Tu 6/13

Optimization

Applications to real life problem: make some quantity as large/small as possible

### Steps In Solving Optimization Problems

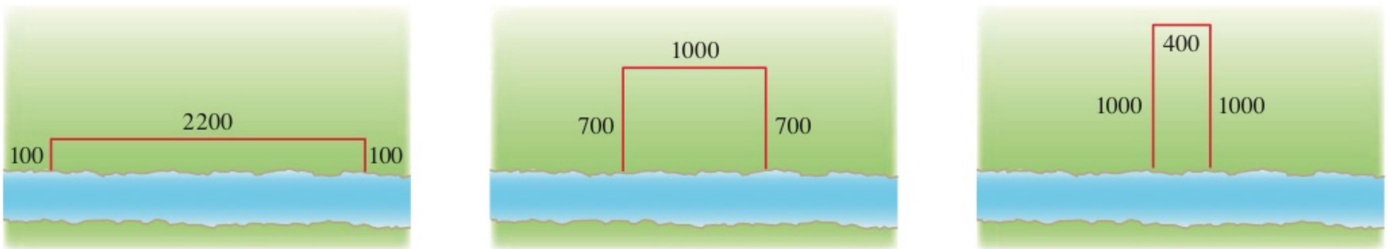
1. **Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
2. **Draw a Diagram** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.
3. **Introduce Notation** Assign a symbol to the quantity that is to be maximized or minimized (let's call it $Q$ for now). Also select symbols ($a, b, c, \ldots, x, y$) for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, $A$ for area, $h$ for height, $t$ for time.
4. Express $Q$ in terms of some of the other symbols from Step 3.
5. If $Q$ has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for $Q$. Thus $Q$ will be expressed as a function of *one* variable $x$, say, $Q = f(x)$. Write the domain of this function in the given context.
6. Use the methods of Sections 4.1 and 4.3 to find the *absolute* maximum or minimum value of $f$. In particular, if the domain of $f$ is a closed interval, then the Closed Interval Method in Section 4.1 can be used.

**EXAMPLE 1** A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

of the field that has the largest area?

PS Understand the problem
PS Analogy: Try special cases
PS Draw diagrams

**SOLUTION** In order to get a feeling for what is happening in this problem, let's experiment with some specific cases. Figure 1 (not to scale) shows three possible ways of laying out the 2400 ft of fencing.




Area = 100 · 2200 = 220,000 ft$^2$ Area = 700 · 1000 = 700,000 ft$^2$ Area = 1000 · 400 = 400,000 ft$^2$

**FIGURE 1**

We see that when we try shallow, wide fields or deep, narrow fields, we get relatively small areas. It seems plausible that there is some intermediate configuration that produces the largest area.

PS Introduce notation

Figure 2 illustrates the general case. We wish to maximize the area $A$ of the rectangle. Let $x$ and $y$ be the depth and width of the rectangle (in feet). Then we express $A$ in terms of $x$ and $y$:

$$A = xy$$

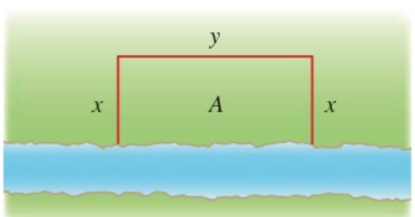



**FIGURE 2**

We want to express $A$ as a function of just one variable, so we eliminate $y$ by expressing it in terms of $x$. To do this we use the given information that the total length of the fencing is 2400 ft. Thus

$$2x + y = 2400$$

From this equation we have $y = 2400 - 2x$, which gives

$$A = xy = x(2400 - 2x) = 2400x - 2x^2$$

Note that the largest $x$ can be is 1200 (this uses all the fence for the depth and none for the width) and $x$ can't be negative, so the function that we wish to maximize is

$$A(x) = 2400x - 2x^2 \qquad 0 \leqslant x \leqslant 1200$$

The derivative is $A'(x) = 2400 - 4x$, so to find the critical numbers we solve the equation

$$2400 - 4x = 0$$

which gives $x = 600$. The maximum value of $A$ must occur either at this critical number or at an endpoint of the interval. Since $A(0) = 0$, $A(600) = 720,000$, and $A(1200) = 0$, the Closed Interval Method gives the maximum value as $A(600) = 720,000$.

[Alternatively, we could have observed that $A''(x) = -4 < 0$ for all $x$, so $A$ is always concave downward and the local maximum at $x = 600$ must be an absolute maximum.]

The corresponding $y$-value is $y = 2400 - 2(600) = 1200$, so the rectangular field should be 600 ft deep and 1200 ft wide. ■

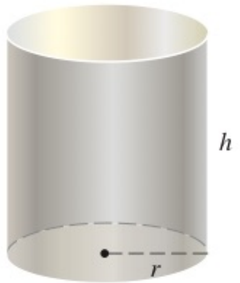



**FIGURE 3**

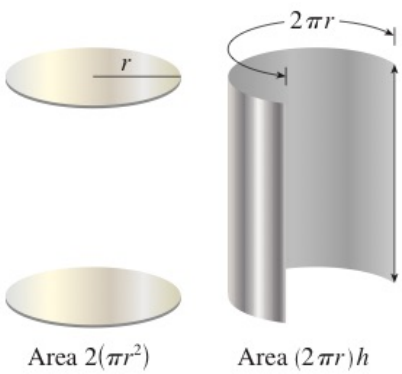



**FIGURE 4**

**EXAMPLE 2** A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

**SOLUTION** Draw a diagram as in Figure 3, where $r$ is the radius and $h$ the height (both in centimeters). In order to minimize the cost of the metal, we minimize the total surface area of the cylinder (top, bottom, and sides). From Figure 4 we see that the sides are made from a rectangular sheet with dimensions $2\pi r$ and $h$. So the surface area is

$$A = 2\pi r^2 + 2\pi rh$$

We would like to express $A$ in terms of one variable, $r$. To eliminate $h$ we use the fact that the volume is given as 1 L, which is equivalent to 1000 cm$^3$. Thus

$$\pi r^2 h = 1000$$

which gives $h = 1000/(\pi r^2)$. Substitution of this into the expression for $A$ gives

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) = 2\pi r^2 + \frac{2000}{r}$$

We know that $r$ must be positive, and there are no limitations on how large $r$ can be. Therefore the function that we want to minimize is

$$A(r) = 2\pi r^2 + \frac{2000}{r} \qquad r > 0$$

To find the critical numbers, we differentiate:

$$A'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}$$

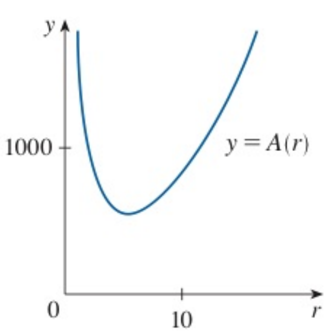



**FIGURE 5**

Then $A'(r) = 0$ when $\pi r^3 = 500$, so the only critical number is $r = \sqrt[3]{500/\pi}$.

Since the domain of $A$ is $(0, \infty)$, we can't use the argument of Example 1 concerning endpoints. But we can observe that $A'(r) < 0$ for $r < \sqrt[3]{500/\pi}$ and $A'(r) > 0$ for $r > \sqrt[3]{500/\pi}$, so $A$ is decreasing for *all* $r$ to the left of the critical number and increasing for *all* $r$ to the right. Thus $r = \sqrt[3]{500/\pi}$ must give rise to an *absolute* minimum.

[Alternatively, we could argue that $A(r) \to \infty$ as $r \to 0^+$ and $A(r) \to \infty$ as $r \to \infty$, so there must be a minimum value of $A(r)$, which must occur at the critical number. See Figure 5.]

The value of $h$ corresponding to $r = \sqrt[3]{500/\pi}$ is

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi(500/\pi)^{2/3}} = 2\sqrt[3]{\frac{500}{\pi}} = 2r$$

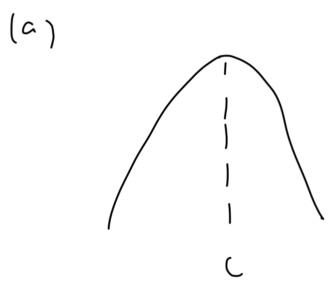
Thus, to minimize the cost of the can, the radius should be $\sqrt[3]{500/\pi}$ cm and the height should be equal to twice the radius, namely, the diameter. ■

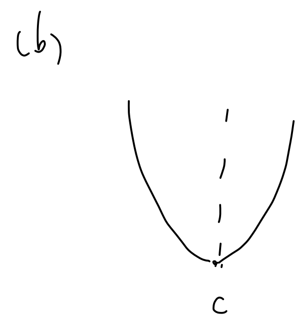
In the Applied Project following this section we investigate the most economical shape for a can by taking into account other manufacturing costs.

**First Derivative Test for Absolute Extreme Values** Suppose that $c$ is a critical number of a continuous function $f$ defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of $f$.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of $f$.



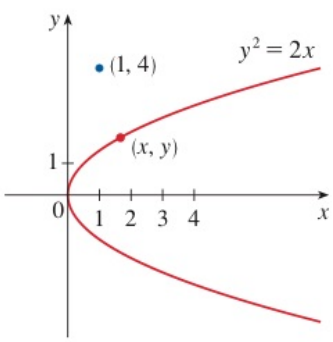




**FIGURE 6**

**EXAMPLE 3** Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

**SOLUTION** The distance between the point $(1, 4)$ and the point $(x, y)$ is

$$d = \sqrt{(x - 1)^2 + (y - 4)^2}$$

(See Figure 6.) But if $(x, y)$ lies on the parabola, then $x = \frac{1}{2}y^2$, so the expression for $d$ becomes

$$d = \sqrt{\left(\tfrac{1}{2}y^2 - 1\right)^2 + (y - 4)^2}$$

(Alternatively, we could have substituted $y = \sqrt{2x}$ to get $d$ in terms of $x$ alone.)

Instead of minimizing $d$, we minimize its square:

$$d^2 = f(y) = \left(\tfrac{1}{2}y^2 - 1\right)^2 + (y - 4)^2$$

(You should convince yourself that the minimum of $d$ occurs at the same point as the minimum of $d^2$, but $d^2$ is easier to work with.) Note that there is no restriction on $y$, so the domain is all real numbers. Differentiating, we obtain

$$f'(y) = 2\left(\tfrac{1}{2}y^2 - 1\right)y + 2(y - 4) = y^3 - 8$$

so $f'(y) = 0$ when $y = 2$. Observe that $f'(y) < 0$ when $y < 2$ and $f'(y) > 0$ when $y > 2$, so by the First Derivative Test for Absolute Extreme Values, the absolute minimum occurs when $y = 2$. (Or we could simply say that because of the geometric nature of the problem, it's obvious that there is a closest point but not a farthest point.) The corresponding value of $x$ is $x = \frac{1}{2}y^2 = 2$. Thus the point on $y^2 = 2x$ closest to $(1, 4)$ is $(2, 2)$. [The distance between the points is $d = \sqrt{f(2)} = \sqrt{5}$.] ■

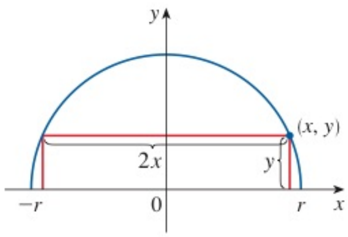



FIGURE 9

**EXAMPLE 5** Find the area of the largest rectangle that can be inscribed in a semicircle of radius $r$.

**SOLUTION 1** Let's take the semicircle to be the upper half of the circle $x^2 + y^2 = r^2$ with center the origin. Then the word *inscribed* means that the rectangle has two vertices on the semicircle and two vertices on the $x$-axis as shown in Figure 9.

Let $(x, y)$ be the vertex that lies in the first quadrant. Then the rectangle has sides of lengths $2x$ and $y$, so its area is

$$A = 2xy$$





To eliminate $y$ we use the fact that $(x, y)$ lies on the circle $x^2 + y^2 = r^2$ and so $y = \sqrt{r^2 - x^2}$. Thus

$$A = 2x\sqrt{r^2 - x^2}$$

The domain of this function is $0 \leqslant x \leqslant r$. Its derivative is

$$A' = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$$

which is 0 when $2x^2 = r^2$, that is, $x = r/\sqrt{2}$ (since $x \geqslant 0$). This value of $x$ gives a maximum value of $A$ since $A(0) = 0$ and $A(r) = 0$. Therefore the area of the largest inscribed rectangle is

$$A\left(\frac{r}{\sqrt{2}}\right) = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = r^2$$

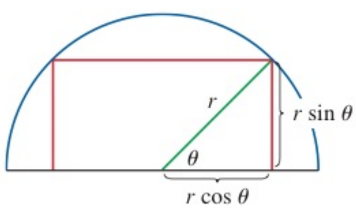



FIGURE 10

**SOLUTION 2** A simpler solution is possible if we think of using an angle as a variable. Let $\theta$ be the angle shown in Figure 10. Then the area of the rectangle is

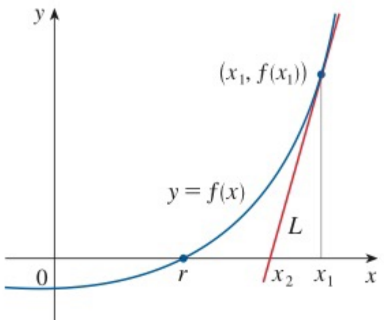
$$A(\theta) = (2r\cos\theta)(r\sin\theta) = r^2(2\sin\theta\cos\theta) = r^2\sin 2\theta$$

We know that $\sin 2\theta$ has a maximum value of 1 and it occurs when $2\theta = \pi/2$. So $A(\theta)$ has a maximum value of $r^2$ and it occurs when $\theta = \pi/4$.

Notice that this trigonometric solution doesn't involve differentiation. In fact, we didn't need to use calculus at all. ■

# Newton's Method :

Approximate solutions $f(x) = 0$




The geometry behind Newton's method is shown in Figure 2. We wish to solve an equation of the form $f(x) = 0$, so the solutions of the equation correspond to the $x$-intercepts of the graph of $f$. The solution that we are trying to find is labeled $r$ in the figure. We start with a first approximation $x_1$, which is obtained by guessing, or from a rough sketch of the graph of $f$, or from a computer-generated graph of $f$. Consider the tangent line $L$ to the curve $y = f(x)$ at the point $(x_1, f(x_1))$ and look at the $x$-intercept of $L$, labeled $x_2$. The idea behind Newton's method is that the tangent line is close to the curve and so its $x$-intercept, $x_2$, is close to the $x$-intercept of the curve (namely, the solution $r$ that we are seeking). Because the tangent is a line, we can easily find its $x$-intercept.

To find a formula for $x_2$ in terms of $x_1$ we use the fact that the slope of $L$ is $f'(x_1)$, so its equation is

$$y - f(x_1) = f'(x_1)(x - x_1)$$

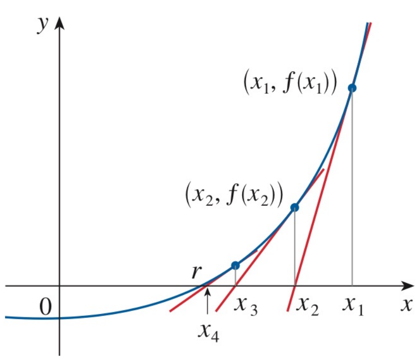
Since the $x$-intercept of $L$ is $x_2$, we know that the point $(x_2, 0)$ is on the line, and so

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

If $f'(x_1) \neq 0$, we can solve this equation for $x_2$:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We use $x_2$ as a second approximation to $r$.




Next we repeat this procedure with $x_1$ replaced by the second approximation $x_2$, using the tangent line at $(x_2, f(x_2))$. This gives a third approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

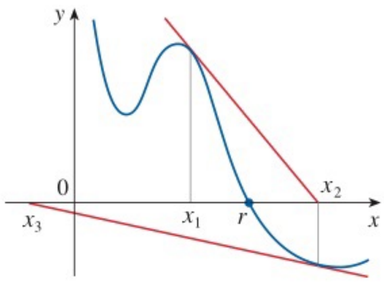
If we keep repeating this process, we obtain a sequence of approximations $x_1, x_2, x_3, x_4, \ldots$ as shown in Figure 3. In general, if the $n$th approximation is $x_n$ and $f'(x_n) \neq 0$, then the next approximation is given by

**2**
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If the numbers $x_n$ become closer and closer to $r$ as $n$ becomes large, then we say that the sequence *converges* to $r$ and we write

$$\lim_{n \to \infty} x_n = r$$

WARNING:




Although the sequence of successive approximations converges to the desired solution for functions of the type illustrated in Figure 3, in certain circumstances the sequence may not converge. For example, consider the situation shown in Figure 4. You can see that $x_2$ is a worse approximation than $x_1$. This is likely to be the case when $f'(x_1)$ is close to 0. It might even happen that an approximation (such as $x_3$ in Figure 4) falls outside the domain of $f$. Then Newton's method fails and a better initial approximation $x_1$ should be chosen. See Exercises 31–34 for specific examples in which Newton's method works very slowly or does not work at all.

**EXAMPLE 1** Starting with $x_1 = 2$, find the third approximation $x_3$ to the solution of the equation $x^3 - 2x - 5 = 0$.

**SOLUTION** We apply Newton's method with

$$f(x) = x^3 - 2x - 5 \qquad \text{and} \qquad f'(x) = 3x^2 - 2$$

Newton himself used this equation to illustrate his method and he chose $x_1 = 2$ after some experimentation because $f(1) = -6$, $f(2) = -1$, and $f(3) = 16$. Equation 2 becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

With $n = 1$ we have

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2}$$

$$= 2 - \frac{2^3 - 2(2) - 5}{3(2)^2 - 2} = 2.1$$

Then with $n = 2$ we obtain

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2} = 2.1 - \frac{(2.1)^3 - 2(2.1) - 5}{3(2.1)^2 - 2} \approx 2.0946$$

It turns out that this third approximation $x_3 \approx 2.0946$ is accurate to four decimal places. ■

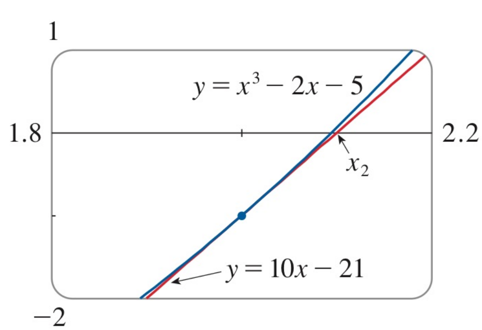



**FIGURE 5**

**EXAMPLE 2** Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places.

**SOLUTION** First we observe that finding $\sqrt[6]{2}$ is equivalent to finding the positive solution of the equation

$$x^6 - 2 = 0$$

so we take $f(x) = x^6 - 2$. Then $f'(x) = 6x^5$ and Formula 2 (Newton's method) becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^6 - 2}{6x_n^5}$$

If we choose $x_1 = 1$ as the initial approximation, then we obtain

$$x_2 \approx 1.16666667$$

$$x_3 \approx 1.12644368$$

$$x_4 \approx 1.12249707$$

$$x_5 \approx 1.12246205$$

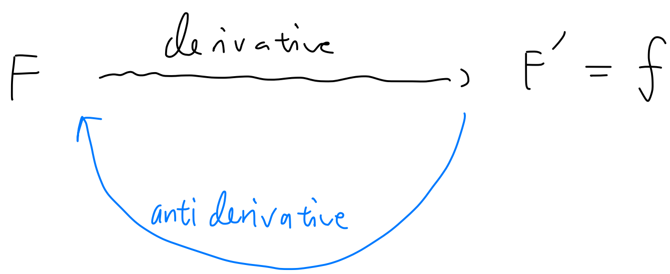
$$x_6 \approx 1.12246205$$

Since $x_5$ and $x_6$ agree to eight decimal places, we conclude that

$$\sqrt[6]{2} \approx 1.12246205$$

to eight decimal places. ■

# Antiderivatives




If we have a function $F$ whose derivative is the function $f$, then $F$ is called an *antiderivative* of $f$.

> **Definition** A function $F$ is called an **antiderivative** of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

e.g. $f(x) = x^2 \Rightarrow F(x) = \frac{1}{3}x^3$

$F(x) = \frac{1}{3}x^3 + 1$

$\frac{1}{3}x^3 + 100$ } are antiderivatives of $f$

$\rightsquigarrow \frac{1}{3}x^3 + C$ ($C$ is a constant)

Q: Are there others?

> **1 Theorem** If $F$ is an antiderivative of $f$ on an interval $I$, then the most general antiderivative of $f$ on $I$ is
>
> $$F(x) + C$$
>
> where $C$ is an arbitrary constant.

$\Rightarrow$ taking antiderivatives give a "family of functions", not just one function

**EXAMPLE 1** Find the most general antiderivative of each of the following functions.

(a) $f(x) = \sin x$ (b) $f(x) = 1/x$ (c) $f(x) = x^n, \quad n \neq -1$

## Antidifferentiation Formulas

As in Example 1, every differentiation formula, when read from right to left, gives rise to an antidifferentiation formula. In Table 2 we list some particular antiderivatives. Each formula in the table is true because the derivative of the function in the right column appears in the left column. In particular, the first formula says that the antiderivative of a constant times a function is the constant times the antiderivative of the function. The second formula says that the antiderivative of a sum is the sum of the antiderivatives. (We use the notation $F' = f$, $G' = g$.)

**2 Table of Antidifferentiation Formulas**

| Function | Particular antiderivative | Function | Particular antiderivative |
|---|---|---|---|
| $cf(x)$ | $cF(x)$ | $\sin x$ | $-\cos x$ |
| $f(x) + g(x)$ | $F(x) + G(x)$ | $\sec^2 x$ | $\tan x$ |
| $x^n \ (n \neq -1)$ | $\dfrac{x^{n+1}}{n+1}$ | $\sec x \tan x$ | $\sec x$ |
| $\dfrac{1}{x}$ | $\ln \lvert x \rvert$ | $\dfrac{1}{\sqrt{1-x^2}}$ | $\sin^{-1} x$ |
| $e^x$ | $e^x$ | $\dfrac{1}{1+x^2}$ | $\tan^{-1} x$ |
| $b^x$ | $\dfrac{b^x}{\ln b}$ | $\cosh x$ | $\sinh x$ |
| $\cos x$ | $\sin x$ | $\sinh x$ | $\cosh x$ |

**EXAMPLE 2** Find all functions $g$ such that

$$g'(x) = 4 \sin x + \frac{2x^5 - \sqrt{x}}{x}$$

**SOLUTION** We first rewrite the given function as follows:

$$g'(x) = 4 \sin x + \frac{2x^5}{x} - \frac{\sqrt{x}}{x} = 4 \sin x + 2x^4 - \frac{1}{\sqrt{x}}$$

Thus we want to find an antiderivative of

$$g'(x) = 4 \sin x + 2x^4 - x^{-1/2}$$

Using the formulas in Table 2 together with Theorem 1, we obtain

$$g(x) = 4(-\cos x) + 2\,\frac{x^5}{5} - \frac{x^{1/2}}{\frac{1}{2}} + C$$

$$= -4 \cos x + \tfrac{2}{5}x^5 - 2\sqrt{x} + C$$

■

**EXAMPLE 3** Find $f$ if $f'(x) = e^x + 20(1 + x^2)^{-1}$ and $f(0) = -2$.

**SOLUTION** The general antiderivative of

$$f'(x) = e^x + \frac{20}{1 + x^2}$$

is

$$f(x) = e^x + 20 \tan^{-1}x + C$$

To determine $C$ we use the fact that $f(0) = -2$:

$$f(0) = e^0 + 20 \tan^{-1} 0 + C = -2$$

Thus we have $C = -2 - 1 = -3$, so the particular solution is

$$f(x) = e^x + 20 \tan^{-1}x - 3$$

■

**EXAMPLE 4** Find $f$ if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, and $f(1) = 1$.

**SOLUTION** The general antiderivative of $f''(x) = 12x^2 + 6x - 4$ is

$$f'(x) = 12\frac{x^3}{3} + 6\frac{x^2}{2} - 4x + C = 4x^3 + 3x^2 - 4x + C$$

Using the antidifferentiation rules once more, we find that

$$f(x) = 4\frac{x^4}{4} + 3\frac{x^3}{3} - 4\frac{x^2}{2} + Cx + D = x^4 + x^3 - 2x^2 + Cx + D$$

To determine $C$ and $D$ we use the given conditions that $f(0) = 4$ and $f(1) = 1$. Since $f(0) = 0 + D = 4$, we have $D = 4$. Since

$$f(1) = 1 + 1 - 2 + C + 4 = 1$$

we have $C = -3$. Therefore the required function is

$$f(x) = x^4 + x^3 - 2x^2 - 3x + 4$$

■

<u>Remark</u>: Given a function involving

elementary function { Algebraic operations between polynomials, exponential / log, trigonometric / inverse-trigonometric; A composition

$\Rightarrow$ we can find its derivative

But in general, find antiderivative of elementary functions is very difficult! and anti-derivative may not be elementary any more, so we don't know how to express it

Applications

**EXAMPLE 6** A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

**SOLUTION** Since $v'(t) = a(t) = 6t + 4$, antidifferentiation gives

$$v(t) = 6\frac{t^2}{2} + 4t + C = 3t^2 + 4t + C$$

Note that $v(0) = C$. But we are given that $v(0) = -6$, so $C = -6$ and

$$v(t) = 3t^2 + 4t - 6$$

Since $v(t) = s'(t)$, $s$ is the antiderivative of $v$:

$$s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D$$

This gives $s(0) = D$. We are given that $s(0) = 9$, so $D = 9$ and the required position function is

$$s(t) = t^3 + 2t^2 - 6t + 9$$

■

An object near the surface of the earth is subject to a gravitational force that produces a downward acceleration denoted by $g$. For motion close to the ground we may assume that $g$ is constant, its value being about $9.8\ \text{m/s}^2$ (or $32\ \text{ft/s}^2$). It is remarkable that from the single fact that the acceleration due to gravity is constant, we can use calculus to deduce the position and velocity of any object moving under the force of gravity, as illustrated in the next example.

**EXAMPLE 7** A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff, 432 ft above the ground. Find its height above the ground $t$ seconds later. When does it reach its maximum height? When does it hit the ground?

**SOLUTION** The motion is vertical and we choose the positive direction to be upward. At time $t$ the distance above the ground is $s(t)$ and the velocity $v(t)$ is decreasing. Therefore the acceleration must be negative and we have

$$a(t) = \frac{dv}{dt} = -32$$

Taking antiderivatives, we have

$$v(t) = -32t + C$$

To determine $C$ we use the given information that $v(0) = 48$. This gives $48 = 0 + C$, so

$$v(t) = -32t + 48$$

The maximum height is reached when $v(t) = 0$, that is, after 1.5 seconds. Since $s'(t) = v(t)$, we antidifferentiate again and obtain

$$s(t) = -16t^2 + 48t + D$$

Using the fact that $s(0) = 432$, we have $432 = 0 + D$ and so

$$s(t) = -16t^2 + 48t + 432$$

The expression for $s(t)$ is valid until the ball hits the ground. This happens when $s(t) = 0$, that is, when

$$-16t^2 + 48t + 432 = 0$$

or, equivalently,

$$t^2 - 3t - 27 = 0$$

Using the quadratic formula to solve this equation, we get

$$t = \frac{3 \pm 3\sqrt{13}}{2}$$

We reject the solution with the minus sign because it gives a negative value for $t$. Therefore the ball hits the ground after $3(1 + \sqrt{13})/2 \approx 6.9$ seconds. ■

**11.** Consider the following problem: a farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?

(a) Draw several diagrams illustrating the situation, some with shallow, wide pens and some with deep, narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate it.

(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.

(c) Write an expression for the total area.

(d) Use the given information to write an equation that relates the variables.

(e) Use part (d) to write the total area as a function of one variable.

(f) Finish solving the problem and compare the answer with your estimate in part (a).