Tu 5/28

Review on Limit & Continuity

1 Intuitive Definition of a Limit Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

Left limit & Right limit

$x \to a^-$ $\qquad$ $x \to a^+$

L can be $+\infty$ or $-\infty$, e.g. $f(x) = \frac{1}{x}$, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$\lim_{x \to 0^+} \frac{1}{x} = +\infty$

$\Rightarrow$ vertical asymptote

6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

a can be $+\infty$ or $-\infty$, e.g. $f(x) = \frac{1}{x}$, $\lim_{x \to +\infty} \frac{1}{x} = 0$, $\lim_{x \to -\infty} \frac{1}{x} = 0$

$\Rightarrow$ horizontal asymptote

3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$

Continuous function

Continuity of a Function

We noticed in Section 2.3 that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions having this property are called *continuous at a*. We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place without interruption.)

1 Definition A function f is **continuous at a number *a*** if

$$\lim_{x \to a} f(x) = f(a)$$

Notice that Definition 1 implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

Examples:
- Polynomial functions are continuous over $\mathbb{R}$
- Rational functions are continuous over its domain
- sin / cos continuous over $\mathbb{R}$

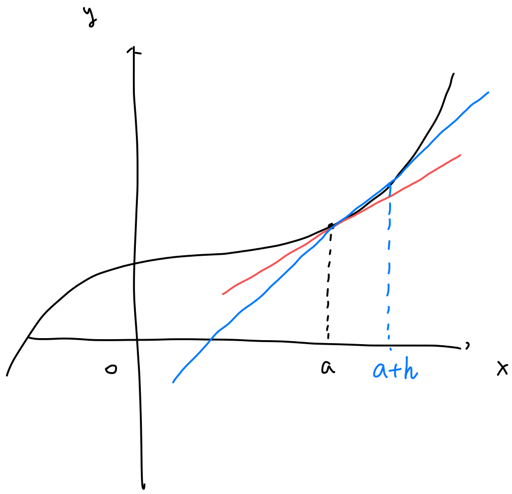
  trigonometric functions are continuous over its domain
  (inverse-)
- exponential ... over $\mathbb{R}$

  logarithmic ... over $\mathbb{R}_{>0}$

Derivatives & Rates of change

Tangent problem

Let $f(x)$ be a function. Let $a \in D$ be a pt in the domain



Tangent line: among all the lines passing through $(a, f(a))$, tangent line is closest to the function at the pt a than any other lines

Since we know the line passes through $(a, f(a))$

We only need to determine the slope!

Given another point $(x, f(x))$

sectant line $\Longrightarrow$ tangent line

slope of sectant line $\Longrightarrow$ slope of the tan-- line

$\Downarrow$

$$\boxed{\frac{f(x)-f(a)}{x-a}}$$

, we want to find this when x is apporaching a

we write it as $\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

there fore we have found the tangent line:

passing through $(x, f(x)) = P$, with slope $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

There is another expression for the slope of a tangent line that is sometimes easier to use. If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$

(See Figure 3 where the case $h > 0$ is illustrated and Q is located to the right of P. If it happened that $h < 0$, however, Q would be to the left of P.)

Notice that as x approaches a, h approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line in Definition 1 becomes

2

$$\boxed{m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}}$$

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at $(1, 1)$ is

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$ ■

Velocity Problem

$f(t)$ = the position of the pt at the time t

from a & t,

$$\text{average velocity between } a \text{ \& } t = \frac{\text{change in position}}{\text{change in time}} = \frac{f(t)-f(a)}{t-a}$$

· What is the instantaneous velocity at a?

Intuition :

instantaneous velocity = $\boxed{\text{limit of}}$ average velocity

$$v(a) = \lim_{t \to a} \frac{f(t)-f(a)}{t-a}$$

Derivatives

4 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

EXAMPLE 4 Use Definition 4 to find the derivative of the function $f(x) = x^2 - 8x + 9$ at the numbers (a) 2 and (b) a.

(a) From Definition 4 we have

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} \frac{(2+h)^2 - 8(2+h) + 9 - (-3)}{h} \\ &= \lim_{h \to 0} \frac{4 + 4h + h^2 - 16 - 8h + 9 + 3}{h} \\ &= \lim_{h \to 0} \frac{h^2 - 4h}{h} = \lim_{h \to 0} \frac{h(h-4)}{h} = \lim_{h \to 0} (h-4) = -4 \end{aligned}$$

$$\begin{aligned} f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h \to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h} \\ &= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\ &= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h \to 0} (2a + h - 8) = 2a - 8 \end{aligned}$$

EXAMPLE 5 Use Equation 5 to find the derivative of the function $f(x) = 1/\sqrt{x}$ at the number a $(a > 0)$.

SOLUTION From Equation 5 we get

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{a}}}{x - a} = \lim_{x \to a} \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{a}}}{x - a} \cdot \frac{\sqrt{x}\sqrt{a}}{\sqrt{x}\sqrt{a}}$$

$$= \lim_{x \to a} \frac{\sqrt{a} - \sqrt{x}}{\sqrt{ax}(x - a)} = \lim_{x \to a} \frac{\sqrt{a} - \sqrt{x}}{\sqrt{ax}(x - a)} \cdot \frac{\sqrt{a} + \sqrt{x}}{\sqrt{a} + \sqrt{x}}$$

$$= \lim_{x \to a} \frac{-(x - a)}{\sqrt{ax}(x - a)(\sqrt{a} + \sqrt{x})} = \lim_{x \to a} \frac{-1}{\sqrt{ax}(\sqrt{a} + \sqrt{x})}$$

$$= \frac{-1}{\sqrt{a^2}(\sqrt{a} + \sqrt{a})} = \frac{-1}{a \cdot 2\sqrt{a}} = -\frac{1}{2a^{3/2}}$$

Counter-example: $f(x) = |x|$ at $x = 0$

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \begin{cases} 1, & h > 0 \\ -1, & h < 0 \end{cases} \Rightarrow \text{doesn't limit}$$

Explaination:

> The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

$|x|$ has no tangent line at $x = 0$

Rates of change

Philosophy: derivative $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x=a$

$$\text{rate of change of } f \text{ over } [a, a+h] = \frac{\Delta f}{\Delta x} = \frac{f(a+h)-f(a)}{a+h-a}$$

$$= \frac{f(a+h)-f(a)}{h}$$

take $h \to 0 \Rightarrow$ instantaneous rate of change $= f'(a)$

Derivative as a function

Starting from f, we can define

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

provided this limit exists

$f'(x)$ is called the derivative function of $f(x)$

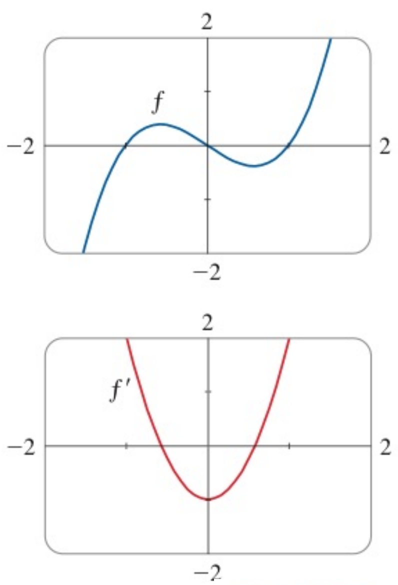
other notations: $\dfrac{df}{dx}$,

$y = f(x)$, we use y', $\dfrac{dy}{dx}$

e-g.

EXAMPLE 2

(a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.
(b) Illustrate this formula by comparing the graphs of f and f'.



is h and that x is temporarily regarded as a constant during the calculation of the limit.

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \\
&= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h} \\
&= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\
&= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1) = 3x^2 - 1
\end{aligned}
$$

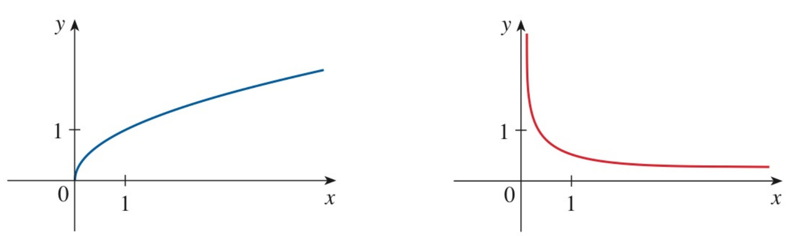
(b) We use a calculator to graph f and f' in Figure 3. Notice that $f'(x) = 0$ when f has horizontal tangents and $f'(x)$ is positive when the tangents have positive slope. So these graphs serve as a check on our work in part (a). ■

EXAMPLE 3 If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

SOLUTION

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \qquad \text{(Rationalize the numerator.)} \\
&= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
\end{aligned}
$$

We see that $f'(x)$ exists if $x > 0$, so the domain of f' is $(0, \infty)$. This is slightly smaller than the domain of f, which is $[0, \infty)$. ■



(a) $f(x) = \sqrt{x}$ (b) $f'(x) = \dfrac{1}{2\sqrt{x}}$

EXAMPLE 4 Find f' if $f(x) = \dfrac{1-x}{2+x}$.

SOLUTION

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1-(x+h)}{2+(x+h)} - \dfrac{1-x}{2+x}}{h} \\
&= \lim_{h\to 0} \frac{(1-x-h)(2+x) - (1-x)(2+x+h)}{h(2+x+h)(2+x)} \\
&= \lim_{h\to 0} \frac{(2-x-2h-x^2-xh) - (2-x+h-x^2-xh)}{h(2+x+h)(2+x)} \\
&= \lim_{h\to 0} \frac{-3h}{h(2+x+h)(2+x)} \\
&= \lim_{h\to 0} \frac{-3}{(2+x+h)(2+x)} = -\frac{3}{(2+x)^2}
\end{aligned}
$$

■

Differentiable functions

3 Definition A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

EXAMPLE 5 Where is the function $f(x) = |x|$ differentiable?

4 Theorem If f is differentiable at a, then f is continuous at a.

PROOF To prove that f is continuous at a, we have to show that $\lim_{x \to a} f(x) = f(a)$. We will do this by showing that the difference $f(x) - f(a)$ approaches 0.

The given information is that f is differentiable at a, that is,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists (see Equation 2.7.5). To connect the given and the unknown, we divide and multiply $f(x) - f(a)$ by $x - a$ (which we can do when $x \neq a$):

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a)$$

Thus, using Limit Law 4, we can write

$$\begin{aligned} \lim_{x \to a} [f(x) - f(a)] &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a) \\ &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a) \\ &= f'(a) \cdot 0 = 0 \end{aligned}$$

To use what we have just proved, we start with $f(x)$ and add and subtract $f(a)$:

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} [f(a) + (f(x) - f(a))] \\ &= \lim_{x \to a} f(a) + \lim_{x \to a} [f(x) - f(a)] \\ &= f(a) + 0 = f(a) \end{aligned}$$

NOTE The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0)$$

(See Example 2.3.7.) But in Example 5 we showed that f is not differentiable at 0.

Astonishing fact. There is a continuous function f over $\mathbb{R}$ is not differentiable everywhere!

Higher derivatives

$$f''(x) = (f')'(x) \quad \text{or} \quad f^{(2)}(x)$$

$$\frac{d^2 f}{dx^2}$$

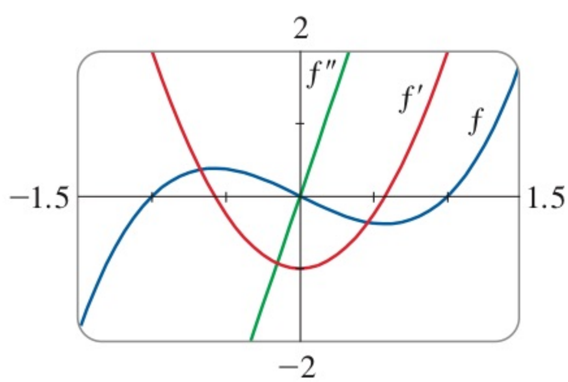
$$y = f(x) < \begin{array}{l} y'' \\ \dfrac{d^2y}{dx^2} \end{array}$$

EXAMPLE 6 If $f(x) = x^3 - x$, find and interpret $f''(x)$.

SOLUTION In Example 2 we found that the first derivative is $f'(x) = 3x^2 - 1$. So the second derivative is

$$\begin{aligned} f''(x) = (f')'(x) &= \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h \to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h} \\ &= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h} \\ &= \lim_{h \to 0} (6x + 3h) = 6x \end{aligned}$$

The graphs of f, f', and f'' are shown in Figure 10.



Differentiation Rules

Constant Functions

Let's start with the simplest of all functions, the constant function $f(x) = c$. The graph of this function is the horizontal line $y = c$, which has slope 0, so we must have $f'(x) = 0$. (See Figure 1.) A formal proof, from the definition of a derivative, is also easy:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0$$

In Leibniz notation, we write this rule as follows.

Derivative of a Constant Function

$$\frac{d}{dx}(c) = 0$$

Power Functions

We next look at the functions $f(x) = x^n$, where n is a positive integer. If $n = 1$, the graph of $f(x) = x$ is the line $y = x$, which has slope 1. (See Figure 2.) So

1
$$\boxed{\frac{d}{dx}(x) = 1}$$

(You can also verify Equation 1 from the definition of a derivative.) We have already investigated the cases $n = 2$ and $n = 3$. In fact, in Section 2.8 (Exercises 19 and 20) we found that

2
$$\frac{d}{dx}(x^2) = 2x \qquad \frac{d}{dx}(x^3) = 3x^2$$

For $n = 4$ we find the derivative of $f(x) = x^4$ as follows:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^4 - x^4}{h} \\ &= \lim_{h \to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \\ &= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\ &= \lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3 \end{aligned}$$

Thus

3
$$\frac{d}{dx}(x^4) = 4x^3$$

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

SECOND PROOF

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

In finding the derivative of x^4 we had to expand $(x + h)^4$. Here we need to expand $(x + h)^n$ and we use the Binomial Theorem to do so:

$$f'(x) = \lim_{h \to 0} \frac{\left[x^n + nx^{n-1}h + \dfrac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h}$$

$$= \lim_{h \to 0} \frac{nx^{n-1}h + \dfrac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h}$$

$$= \lim_{h \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right]$$

$$= nx^{n-1}$$

because every term except the first has h as a factor and therefore approaches 0.

EXAMPLE 1

(a) If $f(x) = x^6$, then $f'(x) = 6x^5$.

(b) If $y = x^{1000}$, then $y' = 1000x^{999}$.

(c) If $y = t^4$, then $\dfrac{dy}{dt} = 4t^3$.

(d) $\dfrac{d}{dr}(r^3) = 3r^2$ ■

The Power Rule (General Version) If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$n = -1$: $\left(\frac{1}{x}\right)' = -x^{-2}$

$n = \frac{1}{2}$: $(\sqrt{x})' = \frac{1}{2} x^{-\frac{1}{2}}$

Rules:

The Constant Multiple Rule If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

The Sum and Difference Rules If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

PROOF To prove the Sum Rule, we let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned}
F'(x) &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\
&= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right] \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \qquad \text{(by Limit Law 1)} \\
&= f'(x) + g'(x)
\end{aligned}$$

To prove the Difference Rule, we write $f - g$ as $f + (-1)g$ and apply the Sum Rule and the Constant Multiple Rule. ■

EXAMPLE 6 Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

SOLUTION Horizontal tangents occur where the derivative is zero. We have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^4) - 6\frac{d}{dx}(x^2) + \frac{d}{dx}(4)\\ &= 4x^3 - 12x + 0 = 4x(x^2 - 3)\end{aligned}$$

Thus $dy/dx = 0$ if $x = 0$ or $x^2 - 3 = 0$, that is, $x = \pm\sqrt{3}$. So the given curve has horizontal tangents when $x = 0$, $\sqrt{3}$, and $-\sqrt{3}$. The corresponding points are $(0, 4)$, $(\sqrt{3}, -5)$, and $(-\sqrt{3}, -5)$. (See Figure 5.) ■

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Before stating the Product Rule, let's see how we might discover it. We start by assuming that $u = f(x)$ and $v = g(x)$ are both positive differentiable functions. Then we can interpret the product uv as an area of a rectangle (see Figure 1). If x changes by an amount Δx, then the corresponding changes in u and v are

$$\Delta u = f(x + \Delta x) - f(x) \qquad \Delta v = g(x + \Delta x) - g(x)$$

and the new value of the product, $(u + \Delta u)(v + \Delta v)$, can be interpreted as the area of the large rectangle in Figure 1 (provided that Δu and Δv happen to be positive).

The change in the area of the rectangle is

1 $$\begin{aligned}\Delta(uv) &= (u + \Delta u)(v + \Delta v) - uv = u\,\Delta v + v\,\Delta u + \Delta u\,\Delta v\\ &= \text{the sum of the three shaded areas}\end{aligned}$$

If we divide by Δx, we get

$$\frac{\Delta(uv)}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}$$

If we now let $\Delta x \to 0$, we get the derivative of uv:

$$\begin{aligned}\frac{d}{dx}(uv) &= \lim_{\Delta x\to 0}\frac{\Delta(uv)}{\Delta x} = \lim_{\Delta x\to 0}\left(u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}\right)\\ &= u\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x} + v\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} + \left(\lim_{\Delta x\to 0}\Delta u\right)\left(\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}\right)\\ &= u\frac{dv}{dx} + v\frac{du}{dx} + 0\cdot\frac{dv}{dx}\end{aligned}$$

2 $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

e.g.

EXAMPLE 2 Differentiate the function $f(t) = \sqrt{t}\,(a + bt)$.

SOLUTION 1 Using the Product Rule, we have

$$\begin{aligned} f'(t) &= \sqrt{t}\,\frac{d}{dt}(a + bt) + (a + bt)\frac{d}{dt}(\sqrt{t}\,) \\ &= \sqrt{t} \cdot b + (a + bt) \cdot \tfrac{1}{2}t^{-1/2} \\ &= b\sqrt{t} + \frac{a + bt}{2\sqrt{t}} = \frac{a + 3bt}{2\sqrt{t}} \end{aligned}$$

SOLUTION 2 If we first use the laws of exponents to rewrite $f(t)$, then we can proceed directly without using the Product Rule.

$$f(t) = a\sqrt{t} + bt\sqrt{t} = at^{1/2} + bt^{3/2}$$

$$f'(t) = \tfrac{1}{2}at^{-1/2} + \tfrac{3}{2}bt^{1/2}$$

which is equivalent to the answer given in Solution 1. ■

EXAMPLE 3 If $f(x) = \sqrt{x}\,g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

SOLUTION Applying the Product Rule, we get

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\sqrt{x}\,g(x)\right] = \sqrt{x}\,\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}\left[\sqrt{x}\right] \\ &= \sqrt{x}\,g'(x) + g(x) \cdot \tfrac{1}{2}x^{-1/2} \\ &= \sqrt{x}\,g'(x) + \frac{g(x)}{2\sqrt{x}} \end{aligned}$$

So $$f'(4) = \sqrt{4}\,g'(4) + \frac{g(4)}{2\sqrt{4}} = 2 \cdot 3 + \frac{2}{2 \cdot 2} = 6.5$$

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,\dfrac{d}{dx}[f(x)] - f(x)\,\dfrac{d}{dx}[g(x)]}{[g(x)]^2}$$

$$\Delta\left(\frac{u}{v}\right) = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{(u + \Delta u)v - u(v + \Delta v)}{v(v + \Delta v)}$$

$$= \frac{v\Delta u - u\,\Delta v}{v(v + \Delta v)}$$

so $$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{\Delta x \to 0} \frac{\Delta(u/v)}{\Delta x} = \lim_{\Delta x \to 0} \frac{v\,\dfrac{\Delta u}{\Delta x} - u\,\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}$$

As $\Delta x \to 0$, $\Delta v \to 0$ also, because $v = g(x)$ is differentiable and therefore continuous. Thus, using the Limit Laws, we get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \displaystyle\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} - u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}}{v \displaystyle\lim_{\Delta x \to 0} (v + \Delta v)} = \frac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}$$

EXAMPLE 4 Let $y = \dfrac{x^2 + x - 2}{x^3 + 6}$. Then

$$y' = \frac{(x^3 + 6)\,\dfrac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\,\dfrac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

■

Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf' \qquad (f + g)' = f' + g' \qquad (f - g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

we can get derivative of any <u>algebraic functions</u> by these laws

Polynomial P

$\Downarrow$

Rational $\frac{P}{Q}$

$\Downarrow$

algebraic

Exponential functions

Let's try to compute the derivative of the exponential function $f(x) = b^x$ using the definition of a derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}$$
$$= \lim_{h \to 0} \frac{b^x b^h - b^x}{h} = \lim_{h \to 0} \frac{b^x(b^h - 1)}{h}$$

The factor b^x doesn't depend on h, so we can take it in front of the limit:

$$f'(x) = b^x \lim_{h \to 0} \frac{b^h - 1}{h}$$

Notice that the limit is the value of the derivative of f at 0, that is,

$$\lim_{h \to 0} \frac{b^h - 1}{h} = f'(0)$$

Therefore we have shown that if the exponential function $f(x) = b^x$ is differentiable at 0, then it is differentiable everywhere and

4 $$f'(x) = f'(0)\, b^x$$

This equation says that *the rate of change of any exponential function is proportional to the function itself*. (The slope is proportional to the height.)

Numerical evidence for the existence of $f'(0)$ is given in the table at the left for the cases $b = 2$ and $b = 3$. (Values are stated correct to four decimal places.) It can be proved that the limits exist and

$$\text{for } b = 2, \quad f'(0) = \lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.693$$

$$\text{for } b = 3, \quad f'(0) = \lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.099$$

Thus, from Equation 4 we have

5 $$\frac{d}{dx}(2^x) \approx (0.693)2^x \qquad \frac{d}{dx}(3^x) \approx (1.099)3^x$$

Definition of the Number e

$$e \text{ is the number such that } \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

EXAMPLE 8 If $f(x) = e^x - x$, find f' and f''. Compare the graphs of f and f'.

SOLUTION Using the Difference Rule, we have

$$f'(x) = \frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x) = e^x - 1$$

In Section 2.8 we defined the second derivative as the derivative of f', so

$$f''(x) = \frac{d}{dx}(e^x - 1) = \frac{d}{dx}(e^x) - \frac{d}{dx}(1) = e^x$$

The function f and its derivative f' are graphed in Figure 8. Notice that f has a horizontal tangent when $x = 0$; this corresponds to the fact that $f'(0) = 0$. Notice also that, for $x > 0$, $f'(x)$ is positive and f is increasing. When $x < 0$, $f'(x)$ is negative and f is decreasing. ■

EXAMPLE 9 At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

SOLUTION Since $y = e^x$, we have $y' = e^x$. Let the x-coordinate of the point in question be a. Then the slope of the tangent line at that point is e^a. This tangent line will be parallel to the line $y = 2x$ if it has the same slope, that is, 2. Equating slopes, we get

$$e^a = 2 \qquad a = \ln 2$$

Therefore the required point is $(a, e^a) = (\ln 2, 2)$. (See Figure 9.) ■

In general: $(b^x)' = b^x \cdot (\ln b)$ (will be proved later)

EXAMPLE 1

(a) If $f(x) = xe^x$, find $f'(x)$.

(b) Find the nth derivative, $f^{(n)}(x)$.

SOLUTION

(a) By the Product Rule, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}(xe^x) \\ &= x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x) \\ &= xe^x + e^x \cdot 1 = (x+1)e^x \end{aligned}$$

(b) Using the Product Rule a second time, we get

$$\begin{aligned} f''(x) &= \frac{d}{dx}[(x+1)e^x] \\ &= (x+1)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+1) \\ &= (x+1)e^x + e^x \cdot 1 = (x+2)e^x \end{aligned}$$

Further applications of the Product Rule give

$$f'''(x) = (x+3)e^x \qquad f^{(4)}(x) = (x+4)e^x$$

In fact, each successive differentiation adds another term e^x, so

$$f^{(n)}(x) = (x+n)e^x$$

■