Tu 6/4

· Inverse derivative

Thm: Let f be a one-to-one, differentiable function, then f^{-1} is also differentiable, and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

pf: we only prove the differentiation part

$$y = f^{-1}(x) \Leftrightarrow f(y) = x$$

$\downarrow$ implicit differentiation

$$f'(y) \cdot y' = 1$$

$$y' = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

e.g.: $f(x) = e^x$, $f^{-1}(x) = \ln x$

$$(\ln x)' = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{f^{-1}(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Derivatives of logarithmic functions

Because the logarithmic function $y = \log_b x$ is the inverse of the exponential function $y = b^x$, which we know is differentiable from Section 3.1, it follows that the logarithmic function is also differentiable. We now state and prove the formula for the derivative of a logarithmic function.

1
$$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$$

pf: Using implicit differentiation

PROOF Let $y = \log_b x$. Then

$$b^y = x$$

Differentiating this equation implicitly with respect to x, and using Formula 3.4.5, we get

$$(b^y \ln b)\frac{dy}{dx} = 1$$

and so
$$\frac{dy}{dx} = \frac{1}{b^y \ln b} = \frac{1}{x \ln b}$$ ■

If we put $b = e$ in Formula 1, then the factor $\ln b$ on the right side becomes $\ln e = 1$ and we get the formula for the derivative of the natural logarithmic function $\log_e x = \ln x$:

2
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

By comparing Formulas 1 and 2, we see one of the main reasons that natural logarithms (logarithms with base e) are used in calculus: the differentiation formula is simplest when $b = e$ because $\ln e = 1$.

e.g.

EXAMPLE 1 Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^3 + 1}(3x^2) = \frac{3x^2}{x^3 + 1}$$ ■

$$\text{Rational functions} \xrightarrow{\text{taking derivatives}} \text{Rational functions}$$

false!

e.g. $\dfrac{3x^2}{x^3+1} \longleftarrow \ln(x^3+1)$

EXAMPLE 2 Find $\dfrac{d}{dx}\ln(\sin x)$.

SOLUTION Using (3), we have

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x$$ ■

EXAMPLE 6 Find $f'(x)$ if $f(x) = \ln|x|$.

SOLUTION Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus $f'(x) = 1/x$ for all $x \neq 0$. ■

The result of Example 6 is worth remembering:

4
$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}}$$

· logarithmic differentiation method: find derivative quickly

EXAMPLE 7 Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \tfrac{3}{4}\ln x + \tfrac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$ ■

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to expand the expression.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y' and replace y by $f(x)$.

PROOF OF THE POWER RULE (GENERAL VERSION) Let $y = x^n$ and use logarithmic differentiation:

$$\ln|y| = \ln|x|^n = n\ln|x| \qquad x \neq 0$$

Therefore
$$\frac{y'}{y} = \frac{n}{x}$$

Hence
$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$ ■

EXAMPLE 8 Differentiate $y = x^{\sqrt{x}}$.

SOLUTION 1 Since both the base and the exponent are variable, we use logarithmic differentiation:

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\,\ln x$$

$$\frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + (\ln x)\,\frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right)$$

SOLUTION 2 Another method is to use Equation 1.5.10 to write $x^{\sqrt{x}} = e^{\sqrt{x}\,\ln x}$:

$$\frac{d}{dx}\left(x^{\sqrt{x}}\right) = \frac{d}{dx}\left(e^{\sqrt{x}\,\ln x}\right) = e^{\sqrt{x}\,\ln x}\,\frac{d}{dx}\left(\sqrt{x}\,\ln x\right)$$

$$= x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right) \qquad \text{(as in Solution 1)}$$

■

The Number e as a Limit

We have shown that if $f(x) = \ln x$, then $f'(x) = 1/x$. Thus $f'(1) = 1$. We now use this fact to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{x\to 0}\frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x\to 0}\frac{\ln(1+x) - \ln 1}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x)$$

$$= \lim_{x\to 0}\ln(1+x)^{1/x}$$

Because $f'(1) = 1$, we have

$$\lim_{x\to 0}\ln(1+x)^{1/x} = 1$$

Then, by Theorem 2.5.8 and the continuity of the exponential function, we have

$$e = e^1 = e^{\lim_{x\to 0}\ln(1+x)^{1/x}} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}} = \lim_{x\to 0}(1+x)^{1/x}$$

5 $$\boxed{e = \lim_{x\to 0}(1+x)^{1/x}}$$

Formula 5 is illustrated by the graph of the function $y = (1+x)^{1/x}$ in Figure 4 and a table of values for small values of x. This illustrates the fact that, correct to seven decimal places,

$$e \approx 2.7182818$$

If we put $n = 1/x$ in Formula 5, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for e is

6 $$\boxed{e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n}$$

Derivatives of Inverse trigonometric functions

$$f(x) = \sin x : \quad [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \longrightarrow [-1, 1]$$

$$f^{-1}(x) = \sin^{-1} x : \quad [-1, 1] \longrightarrow [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

$$(\sin^{-1} x)' = \frac{1}{\cos(f^{-1}(x))} = \frac{1}{\cos(\sin^{-1} x)}$$

Notice that: $\quad (\sin(\theta))^2 + (\cos(\theta))^2 = 1$

$$\downarrow \theta = \sin^{-1} x$$

$$x^2 + \cos(\sin^{-1} x)^2 = 1$$

$$\Rightarrow \quad \cos(\sin^{-1} x) = \sqrt{1 - x^2} > 0 \quad \text{when} \quad x \in [-1, 1]$$

$$\theta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

$$\Rightarrow \quad (\sin^{-1} x)' = \frac{1}{\sqrt{1 - x^2}}$$

Recall the definition of the arcsine function:

$$y = \sin^{-1} x \qquad \text{means} \qquad \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

Differentiating $\sin y = x$ implicitly with respect to x, we obtain

$$\cos y \frac{dy}{dx} = 1 \qquad \text{or} \qquad \frac{dy}{dx} = \frac{1}{\cos y}$$

Now $\cos y \geqslant 0$ because $-\pi/2 \leqslant y \leqslant \pi/2$, so

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2} \qquad (\cos^2 y + \sin^2 y = 1)$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

$$\boxed{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}}$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1 + x^2}$$

$$\begin{tikzcd} \text{algebraic functions} \arrow[r, "\text{derivative}"] & \text{algebraic functions} \arrow[l, bend left, "\text{false!}"] \end{tikzcd}$$

e.g. $\sin^{-1} x$

EXAMPLE 9 Differentiate (a) $y = \dfrac{1}{\sin^{-1}x}$ and (b) $f(x) = x \arctan\sqrt{x}$.

SOLUTION

(a)
$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^{-1} = -(\sin^{-1}x)^{-2}\frac{d}{dx}(\sin^{-1}x)$$

$$= -\frac{1}{(\sin^{-1}x)^2\sqrt{1 - x^2}}$$

(b)
$$f'(x) = x\frac{1}{1 + \left(\sqrt{x}\right)^2}\left(\tfrac{1}{2}x^{-1/2}\right) + \arctan\sqrt{x}$$

$$= \frac{\sqrt{x}}{2(1 + x)} + \arctan\sqrt{x}$$

Linear approximation

The idea of derivative: instantaneous rate of change

Using lines to approximate complicated functions

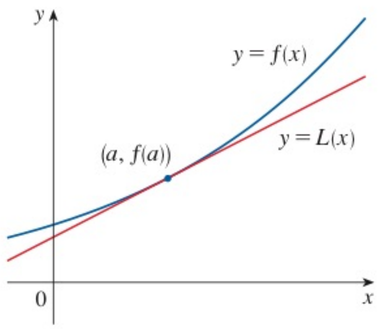


FIGURE 1

It might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f. So we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$. (See Figure 1.)

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

The linear function whose graph is this tangent line, that is,

$$\boxed{1} \qquad \boxed{L(x) = f(a) + f'(a)(x - a)}$$

is called the **linearization** of f at a. The approximation $f(x) \approx L(x)$ or

$$\boxed{2} \qquad \boxed{f(x) \approx f(a) + f'(a)(x - a)}$$

is called the **linear approximation** or **tangent line approximation** of f at a.

EXAMPLE 1 Find the linearization of the function $f(x) = \sqrt{x + 3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

SOLUTION The derivative of $f(x) = (x + 3)^{1/2}$ is

$$f'(x) = \tfrac{1}{2}(x + 3)^{-1/2} = \frac{1}{2\sqrt{x + 3}}$$

and so we have $f(1) = 2$ and $f'(1) = \frac{1}{4}$. Putting these values into Equation 1, we see that the linearization is

$$L(x) = f(1) + f'(1)(x - 1) = 2 + \tfrac{1}{4}(x - 1) = \frac{7}{4} + \frac{x}{4}$$

The corresponding linear approximation (2) is

$$\sqrt{x + 3} \approx \frac{7}{4} + \frac{x}{4} \qquad \text{(when } x \text{ is near 1)}$$

In particular, we have

$$\sqrt{3.98} \approx \tfrac{7}{4} + \tfrac{0.98}{4} = 1.995 \qquad \text{and} \qquad \sqrt{4.05} \approx \tfrac{7}{4} + \tfrac{1.05}{4} = 2.0125$$

The linear approximation is illustrated in Figure 2. We see that, indeed, the tangent line approximation is a good approximation to the given function when x is near 1. We also see that our approximations are overestimates because the tangent line lies above the curve.

Of course, a calculator could give us approximations for $\sqrt{3.98}$ and $\sqrt{4.05}$, but the linear approximation gives an approximation *over an entire interval*. ■

Review:

limit: $\lim_{x \to a} f(x) = L$

$a = +\infty / -\infty \Rightarrow$ horizontal asympotote

$L = +\infty / -\infty \Rightarrow$ vertical asympotote

Rules: Constant multiple: $C \cdot f(x)$

Sum / difference $f \pm g$

product $f \cdot g$

quotient $\frac{f}{g}$

Sequeeze thm: $h(x) \le f(x) \le g(x)$

continuous functions: Def: $\lim_{x \to c} f(x) = f(a)$

Basic examples:
- Polynomials
- Rational function
- Algebraic functions
- Trigonometric / Inverse trigonometric
- exponential / logarithmic

Basic properties:

- $c \cdot f$
- $f \pm g$
- $f \cdot g$
- $\dfrac{f}{g}$
- $\boxed{f \circ g}$: g ... at a

  f ... at $g(a)$

  $f \circ g$ ... at a

Practice test

1. (1) $$\lim_{x \to a} f(x) = +\infty$$

the values of $f(x)$ can be as large as you want if you restrict x to a sufficiently small interval containing a

(2) $$\lim_{x \to +\infty} f(x) = L$$

the values of $f(x)$ can be as close to L as you want if the variable is sufficiently large

(3)

$$\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

(4) $$(f \cdot g)' = f' \cdot g + f \cdot g'$$

2. (1) continuous functions

(2) $\lim_{t\to 0} \frac{\sin(t)^2}{t} = \lim_{t\to 0} \sin t \cdot \frac{\sin t}{t} = 0 \cdot 1 = 0$

(3) $f'(x) = \frac{1}{3} \cdot \left(1+\tan(x)\right)^{-\frac{2}{3}} \cdot \left(\sec^2 x\right)$

(4) $g'(x) = e^{\sin x} \cdot \cos x - \sin(e^x) \cdot e^x$

3. (1) ×

(2) ×

(3) ✓

(4) ✓

4. $y = \sqrt{x\ln(x^4)} \Rightarrow \ln y = \frac{1}{2}\ln x + \frac{1}{2}\ln\left(4\ln x\right)$

$$\frac{y'}{y} = \frac{1}{2x} + \frac{1}{2} \cdot \frac{1}{4\ln x} \cdot \frac{4}{x}$$

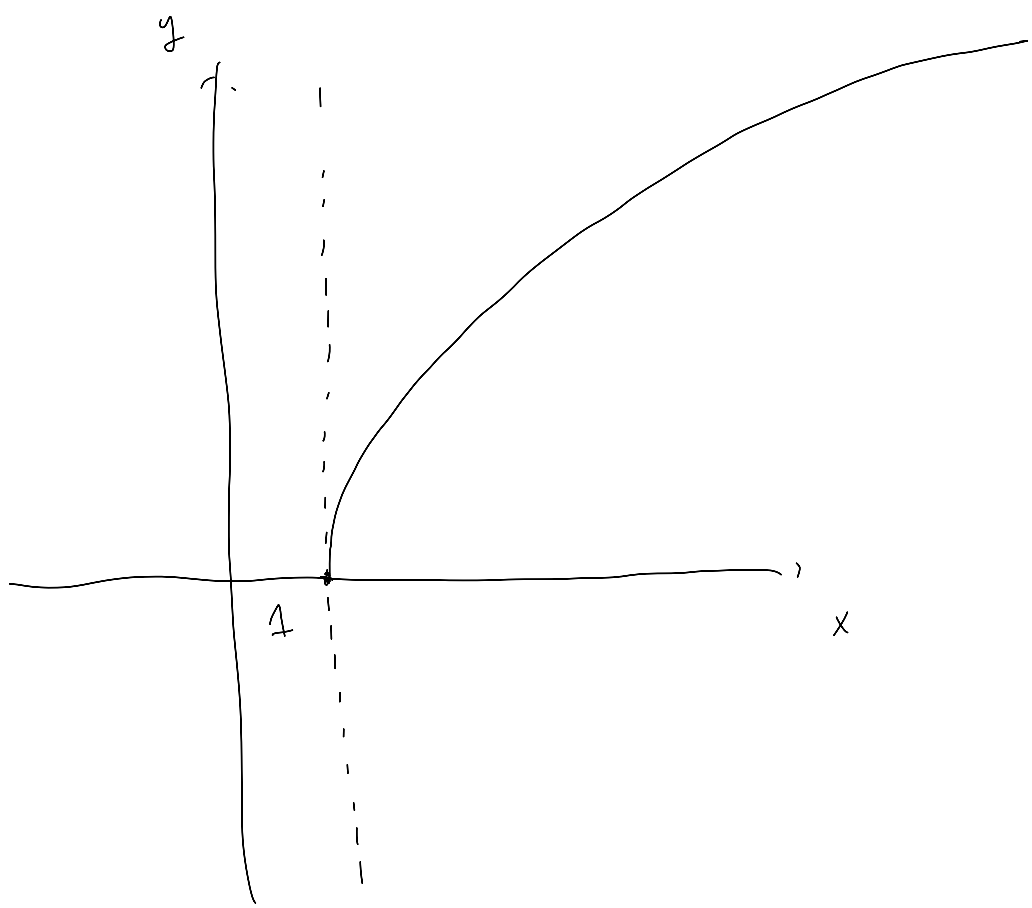
$$= \frac{1}{2x} + \frac{1}{2x\ln x}$$

$$y' = \sqrt{x\ln(x^4)} \cdot \frac{\ln x + 1}{2x\ln x}$$

$$f'(x) = \sqrt{x\ln(x^4)} \cdot \frac{\ln x + 1}{2x\ln x}$$

$$f'(x) = \frac{1}{2}\frac{1}{\sqrt{x\ln(x^4)}} \cdot \left(4 \cdot (\ln x + 1)\right)$$

$$= \frac{2(\ln x + 1)}{\sqrt{x\ln(x^4)}} = \frac{\ln x + 1}{\sqrt{x\ln x}}$$



5. (1) $2x + 4y + 4xy' + 2y \cdot y' = 0$

$$x + 2y + 2xy' + y \cdot y' = 0$$

$$y'(2x + y) = -(2y + x)$$

$$y' = -\frac{x+2y}{2x+y}$$