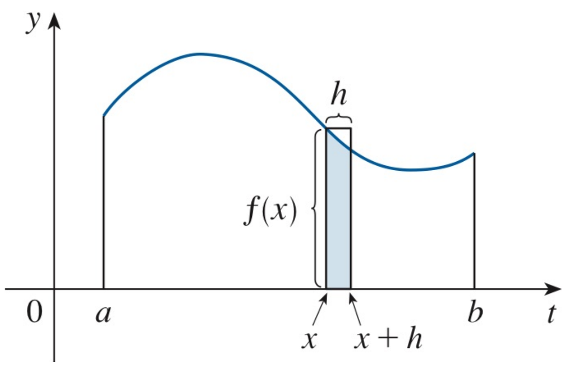
$$= \lim_{N\to+\infty} \frac{x}{N}\cdot\sum_{i=1}^{N}\frac{ix}{N} = x^2\cdot\lim_{N\to+\infty}\frac{1}{N^2}\cdot\sum_{i=1}^{N} i$$

$$= x^2\cdot\lim_{N\to\infty}\left(\frac{1}{2}+\frac{1}{2N}\right) = \frac{1}{2}x^2$$

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.



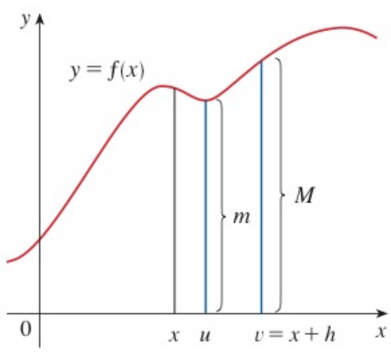


FIGURE 6

and so, for $h \neq 0$,

$$\boxed{2} \qquad \frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

For now let's assume that $h > 0$. Since f is continuous on $[x, x + h]$, the Extreme Value Theorem says that there are numbers u and v in $[x, x + h]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x, x + h]$. (See Figure 6.)

By Property 8 of integrals, we have

$$mh \leqslant \int_x^{x+h} f(t)\,dt \leqslant Mh$$

that is,

$$f(u)h \leqslant \int_x^{x+h} f(t)\,dt \leqslant f(v)h$$

Since $h > 0$, we can divide this inequality by h:

$$f(u) \leqslant \frac{1}{h}\int_x^{x+h} f(t)\,dt \leqslant f(v)$$

Now we use Equation 2 to replace the middle part of this inequality:

$$\boxed{3} \qquad f(u) \leqslant \frac{g(x+h) - g(x)}{h} \leqslant f(v)$$

Inequality 3 can be proved in a similar manner for the case where $h < 0$. (See Exercise 87.)

Now we let $h \to 0$. Then $u \to x$ and $v \to x$ because u and v lie between x and $x + h$. Therefore

$$\lim_{h \to 0} f(u) = \lim_{u \to x} f(u) = f(x) \qquad \text{and} \qquad \lim_{h \to 0} f(v) = \lim_{v \to x} f(v) = f(x)$$

because f is continuous at x. We conclude, from (3) and the Squeeze Theorem, that

$$\boxed{4} \qquad g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x)$$

If $x = a$ or b, then Equation 4 can be interpreted as a one-sided limit. Then Theorem 2.8.4 (modified for one-sided limits) shows that g is continuous on $[a, b]$. ■

Review on anti-derivative

if $F' = f \Rightarrow F$ is called a anti-derivative

all the others are of the form $F + C$

e.g. $x^n \sim \frac{1}{n+1} x^{n+1} + C \quad (n \neq -1)$

$x^{-1} \sim \ln|x| + C$

$a^x \sim \frac{a^x}{\ln a} + C$

$\sin x \sim -\cos x + C$

$\cos x \sim \sin x + C$

e.g.: $\sqrt{1-x^2} \sim \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2}$

EXAMPLE 2 Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}\, dt$.

SOLUTION Since $f(t) = \sqrt{1 + t^2}$ is continuous, Part 1 of the Fundamental Theorem of Calculus gives

$$g'(x) = \sqrt{1 + x^2}$$

■

EXAMPLE 4 Find $\dfrac{d}{dx} \displaystyle\int_1^{x^4} \sec t\, dt$.

SOLUTION Here we have to be careful to use the Chain Rule in conjunction with FTC1. Let $u = x^4$. Then

$$\begin{aligned}
\frac{d}{dx} \int_1^{x^4} \sec t\, dt &= \frac{d}{dx} \int_1^{u} \sec t\, dt \\
&= \frac{d}{du} \left[\int_1^{u} \sec t\, dt \right] \frac{du}{dx} && \text{(by the Chain Rule)} \\
&= \sec u \, \frac{du}{dx} && \text{(by FTC1)} \\
&= \sec(x^4) \cdot 4x^3
\end{aligned}$$

■

The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function F such that $F' = f$.

PROOF Let $g(x) = \int_a^x f(t)\,dt$. We know from Part 1 that $g'(x) = f(x)$; that is, g is an antiderivative of f. If F is any other antiderivative of f on $[a, b]$, then we know from Corollary 4.2.7 that F and g differ by a constant:

$$F(x) = g(x) + C \tag{6}$$

for $a < x < b$. But both F and g are continuous on $[a, b]$ and so, by taking limits of both sides of Equation 6 (as $x \to a^+$ and $x \to b^-$), we see that it also holds when $x = a$ and $x = b$. So $F(x) = g(x) + C$ for all x in $[a, b]$.

If we put $x = a$ in the formula for $g(x)$, we get

$$g(a) = \int_a^a f(t)\,dt = 0$$

So, using Equation 6 with $x = b$ and $x = a$, we have

$$\begin{aligned} F(b) - F(a) &= [g(b) + C] - [g(a) + C] \\ &= g(b) - g(a) = g(b) = \int_a^b f(t)\,dt \end{aligned}$$

■

<u>e.g.</u> $\int_0^1 x^2\,dx$

$\int_0^2 e^{-x}\,dx$

EXAMPLE 5 Evaluate the integral $\int_1^3 e^x\,dx$.

SOLUTION The function $f(x) = e^x$ is continuous everywhere and we know that an antiderivative is $F(x) = e^x$, so Part 2 of the Fundamental Theorem gives

$$\int_1^3 e^x\,dx = F(3) - F(1) = e^3 - e$$

Notice that FTC2 says we can use *any* antiderivative F of f. So we may as well use the simplest one, namely $F(x) = e^x$, instead of $e^x + 7$ or $e^x + C$. ■

EXAMPLE 7 Evaluate $\displaystyle\int_3^6 \frac{dx}{x}$.

SOLUTION The given integral is another way of writing

$$\int_3^6 \frac{1}{x}\,dx$$

An antiderivative of $f(x) = 1/x$ is $F(x) = \ln|x|$ and, because $3 \leqslant x \leqslant 6$, we can write $F(x) = \ln x$. So

$$\int_3^6 \frac{1}{x}\,dx = \ln x\Big]_3^6 = \ln 6 - \ln 3 = \ln\frac{6}{3} = \ln 2$$ ■

EXAMPLE 8 Find the area under the cosine curve from 0 to b, where $0 \leqslant b \leqslant \pi/2$.

SOLUTION Since an antiderivative of $f(x) = \cos x$ is $F(x) = \sin x$, we have

$$A = \int_0^b \cos x\,dx = \sin x\Big]_0^b = \sin b - \sin 0 = \sin b$$

In particular, taking $b = \pi/2$, we have proved that the area under the cosine curve from 0 to $\pi/2$ is $\sin(\pi/2) = 1$. (See Figure 9.) ■

f continuous on $[a, b]$!

EXAMPLE 9 What is wrong with the following calculation?

$$\int_{-1}^3 \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Bigg]_{-1}^3 = -\frac{1}{3} - 1 = -\frac{4}{3}$$

SOLUTION To start, we notice that this calculation must be wrong because the answer is negative but $f(x) = 1/x^2 \geqslant 0$ and Property 6 of integrals says that $\int_a^b f(x)\,dx \geqslant 0$ when $f \geqslant 0$. The Fundamental Theorem of Calculus applies to continuous functions. It can't be applied here because $f(x) = 1/x^2$ is not continuous on $[-1, 3]$. In fact, f has an infinite discontinuity at $x = 0$, and we will see in Section 7.8 that

$$\int_{-1}^3 \frac{1}{x^2}\,dx \qquad \text{does not exist.}$$ ■

Area of the circle

unit circle: $x^2 + y^2 = 1$.

we only calculate the part in Quadraph I.

$$f(x) = \sqrt{1-x^2}, \quad 0 \leq x \leq 1$$

$$\int_0^1 \sqrt{1-x^2}\, dx = \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} \Bigg|_0^1$$

$$= \frac{1}{2} \times \frac{\pi}{2} + 0 = \frac{\pi}{4}$$