Tu 6/11

Review: Concept: absolute / local maximum / minimum

Externe Value thm:

3 The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

4 Fermat's Theorem If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

6 Definition A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

The Closed Interval Method To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

The mean value theorem

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a) \tag{2}$$

Graphs of functions with the help of derivatives

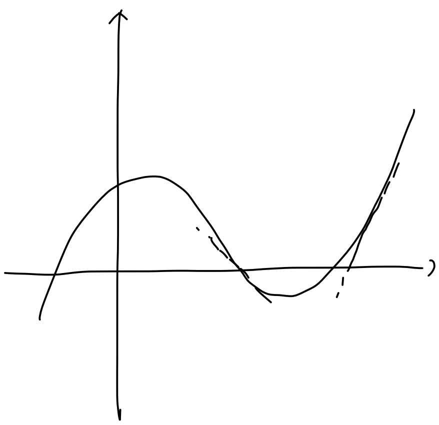
Instantaneous rate of change!

Increasing/Decreasing Test Let $[a,b]$ be an interval s.t. f is differentiable

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Geometric picture:

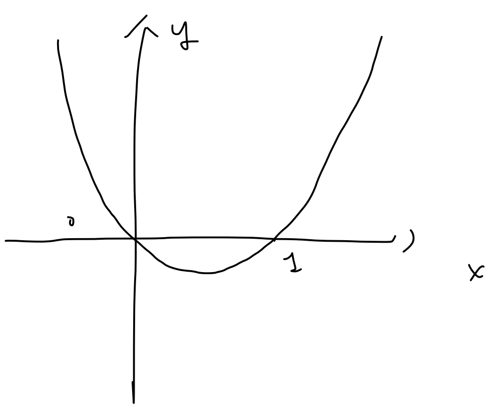


pf: If $x_2 > x_1$, then

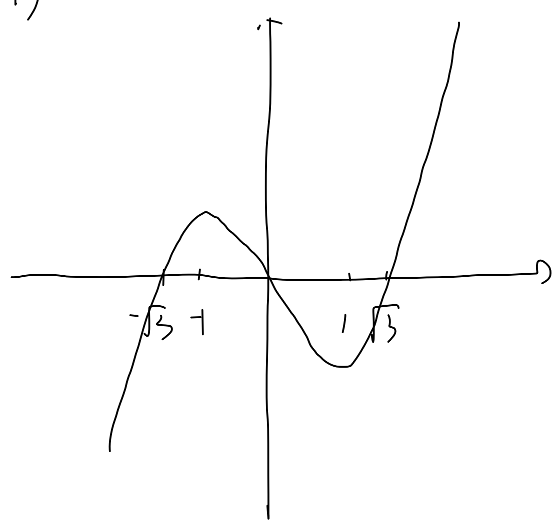
$$f(x_2) - f(x_1) = f'(c) \cdot (x_2 - x_1)$$

e.g. Find where the function $f(x) = x^3 - 3x$ is increasing / decreasing

$$f'(x) = 3x^2 - 3x = 3x(x-1)$$



$\Rightarrow$



EXAMPLE 1 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

$f'(x) < 0$ for $0 < x < 2$, so f is decreasing on $(0, 2)$. (It would also be true to say that f is decreasing on the closed interval $[0, 2]$.)

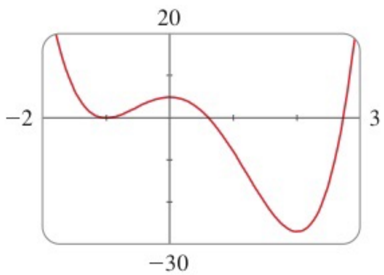


FIGURE 3

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

The graph of f shown in Figure 3 confirms the information in the chart. ■

Locate local max/min

Q: Given a function f & a critical point c of f
How do we know it is a local max/min or not?

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

e.g. $f(x) = x^3$, $f'(x) = 3x^2 \geq 0$

EXAMPLE 2 Find the local minimum and maximum values of the function f in Example 1.

SOLUTION From the chart in the solution to Example 1 we see that $f'(x)$ changes from negative to positive at -1, so $f(-1) = 0$ is a local minimum value by the First Derivative Test. Similarly, f' changes from negative to positive at 2, so $f(2) = -27$ is also a local minimum value. As noted previously, $f(0) = 5$ is a local maximum value because $f'(x)$ changes from positive to negative at 0. ■

EXAMPLE 3 Find the local maximum and minimum values of the function

$$g(x) = x + 2\sin x \qquad 0 \leqslant x \leqslant 2\pi$$

SOLUTION As in Example 1, we start by finding the critical numbers. The derivative is:

$$g'(x) = 1 + 2\cos x$$

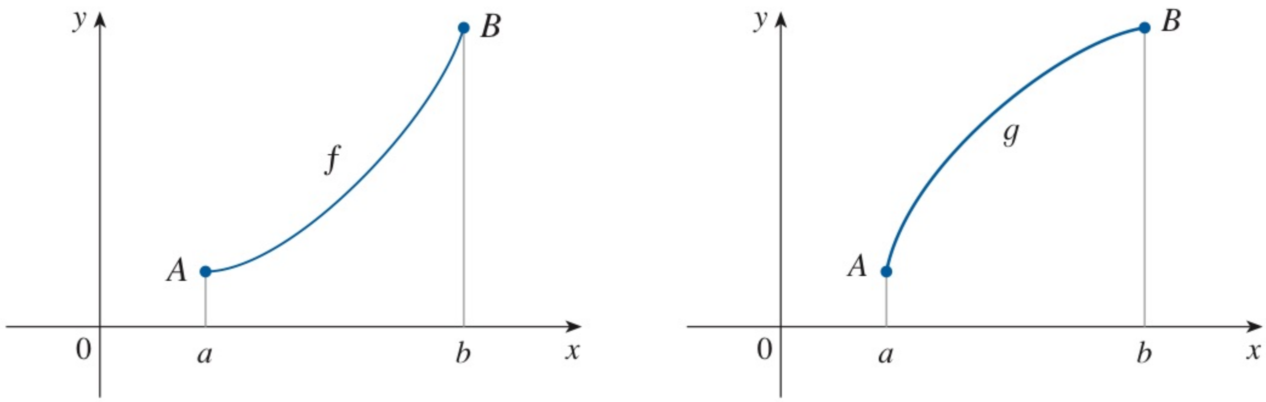
so $g'(x) = 0$ when $\cos x = -\frac{1}{2}$. The solutions of this equation are $2\pi/3$ and $4\pi/3$. Because g is differentiable everywhere, the only critical numbers are $2\pi/3$ and $4\pi/3$. We split the domain into intervals according to the critical numbers. Within each interval, $g'(x)$ is either always positive or always negative and so we analyze g in the following chart.

Interval	$g'(x) = 1 + 2\cos x$	g
$0 < x < 2\pi/3$	$+$	increasing on $(0, 2\pi/3)$
$2\pi/3 < x < 4\pi/3$	$-$	decreasing on $(2\pi/3, 4\pi/3)$
$4\pi/3 < x < 2\pi$	$+$	increasing on $(4\pi/3, 2\pi)$

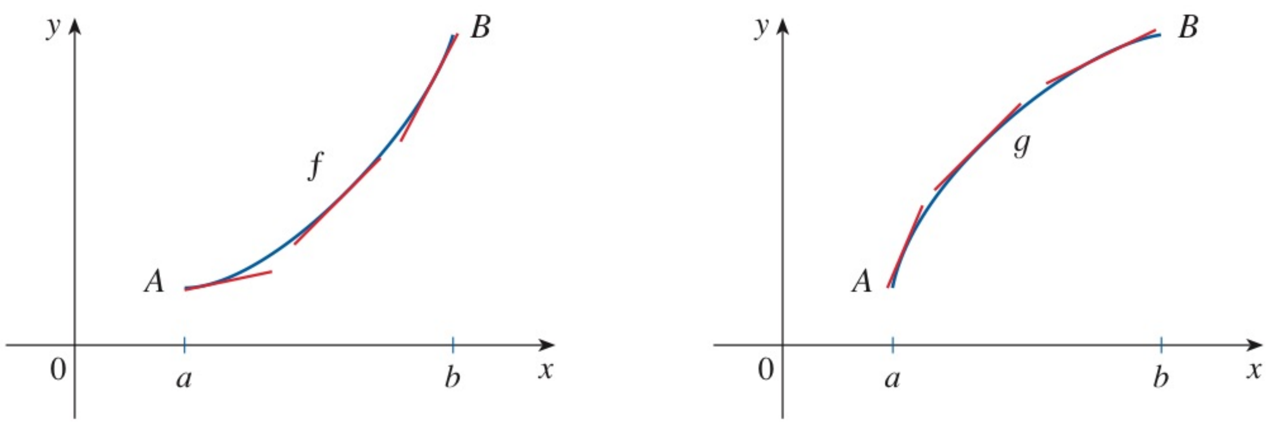
The second derivative f''

- Concavity

Figure 6 shows the graphs of two increasing functions on (a, b). Both graphs join point to point B but they look different because they bend in different directions. How ca we distinguish between these two types of behavior?



Concave upward   Concave downward



(a) Concave upward   (b) Concave downward

Definition If the graph of f lies above all of its tangents on an interval I, then f is called **concave upward** on I. If the graph of f lies below all of its tangents on I, then f is called **concave downward** on I.

or f lies below secant lines $\Rightarrow$ upward

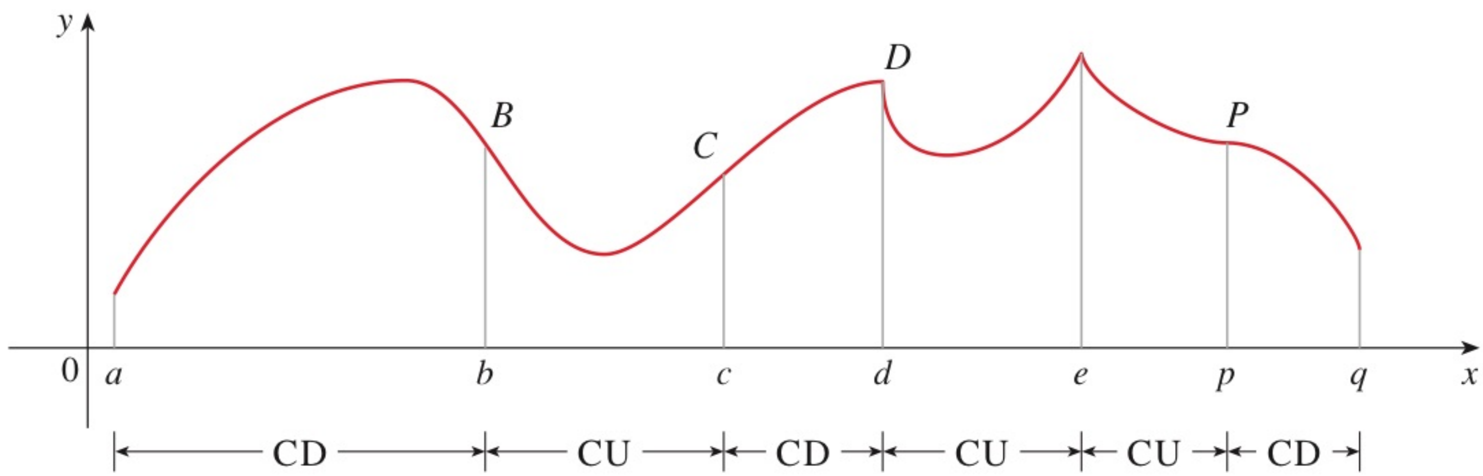
f lies above secant lines $\Rightarrow$ downward

Concavity Test

(a) If $f''(x) > 0$ on an interval I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ on an interval I, then the graph of f is concave downward on I.

Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.



• The second derivative test

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

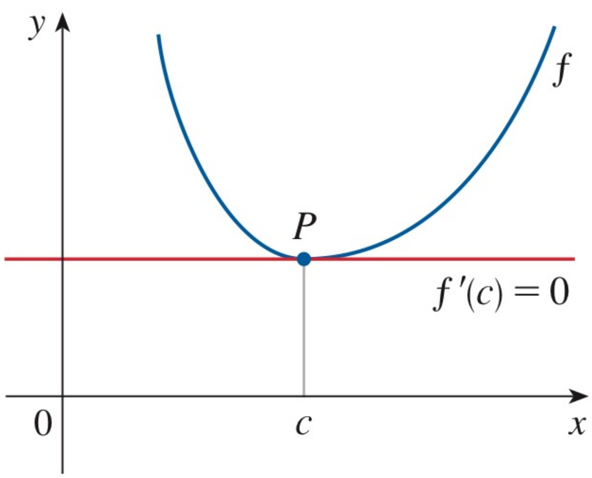
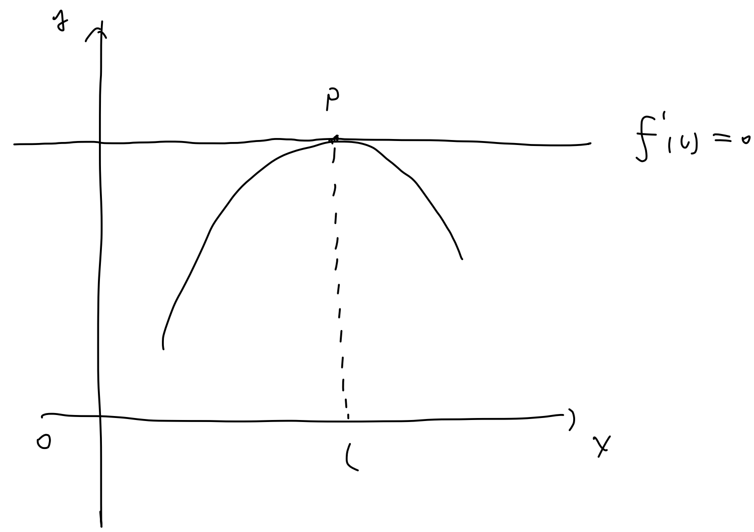


FIGURE 11
$f''(c) > 0$, f is concave upward



$f''(c) < 0$, f is concave down

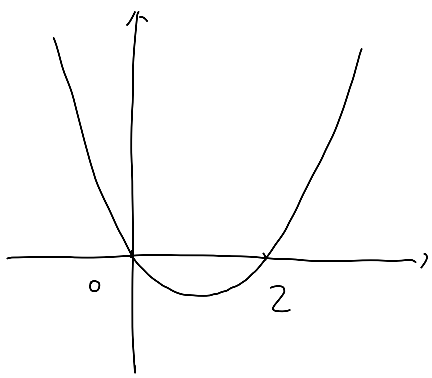
EXAMPLE 6 Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima.

$$f(x) = x^4 - 4x^3$$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

· concavity & inflection pts



$(-\infty, 0)$:	$f'' > 0$	CU
0 :	$f'' = 0$	inflection pt
$(0, 2)$:	$f'' < 0$	CD
2	$f'' = 0$	inflection pt
$(2, +\infty)$:	$f'' > 0$	CU

· local max / min

Critical pt: $f'(x) = 0 \Rightarrow x = 0$ or 3

$\underline{f''(0) = 0},$ $\quad$ $\underline{f''(3) = 36 > 0}$

$\downarrow$

local min

$f'(x)$ doesn't change sign at $x = 0$

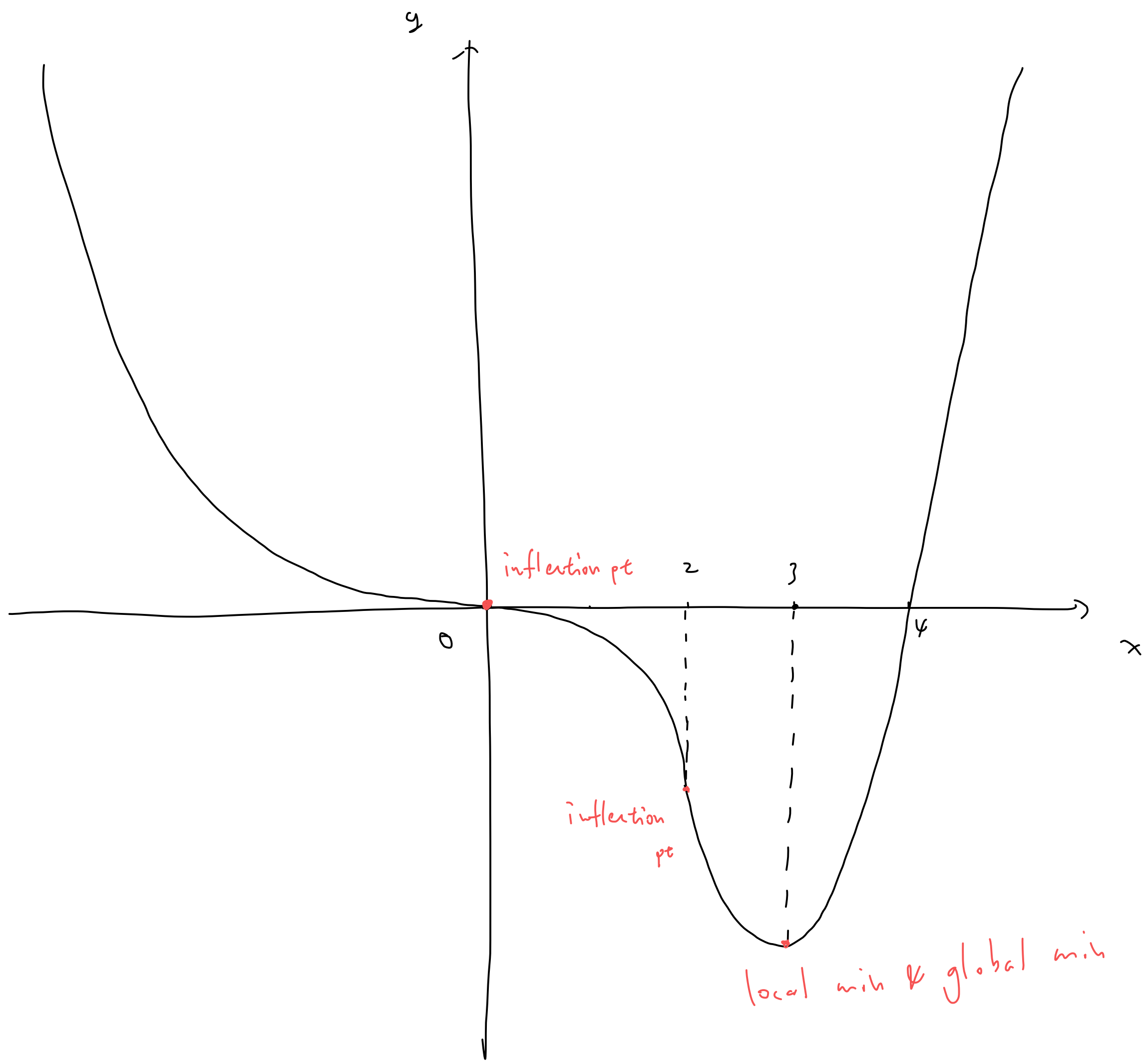
$\Downarrow$

$f'(x) < 0 \Rightarrow$ alway decreasing

$f(3) = 81 - 4 \times 27 = -27$

Sketch the graph

	$(-\infty, 0)$	$(0, 2)$	$(2, 3)$	$(3, +\infty)$
$f'(x)$	$-$	$-$	$-$	$+$
Concavity	CU	CD	CD	CU



Sketch graph of a function:

- Find critical pt: $f'(x)=0$ & $f'(x)$ doesn't exist
- Identify local min/max
- Find inflection pt: $f''(x)=0$
- Analyze concavity

EXAMPLE 7 Sketch the graph of the function $f(x) = x^{2/3}(6-x)^{1/3}$.

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}(6-x)^{\frac{1}{3}} + \frac{1}{3}x^{\frac{2}{3}}\cdot(6-x)^{-\frac{2}{3}}\cdot(-1)$$

$$= x^{-\frac{1}{3}}(6-x)^{-\frac{2}{3}}\left(\frac{2}{3}(6-x) - \frac{1}{3}x\right)$$

$$= \frac{4-x}{x^{\frac{1}{3}}(6-x)^{\frac{2}{3}}} \Rightarrow \text{critical pt: } \underline{\underline{0,\ 6,}}\ \underline{4}$$

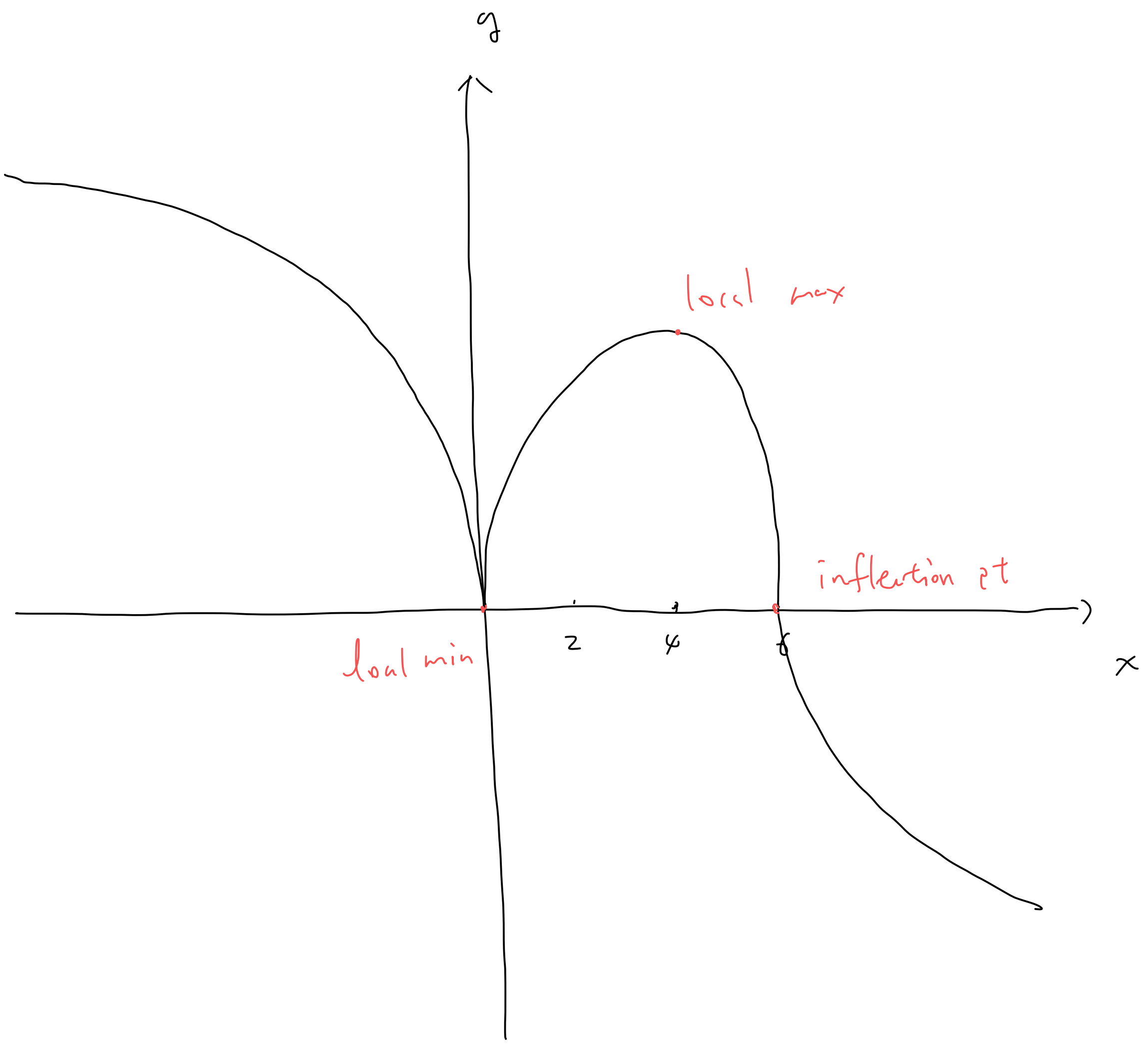
0, 6: doesn't exist; 4: $f'(4)=0$

local max ↓ 4

$$f''(x) = \frac{-8}{x^{\frac{4}{3}}(6-x)^{\frac{5}{3}}}$$

	$(-\infty, 0)$	$(0, 4)$	$(4, 6)$	$(6, +\infty)$
$f'(x)$	$-$	$+$	$-$	$-$
$f''(x)$	$-$	$-$	$-$	$+$
Concavity	CD	CD	CD	CU

$$\lim_{x \to 0^-} f'(x) = -\infty, \quad \lim_{x \to 0^+} f'(x) = +\infty$$



Intermediate form & l'Hospital's rule

L'Hospital's Rule

We now introduce a systematic method, known as *l'Hospital's Rule*, for the evaluation of indeterminate forms of type $\frac{0}{0}$ or type $\frac{\infty}{\infty}$.

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

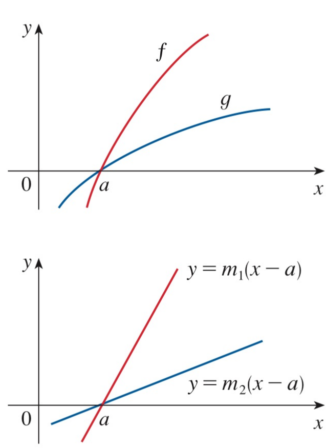
(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

e.g. $F(x) = \dfrac{\ln x}{x-1} = \dfrac{f(x)}{g(x)} \Rightarrow \lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} g(x) = 0$

$$\lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{(\ln x)'}{(x-1)'} = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1$$



idea: linear approximation

$$f(x) \sim f'(a)\cdot(x-a)$$
$$g(x) \sim g'(a)\,(x-a)$$
$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

NOTE 3 For the special case in which $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$, it is easy to see why l'Hospital's Rule is true. In fact, using the alternative form of the definition of a derivative (2.7.5), we have

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\displaystyle\lim_{x \to a} \frac{g(x) - g(a)}{x - a}}$$

$$= \lim_{x \to a} \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x)}{g(x)} \qquad [\text{because } f(a) = g(a) = 0]$$

It is more difficult to prove the general version of l'Hospital's Rule. See Appendix F.

e.g.: $\displaystyle\lim_{x \to +\infty} \frac{x^n}{e^x}$

$n = 1$: $\displaystyle\lim_{x \to +\infty} \frac{x}{e^x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0$

$n = 2$: $\displaystyle\lim_{x \to +\infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to +\infty} \frac{2}{e^x} = 0$

e^x increases much more faster than any polynominals

EXAMPLE 3 Calculate $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$.

SOLUTION Since $\ln x \to \infty$ and $\sqrt{x} \to \infty$ as $x \to \infty$, l'Hospital's Rule applies:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})}$$

Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$. But instead of applying l'Hospital's Rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$ ■

EXAMPLE 4 Find $\displaystyle\lim_{x\to 0} \frac{\tan x - x}{x^3}$. (See Exercise 2.2.48.)

SOLUTION Noting that both $\tan x - x \to 0$ and $x^3 \to 0$ as $x \to 0$, we use l'Hospital's Rule:

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}$$

Since the limit on the right side is still indeterminate of type $\frac{0}{0}$, we apply l'Hospital's Rule again:

$$\lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x}$$

Because $\lim_{x\to 0} \sec^2 x = 1$, we simplify the calculation by writing

$$\lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} = \frac{1}{3} \lim_{x\to 0} \sec^2 x \cdot \lim_{x\to 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x\to 0} \frac{\tan x}{x}$$

We can evaluate this last limit either by using l'Hospital's Rule a third time or by writing $\tan x$ as $(\sin x)/(\cos x)$ and making use of our knowledge of trigonometric limits. Putting together all the steps, we get

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x - x}{x^3} &= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} \\ &= \frac{1}{3} \lim_{x\to 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x\to 0} \frac{\sec^2 x}{1} = \frac{1}{3}\end{aligned}$$ ■

EXAMPLE 7 Compute $\lim_{x\to 1^+}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)$.

SOLUTION First notice that $1/(\ln x)\to\infty$ and $1/(x-1)\to\infty$ as $x\to 1^+$, so the limit is indeterminate of type $\infty-\infty$. Here we can start with a common denominator:

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1^+}\frac{x-1-\ln x}{(x-1)\ln x}$$

Both numerator and denominator have a limit of 0, so l'Hospital's Rule applies, giving

$$\lim_{x\to 1^+}\frac{x-1-\ln x}{(x-1)\ln x}=\lim_{x\to 1^+}\frac{1-\dfrac{1}{x}}{(x-1)\cdot\dfrac{1}{x}+\ln x}=\lim_{x\to 1^+}\frac{x-1}{x-1+x\ln x}$$

Again we have an indeterminate limit of type $\frac{0}{0}$, so we apply l'Hospital's Rule a second time:

$$\lim_{x\to 1^+}\frac{x-1}{x-1+x\ln x}=\lim_{x\to 1^+}\frac{1}{1+x\cdot\dfrac{1}{x}+\ln x}$$

$$=\lim_{x\to 1^+}\frac{1}{2+\ln x}=\frac{1}{2}\qquad\blacksquare$$

Type $0 \cdot \infty$

$$\lim_{x\to 0^+} x^2 = 0, \qquad \lim_{x\to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x\to 0^+} x^2 \cdot \frac{1}{x} = \lim_{x\to 0^+} x = 0$$

$$\lim_{x\to 0^+} x = 0, \qquad \lim_{x\to 0^+} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x\to 0^+} x \cdot \frac{1}{x^2} = \lim_{x\to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x\to 0^+} x = 0, \qquad \lim_{x\to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x\to 0^+} x \cdot \frac{1}{x} = \lim_{x\to 0^+} 1 = 1$$

This kind of limit is called an **indeterminate form of type $0 \cdot \infty$**. We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that we can use l'Hospital's Rule.

EXAMPLE 6 Evaluate $\lim_{x\to 0^+} x \ln x$.

SOLUTION The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor ($\ln x$) approaches $-\infty$. Writing $x = 1/(1/x)$, we have $1/x \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

■

Indeterminate Powers (Types 0^0, ∞^0, 1^∞)

Several indeterminate forms arise from the limit

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0
2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0
3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let} \quad y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x) \ln f(x)$$

or by using Formula 1.5.10 to write the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

(Recall that both of these methods were used in differentiating such functions.) In either method we are led to the indeterminate product $g(x) \ln f(x)$, which is of type $0 \cdot \infty$.

EXAMPLE 9 Calculate $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$.

SOLUTION First notice that as $x \to 0^+$, we have $1 + \sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is indeterminate (type 1^∞). Let

$$y = (1 + \sin 4x)^{\cot x}$$

Then $$\ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot x \ln(1 + \sin 4x) = \frac{\ln(1 + \sin 4x)}{\tan x}$$

so l'Hospital's Rule gives

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x \to 0^+} \frac{\dfrac{4 \cos 4x}{1 + \sin 4x}}{\sec^2 x} = 4$$

So far we have computed the limit of $\ln y$, but what we want is the limit of y. To find this we use the fact that $y = e^{\ln y}$:

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^4$$ ■

EXAMPLE 10 Find $\lim_{x \to 0^+} x^x$.

SOLUTION Notice that this limit is indeterminate since $0^x = 0$ for any $x > 0$ but $x^0 = 1$ for any $x \neq 0$. (Recall that 0^0 is undefined.) We could proceed as in Example 9 or by writing the function as an exponential:

$$x^x = (e^{\ln x})^x = e^{x \ln x}$$

In Example 6 we used l'Hospital's Rule to show that

$$\lim_{x \to 0^+} x \ln x = 0$$

Therefore

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x} = e^0 = 1$$