Th 5/30

Review on derivatives:

Def. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Thm: f is differentiable $\Rightarrow$ f is continuous

Counter-example: $f(x) = |x|$

e.g.:

- $(cf)' = c \cdot f'$
- $(f \pm g)' = f' \pm g'$
- $(f \cdot g)' = f'g + fg'$
- $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$
- $(c)' = 0$
- $(x^a)' = a \cdot x^{a-1}$
- $(a^x)' = a^x \cdot \ln a$ $\left((e^x)' = e^x \right)$

Derivatives of trigonometric functions

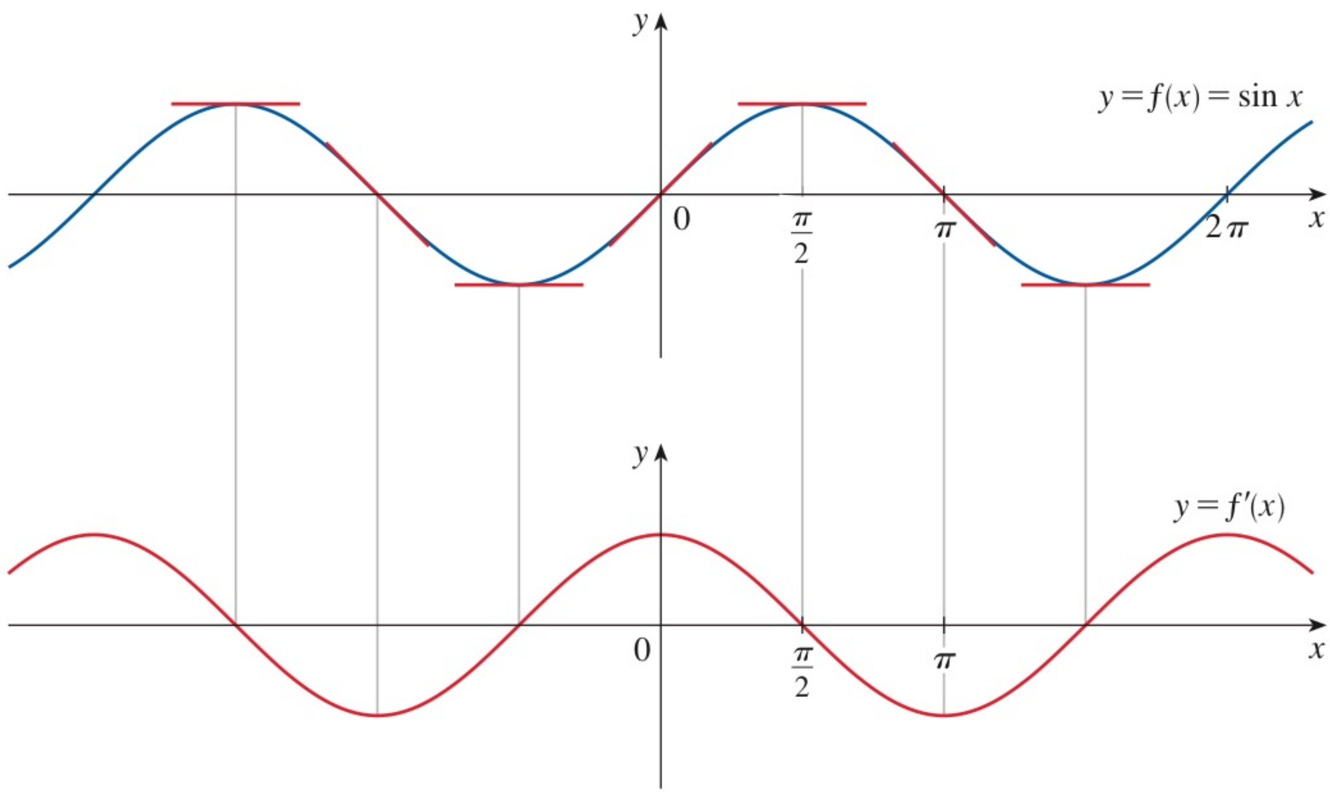
Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Graphs



pf: for sin x

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \left(\sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} \right)$$

$$= \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

Key Limits: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1, \quad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$

In proving the formula for the derivative of sine we used two special limits, which we now prove.

5 $$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

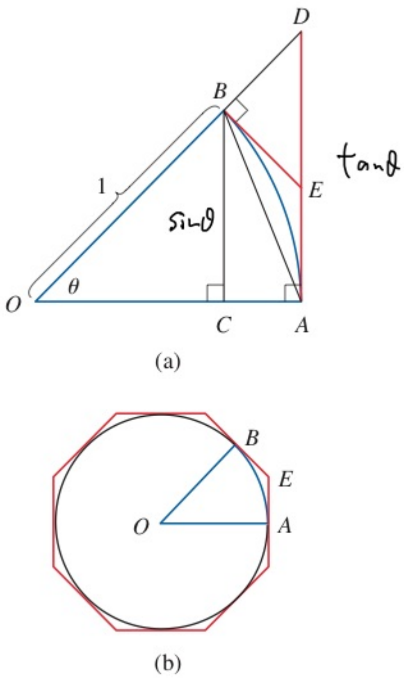


FIGURE 6

PROOF Assume first that θ lies between 0 and $\pi/2$. Figure 6(a) shows a sector of a circle with center O, central angle θ, and radius 1. BC is drawn perpendicular to OA. By the definition of radian measure, we have arc $AB = \theta$. Also $|BC| = |OB| \sin\theta = \sin\theta$. From the diagram we see that

$$|BC| < |AB| < \text{arc } AB$$

Therefore $$\sin\theta < \theta \qquad \text{so} \qquad \frac{\sin\theta}{\theta} < 1$$

Let the tangent lines at A and B intersect at E. You can see from Figure 6(b) that the circumference of a circle is smaller than the length of a circumscribed polygon, and so arc $AB < |AE| + |EB|$. Thus

$$\begin{aligned} \theta = \text{arc } AB &< |AE| + |EB| \\ &< |AE| + |ED| \\ &= |AD| = |OA| \tan\theta \\ &= \tan\theta \end{aligned}$$

(In Appendix F the inequality $\theta \leqslant \tan\theta$ is proved directly from the definition of the length of an arc without resorting to geometric intuition as we did here.) Therefore we have

$$\theta < \frac{\sin\theta}{\cos\theta}$$

so $$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

We know that $\lim_{\theta \to 0} 1 = 1$ and $\lim_{\theta \to 0} \cos\theta = 1$, so by the Squeeze Theorem, we have

$$\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1 \qquad (0 < \theta < \pi/2)$$

But the function $(\sin\theta)/\theta$ is an even function, so its right and left limits must be equal. Hence, we have

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

so we have proved Equation 5. ■

The first special limit we considered concerned the sine function. The following special limit involves cosine.

6

$$\boxed{\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0}$$

PROOF We multiply numerator and denominator by $\cos\theta + 1$ in order to put the function in a form in which we can use limits that we know.

$$\begin{aligned}
\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} &= \lim_{\theta \to 0} \left(\frac{\cos\theta - 1}{\theta} \cdot \frac{\cos\theta + 1}{\cos\theta + 1} \right) = \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} \\
&= \lim_{\theta \to 0} \frac{-\sin^2\theta}{\theta(\cos\theta + 1)} = -\lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{\cos\theta + 1} \right) \\
&= -\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin\theta}{\cos\theta + 1} \\
&= -1 \cdot \left(\frac{0}{1 + 1} \right) = 0 \qquad \text{(by Equation 5)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\sin x)' &= \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\
&= \cos x
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\cos x)' &= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \cos x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\
&= -\sin x
\end{aligned}$$

$\tan x$: $\quad \tan x = \dfrac{\sin x}{\cos x}$

$$(\tan x)' = \frac{\cos x \cdot \cos x - (-\sin x)\cdot \sin x}{\cos^2 x} = \sec^2 x$$

EXAMPLE 4 Find the 27th derivative of $\cos x$.

SOLUTION The first few derivatives of $f(x) = \cos x$ are as follows:

$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$
$$f'''(x) = \sin x$$
$$f^{(4)}(x) = \cos x$$
$$f^{(5)}(x) = -\sin x$$

We see that the successive derivatives occur in a cycle of length 4 and, in particular, $f^{(n)}(x) = \cos x$ whenever n is a multiple of 4. Therefore

$$f^{(24)}(x) = \cos x$$

and, differentiating three more times, we have

$$f^{(27)}(x) = \sin x$$

■

EXAMPLE 2 Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

SOLUTION The Quotient Rule gives

$$\begin{aligned} f'(x) &= \frac{(1 + \tan x)\dfrac{d}{dx}(\sec x) - \sec x\dfrac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2} \\ &= \frac{(1 + \tan x)\sec x \tan x - \sec x \cdot \sec^2 x}{(1 + \tan x)^2} \\ &= \frac{\sec x\,(\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2} \\ &= \frac{\sec x\,(\tan x - 1)}{(1 + \tan x)^2} \qquad (\sec^2 x = \tan^2 x + 1) \end{aligned}$$

Because $\sec x$ is never 0, we see that $f'(x) = 0$ when $\tan x = 1$, and this occurs when $x = \pi/4 + n\pi$, where n is an integer (see Figure 3). ■

EXAMPLE 5 Find $\lim_{x \to 0} \dfrac{\sin 7x}{4x}$.

SOLUTION In order to apply Equation 5, we first rewrite the function by multiplying and dividing by 7:

$$\frac{\sin 7x}{4x} = \frac{7}{4}\left(\frac{\sin 7x}{7x}\right)$$

EXAMPLE 6 Calculate $\lim_{x \to 0} x \cot x$.

SOLUTION Here we divide numerator and denominator by x:

$$\begin{aligned} \lim_{x \to 0} x \cot x &= \lim_{x \to 0} \frac{x \cos x}{\sin x} \\ &= \lim_{x \to 0} \frac{\cos x}{\dfrac{\sin x}{x}} = \frac{\lim\limits_{x \to 0} \cos x}{\lim\limits_{x \to 0} \dfrac{\sin x}{x}} \\ &= \frac{\cos 0}{1} \qquad \text{(by the continuity of cosine and Equation 5)} \\ &= 1 \end{aligned}$$

■

EXAMPLE 7 Find $\lim_{\theta \to 0} \dfrac{\cos \theta - 1}{\sin \theta}$.

SOLUTION In order to use Equations 5 and 6 we divide numerator and denominator by θ:

$$\begin{aligned} \lim_{\theta \to 0} \frac{\cos \theta - 1}{\sin \theta} &= \lim_{\theta \to 0} \frac{\dfrac{\cos \theta - 1}{\theta}}{\dfrac{\sin \theta}{\theta}} \\ &= \frac{\lim\limits_{\theta \to 0} \dfrac{\cos \theta - 1}{\theta}}{\lim\limits_{\theta \to 0} \dfrac{\sin \theta}{\theta}} = \frac{0}{1} = 0 \end{aligned}$$

■

Chain Rule

Motivation: we already know how to differentiate Rational functions

What about Algebraic functions: $\sqrt[n]{\ }$

e.g. $F(x) = \sqrt{x^2+1}$

Notice that $F(x) = f(g(x))$, $f(u) = \sqrt{u}$

$g(x) = x^2+1$

it is a composition of functions!

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

1 $$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

2 $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

e.g. In the above example:

$$f'(u) = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$$

$$f'(g(x)) = \frac{1}{2\sqrt{g(x)}} = \frac{1}{2\sqrt{x^2+1}}$$

$$g'(x) = 2x$$

$$\Rightarrow F'(x) = \frac{x}{\sqrt{x^2+1}}$$

$y = f(u) = F(x)$

$u = x^2+1$

$$\frac{dy}{du} = \frac{1}{2\sqrt{u}}, \quad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{u}} = \frac{x}{\sqrt{x^2+1}}$$

EXAMPLE 2 Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$.

SOLUTION

(a) If $y = \sin(x^2)$, then the outer function is the sine function and the inner function is the squaring function, so the Chain Rule gives

$$\frac{dy}{dx} = \frac{d}{dx} \underbrace{\sin}_{\substack{\text{outer}\\\text{function}}} \underbrace{(x^2)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} = \underbrace{\cos}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}} \underbrace{(x^2)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} \cdot \underbrace{2x}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

$$= 2x\cos(x^2)$$

(b) Note that $\sin^2 x = (\sin x)^2$. Here the outer function is the squaring function and the inner function is the sine function. So

$$\frac{dy}{dx} = \frac{d}{dx} \underbrace{(\sin x)}_{\substack{\text{inner}\\\text{function}}}{}^2 = \underbrace{2}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}} \cdot \underbrace{(\sin x)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} \cdot \underbrace{\cos x}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

The answer can be left as $2 \sin x \cos x$ or written as $\sin 2x$ (by a trigonometric identity known as the double-angle formula). ■

EXAMPLE 3 Differentiate $y = (x^3 - 1)^{100}$.

SOLUTION Taking $u = g(x) = x^3 - 1$ and $n = 100$ in (4), we have

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 1)^{100} = 100(x^3 - 1)^{99} \frac{d}{dx}(x^3 - 1)$$

$$= 100(x^3 - 1)^{99} \cdot 3x^2 = 300x^2(x^3 - 1)^{99}$$ ■

Can use product rule! but much more complicated!

EXAMPLE 4 Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{x^2 + x + 1}}$.

SOLUTION First rewrite f: $\qquad f(x) = (x^2 + x + 1)^{-1/3}$

Thus

$$f'(x) = -\tfrac{1}{3}(x^2 + x + 1)^{-4/3} \frac{d}{dx}(x^2 + x + 1)$$

$$= -\tfrac{1}{3}(x^2 + x + 1)^{-4/3}(2x + 1)$$

EXAMPLE 7 Differentiate $y = e^{\sin x}$.

SOLUTION Here the inner function is $g(x) = \sin x$ and the outer function is the exponential function $f(x) = e^x$. So, by the Chain Rule,

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\sin x}) = e^{\sin x} \frac{d}{dx}(\sin x) = e^{\sin x} \cos x$$ ■

EXAMPLE 8 If $f(x) = \sin(\cos(\tan x))$, then

$$f'(x) = \cos(\cos(\tan x)) \frac{d}{dx} \cos(\tan x)$$

$$= \cos(\cos(\tan x)) [-\sin(\tan x)] \frac{d}{dx}(\tan x)$$

$$= -\cos(\cos(\tan x)) \sin(\tan x) \sec^2 x$$

Notice that we used the Chain Rule twice. ■

EXAMPLE 9 Differentiate $y = e^{\sec 3\theta}$.

SOLUTION The outer function is the exponential function, the middle function is the secant function, and the inner function is the tripling function. So we have

$$\frac{dy}{d\theta} = e^{\sec 3\theta} \frac{d}{d\theta}(\sec 3\theta)$$

$$= e^{\sec 3\theta} \sec 3\theta \tan 3\theta \frac{d}{d\theta}(3\theta)$$

$$= 3e^{\sec 3\theta} \sec 3\theta \tan 3\theta$$ ■

Let's come back to exponential function

$$F(x) = a^x$$
$$= e^{x \cdot \ln a}$$
$$= f(g(x)), \quad f(u) = e^u, \; g(x) = \ln a \cdot x$$

$$\Rightarrow (a^x)' = F'(x) = f'(g(x)) \cdot g'(x)$$
$$= e^{g(x)} \cdot \ln a = e^{x \cdot \ln a} \cdot \ln a = a^x \cdot \ln a$$

<u>Proof of the Chain rule</u> :

Key input: Let f be a function differentiable at a, then

$$f(x) = f(a) + f'(a)(x-a) + \Delta(x-a)$$

where $\lim_{x \to a} \frac{\varepsilon(x-a)}{x-a} = 0$

or. $f(a+h) = f(a) + f'(a) \cdot h + \Delta(h)$, $\lim_{h \to 0} \frac{\Delta(h)}{h} = 0$

Define

$$\varepsilon(h) = \begin{cases} \frac{\Delta(h)}{h}, & h \neq 0; \\ 0, & h = 0. \end{cases}$$

$\varepsilon(h)$ is continuous at 0

$$f(a+h) = f(a) + \left(f'(a) + \varepsilon(h)\right) \cdot h$$

$$F(x) = f(g(x))$$

$$F(x+h) - F(x) = f(g(x+h)) - f(g(x))$$

$$= f\big(g(x) + \underbrace{(g'(x) + \varepsilon_g(h))\, h}_{h'}\big) - f(g(x))$$

$$f(u + h') = f(u) + (f'(u) + \varepsilon_f(h')) \cdot h'$$

$$= f(g(x)) + (f'(g(x)) + \varepsilon_f(h'))\, h' - f(g(x))$$

$$= (f'(g(x)) + \varepsilon_f(h')) \cdot (g'(x) + \varepsilon_g(h))\, h$$

$$= \big(f'(g(x))\, g'(x) + f'(g(x))\, \varepsilon_g(h) + g'(x)\, \varepsilon_f(h') + \varepsilon_f(h')\varepsilon_g(h)\big)\, h$$

$$\Rightarrow F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \to 0} f'(g(x)) \cdot g'(x) + f'(g(x))\, \varepsilon_g(h) + g'(x)\, \varepsilon_f(h') + \varepsilon_f(h')\varepsilon_g(h)$$

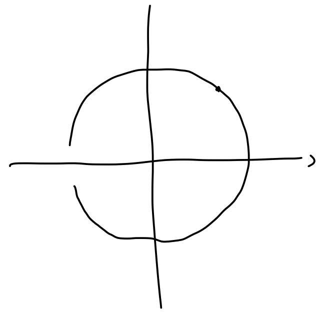
- $\lim_{h \to 0} \varepsilon_g(h) = 0$
- $\lim_{h \to 0} \varepsilon_f(h') = \lim_{h \to 0} \varepsilon_f\big((g'(x) + \varepsilon_g(h))\, h\big) = 0$

  because $\lim_{h \to 0} h' = \lim_{h \to 0} (g'(x) + \varepsilon_g(h))\, h = 0$

$$= f'(g(x)) \cdot g'(x)$$

Implicit Differentiation

Motivity example : $x^2 + y^2 = 1$



What is the tangent line at $\left(\frac{3}{5}, \frac{4}{5}\right)$? or a general pt P on the circle

locally it is a curve ! we want to find y'

but $x^2+y^2=1$ is not of the form $y = f(x)$

How to get y' ?

Assume: $y = y(x)$, we find f'

$$x^2 + \left(y(x)\right)^2 = 1$$

both sides are functions of x , taking derivatives, we get

$$\underbrace{2x + 2y(x)\cdot y'(x)}_{\uparrow \text{ Chain rule}} = 0 \quad \text{or} \quad 2x + 2y \cdot y' = 0$$

$$y \cdot y' + x = 0$$

$$y' = -\frac{x}{y} \quad \text{if } y \neq 0$$

SOLUTION 2
Solving the equation $x^2 + y^2 = 25$ for y, we get $y = \pm\sqrt{25 - x^2}$. The point $(3, 4)$ lies on the upper semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$. Differentiating f using the Chain Rule, we have

$$\begin{aligned} f'(x) &= \tfrac{1}{2}(25 - x^2)^{-1/2} \frac{d}{dx}(25 - x^2) \\ &= \tfrac{1}{2}(25 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{25 - x^2}} \end{aligned}$$

At the point $(3, 4)$ we have

$$f'(3) = -\frac{3}{\sqrt{25 - 3^2}} = -\frac{3}{4}$$

and, as in Solution 1, an equation of the tangent is $3x + 4y = 25$. ■

Key insight: just view y as a function of x, $y = y(x)$
then take derivative of the original equation!

EXAMPLE 3 Find y' if $\sin(x + y) = y^2 \cos x$.

SOLUTION Differentiating implicitly with respect to x and remembering that y is a function of x, we get

$$\cos(x + y) \cdot (1 + y') = y^2(-\sin x) + (\cos x)(2yy')$$

(Note that we have used the Chain Rule on the left side and the Product Rule and Chain Rule on the right side.) If we collect the terms that involve y', we get

$$\cos(x + y) + y^2 \sin x = (2y \cos x)y' - \cos(x + y) \cdot y'$$

So
$$y' = \frac{y^2 \sin x + \cos(x + y)}{2y \cos x - \cos(x + y)}$$

Figure 6, drawn by a computer, shows part of the curve $\sin(x + y) = y^2 \cos x$. As a check on our calculation, notice that $y' = -1$ when $x = y = 0$ and it appears from the graph that the slope is approximately -1 at the origin. ■

EXAMPLE 2

(a) Find y' if $x^3 + y^3 = 6xy$.
(b) Find the tangent to the folium of Descartes $x^3 + y^3 = 6xy$ at the point $(3, 3)$.
(c) At what point in the first quadrant is the tangent line horizontal?

SOLUTION

(a) Differentiating both sides of $x^3 + y^3 = 6xy$ with respect to x, regarding y as a function of x, and using the Chain Rule on the term y^3 and the Product Rule on the term $6xy$, we get

$$3x^2 + 3y^2y' = 6xy' + 6y$$

or

$$x^2 + y^2y' = 2xy' + 2y$$

We now solve for y':

$$y^2y' - 2xy' = 2y - x^2$$

$$(y^2 - 2x)y' = 2y - x^2$$

$$y' = \frac{2y - x^2}{y^2 - 2x}$$

(b) When $x = y = 3$,

$$y' = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3} = -1$$

and a glance at Figure 4 confirms that this is a reasonable value for the slope at $(3, 3)$. So an equation of the tangent to the folium at $(3, 3)$ is

$$y - 3 = -1(x - 3) \qquad \text{or} \qquad x + y = 6$$

(c) The tangent line is horizontal if $y' = 0$. Using the expression for y' from part (a), we see that $y' = 0$ when $2y - x^2 = 0$ (provided that $y^2 - 2x \neq 0$). Substituting $y = \frac{1}{2}x^2$ in the equation of the curve, we get

$$x^3 + \left(\tfrac{1}{2}x^2\right)^3 = 6x\left(\tfrac{1}{2}x^2\right)$$

which simplifies to $x^6 = 16x^3$. Since $x \neq 0$ in the first quadrant, we have $x^3 = 16$. If $x = 16^{1/3} = 2^{4/3}$, then $y = \frac{1}{2}(2^{8/3}) = 2^{5/3}$. Thus the tangent is horizontal at $(2^{4/3}, 2^{5/3})$, which is approximately $(2.5198, 3.1748)$. Looking at Figure 5, we see that our answer is reasonable. ■

54. $y = (1 + \sqrt{x})^3$

56. $y = e^{e^x}$

54. $y' = 3(1+\sqrt{x})^2 \cdot \frac{1}{2\sqrt{x}} = \frac{3(1+\sqrt{x})^2}{2\sqrt{x}}$

56. $y' = e^{e^x} \cdot e^x = e^{x+e^x}$

31. $x^2 - xy - y^2 = 1$, $(2, 1)$ (hyperbola)

$2x - y - xy' - 2yy' = 0$

$4 - 1 - 2y' - 2y' = 0 \Rightarrow y' = \frac{3}{4}$

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43. If $xy + e^y = e$, find the value of y'' at the point where $x = 0$.

$x=0,\ y=1$

$y + xy' + e^y \cdot y' = 0 \Rightarrow 1 + e \cdot y' = 0 \Rightarrow y' = -e^{-1}$

$y'' + y' + xy'' + e^y (y')^2 + e^y \cdot y'' = 0$

$\Downarrow$

$y'' - e^{-1} + e \cdot e^{-2} + e \cdot y'' = 0 \Rightarrow y'' = 0$