Fr 5/17

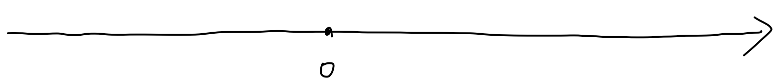
Course Overview

0. Functions
1. Limits & Derivatives
2. Applications of Derivatives
3. Definite integral

Notations:

$\mathbb{Z}$	$0, \pm 1, \pm 2, \pm 3, \ldots$	
$\mathbb{Q}$	$\frac{p}{q}$	set of rationals
$\mathbb{R}$	$\sqrt{2}, \pi, e$	real numbers
$\mathbb{C}$	$x + iy$	complex numbers

usually, we use a line (real line) to represent real numbers



(positive direction)

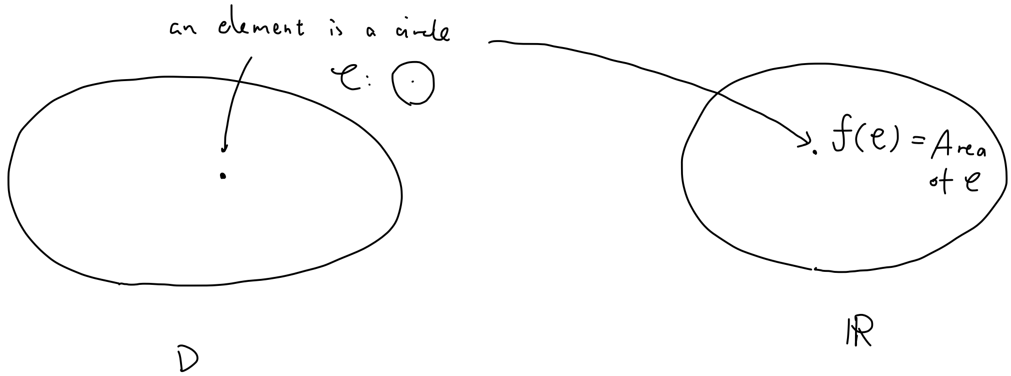
0. Functions

Def: A function consists of $(\underset{\text{domain}}{D}, \underset{\text{range}}{R}, \underset{\text{law}}{f})$

Example: D = the set of all the radius

$R = \mathbb{R}$

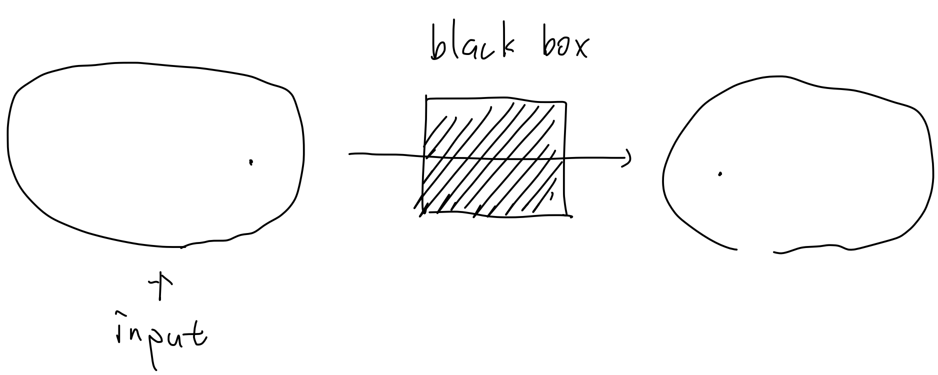
f : Circle $\mathcal{C} \longrightarrow$ Area of $\mathcal{C}$



In our class, function are usually

D : a subset of $\mathbb{R}$, (usually intervals $I = (a, b)$)

R : a subset of $\mathbb{R}$



We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the **domain** of the function. The number $f(x)$ is the **value of f at x** and is read "f of x." The **range** of f is the set of all possible values of $f(x)$ as x varies

$f(x)$: value of f at x,

f of x, or just $f\,x$

- How to represent a function?
  - by formulas

$$f(x) = \pi x^2 \qquad f(1) = \pi \cdot 1^2 = \pi, \quad f(2) = \pi \cdot 2^2 = 4\pi$$

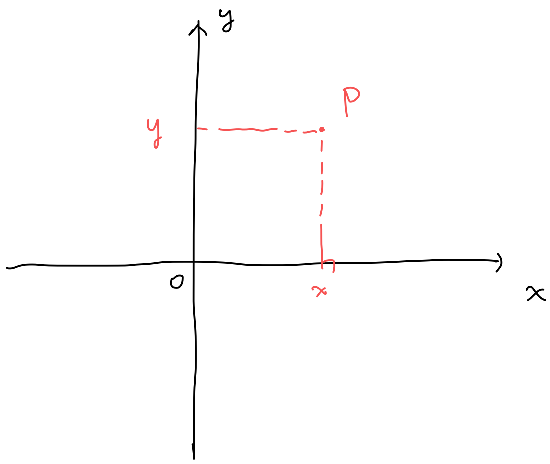
  - Ex. Evaluate: $\dfrac{f(a+h) - f(a)}{h}$ $\quad a, h \in \mathbb{R}$

$$= \frac{\pi (a+h)^2 - \pi a^2}{h}$$

$$= \frac{\pi a^2 + 2\pi a h + \pi h^2 - \pi a^2}{h}$$

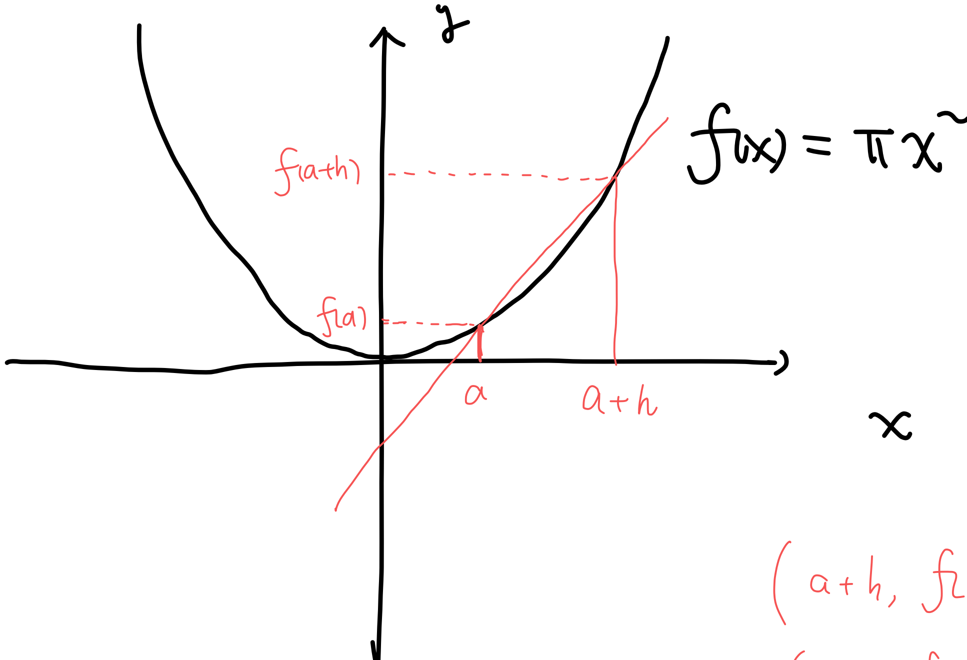
$$= 2\pi a + \pi h$$

· by graph: in rectangular coordinate



$$\{\text{points on the plane}\} \xleftrightarrow{1-1} \{\text{pairs of numbers } (x,y)\}$$

graph of a function: $\{(x, f(x)) : x \in D\}$



$(a+h, f(a+h))$

$(a, f(a))$

$$\text{Slope} = \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h) - f(a)}{h}$$

More Examples of functions

- Absolute Value function

$$|\cdot| : x \longmapsto |x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$$

- Even & Odd function

Even: $\quad f(x) = f(-x), \ \forall x \in D$

$$f(x) = x^2$$

Odd: $\quad f(-x) = -f(x), \ \forall x \in D$

$$f(x) = x, \quad f(x) = x^3$$

- Increasing & Decreasing functions

A function f is called **increasing** on an interval I if

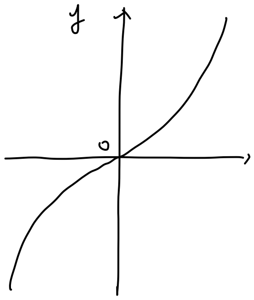
$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on I if

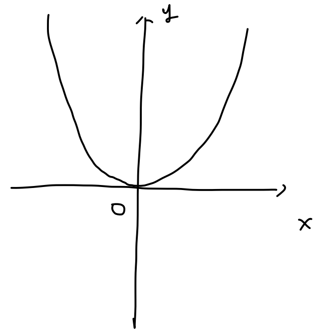
$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

Power functions: $f(x) = x^a$

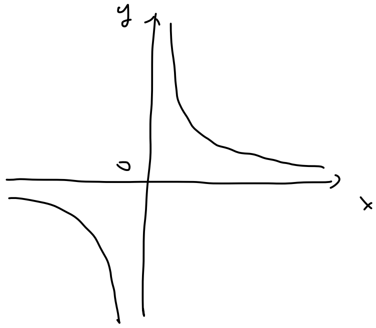
$a > 0$ odd



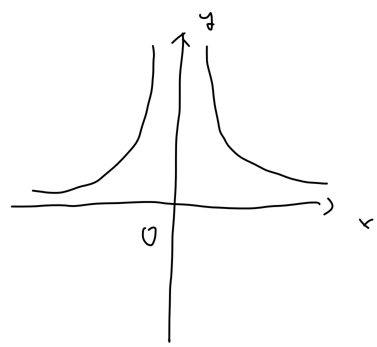
$a > 0$ even



$a < 0$ odd

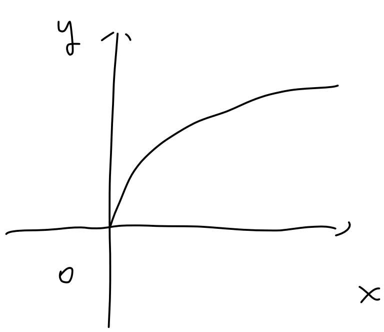


$a < 0$ even

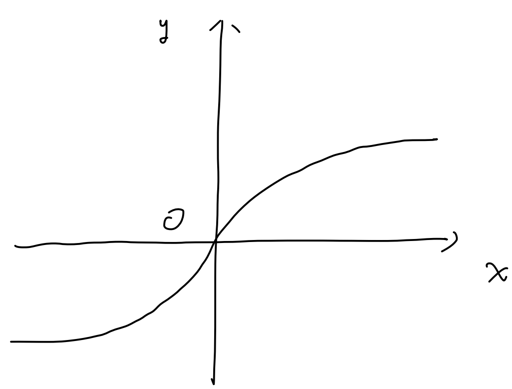


a can even be rational numbers

$a = 1/2$



$a = 1/3$



Polynomial functions

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

$a_i \in \mathbb{R} \Rightarrow$ coefficients of $P(x)$

$a_n \Rightarrow$ leading coefficient, $a_n \neq 0$

$a_0 \Rightarrow$ constant term

Rational functions

$$f(x) = \frac{P(x)}{Q(x)}, \quad P(x) \text{ \& } Q(x) \text{ are polynomial functions}$$

Algebraic functions

A function f is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials. Any rational function is automatically an algebraic function. Here

$$f(x) = \sqrt{x^2 + 1} \qquad g(x) = \frac{x^4 - 16x^2}{x + \sqrt{x}} + (x - 2)\sqrt[3]{x + 1}$$

An example of an algebraic function occurs in the theory of relativity. The mass of a particle with velocity v is

$$m = f(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle and $c = 3.0 \times 10^5$ km/s is the speed of light in a vacuum.

Functions that are not algebraic are called **transcendental**; these include the trigonometric, exponential, and logarithmic functions.

Trigonometric functions

$\sin(x)$

$\cos(x)$

most important!

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

$$\csc x = \frac{1}{\sin x},$$

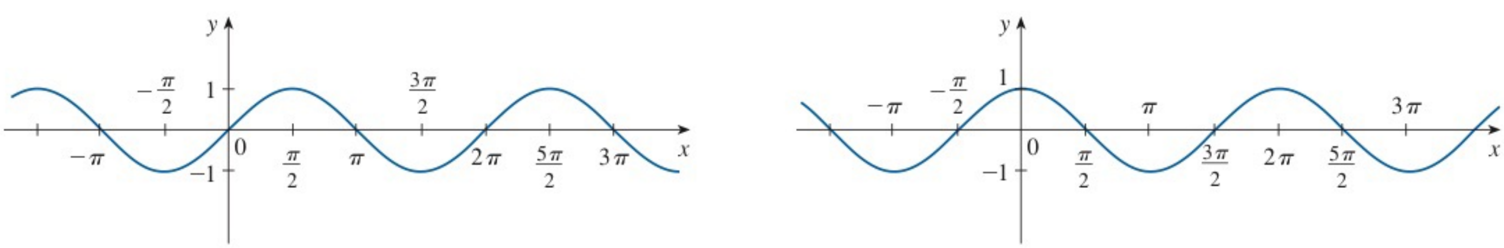
$$\sec x = \frac{1}{\cos x}$$

Recall: definition $\pi \text{ rad} = 180^\circ$

$$x \text{ rad} = \left(\frac{180x}{\pi}\right)^\circ$$

$$\sin(x) = \sin\left(\left(\frac{180x}{\pi}\right)^\circ\right)$$

$$\cos(x) = \cos\left(\left(\frac{180x}{\pi}\right)^\circ\right)$$



(a) $f(x) = \sin x$

(b) $g(x) = \cos x$

Periodicity

An important property of the sine and cosine functions is that they are periodic functions and have period 2π. This means that, for all values of x,

$$\boxed{\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x}$$

EXAMPLE 5 Find the domain of the function $f(x) = \dfrac{1}{1 - 2\cos x}$.

SOLUTION This function is defined for all values of x except for those that make the denominator 0. But

$$1 - 2\cos x = 0 \iff \cos x = \frac{1}{2} \iff x = \frac{\pi}{3} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{3} + 2n\pi$$

where n is any integer (because the cosine function has period 2π). So the domain of f is the set of all real numbers except for the ones noted above. ■

Exponential Functions

We will now study some "non-algebraic" functions

Look back on the "exponent function"

$$x^r$$

we only defined this when r is a rational number

If we fix a number, say 2, then our next goal is to extend

$$2^r$$

from rational exponent r to all the real numbers !

Idea: For any non-rational number r

we can find a sequence of rational number r_n approaches

r as $n \to +\infty$,

then we get a sequence of numbers $\{2^{r_n}\}$

Phenomenon: 2^{r_n} will approach some number in $\mathbb{R}$

i.e. there exists a real number S, s.t.

$$2^{r_n} \longrightarrow S$$

There are holes in the graph in Figure 1 corresponding to irrational values of x. We want to fill in the holes by defining $f(x) = 2^x$, where $x \in \mathbb{R}$, so that f is an increasing function. In particular, since the irrational number $\sqrt{3}$ satisfies

$$1.7 < \sqrt{3} < 1.8$$

we must have

$$2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}$$

and we know what $2^{1.7}$ and $2^{1.8}$ mean because 1.7 and 1.8 are rational numbers. Similarly, if we use better approximations for $\sqrt{3}$, we obtain better approximations for $2^{\sqrt{3}}$:

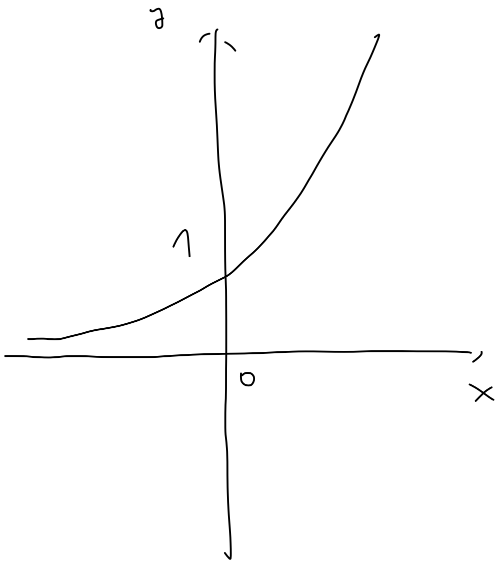
$$\begin{aligned}
1.73 < \sqrt{3} < 1.74 \quad &\Rightarrow \quad 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74} \\
1.732 < \sqrt{3} < 1.733 \quad &\Rightarrow \quad 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733} \\
1.7320 < \sqrt{3} < 1.7321 \quad &\Rightarrow \quad 2^{1.7320} < 2^{\sqrt{3}} < 2^{1.7321} \\
1.73205 < \sqrt{3} < 1.73206 \quad &\Rightarrow \quad 2^{1.73205} < 2^{\sqrt{3}} < 2^{1.73206}
\end{aligned}$$

Laws of Exponents If a and b are positive numbers and x and y are any real numbers, then

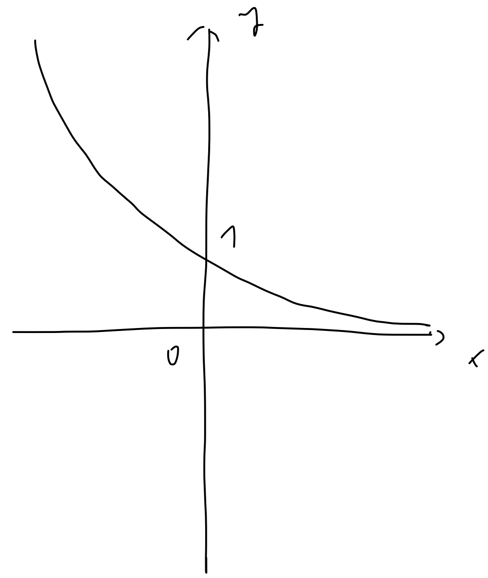
1. $b^{x+y} = b^x b^y$ **2.** $b^{x-y} = \dfrac{b^x}{b^y}$ **3.** $(b^x)^y = b^{xy}$ **4.** $(ab)^x = a^x b^x$

Exponential functions

$$f(x) = a^x \quad , \quad (a > 0, \ a \neq 1)$$

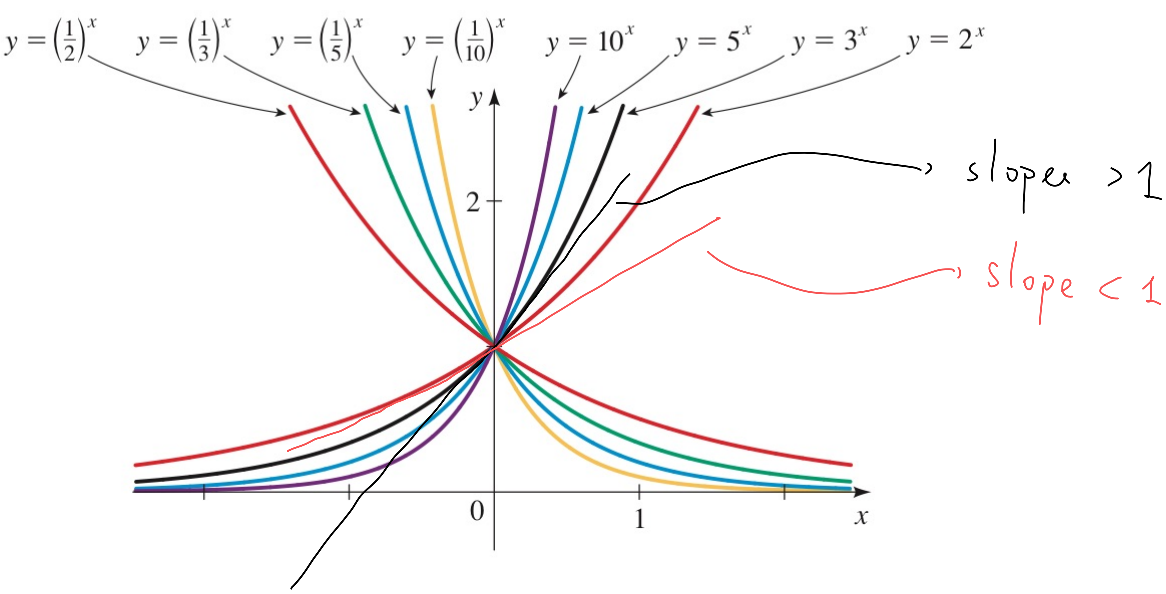


$a > 1$



$0 < a < 1$

tangent line at $(0, 1)$



Natural exponential:

$f(x) = e^x$: its tangent line at $(0,1)$ has slope 1

$$2 < e < 3$$

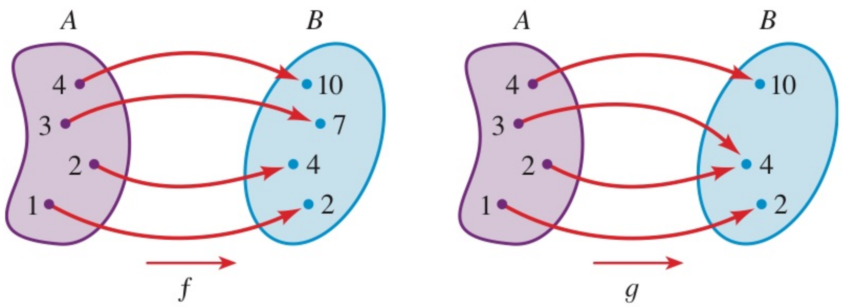
$$e \approx 2.71828 \, 23 \ldots$$

- we will encounter e in future classes repeatedly

One-to-One functions and their inverses

One-to-One Functions

Let's compare the functions f and g whose arrow diagrams are shown in Figure 1. Note that f never takes on the same value twice (any two numbers in A have different images), whereas g does take on the same value twice (both 2 and 3 have the same image, 4). In symbols, $g(2) = g(3)$ but $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. Functions that have this latter property are called *one-to-one*.



f is one-to-one

g is not one-to-one

FIGURE 1

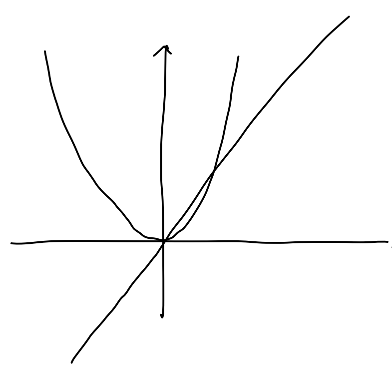
DEFINITION OF A ONE-TO-ONE FUNCTION

A function with domain A is called a **one-to-one function** if no two elements of A have the same image, that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2$$

Ex / Counter-example:

- $f(x) = x$
- $f(x) = x^2$



horizontal line test:

HORIZONTAL LINE TEST

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Inverse function:

The Inverse of a Function

One-to-one functions are important because they are precisely the functions that possess inverse functions according to the following definition.

DEFINITION OF THE INVERSE OF A FUNCTION

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any y in B.

Example: n-th power & principal n-th root

$f(x) = x^n$

$f^{-1}(y) = x$, s.t. $f(x) = y$, i.e. $x^n = y$

$= y^{\frac{1}{n}} = \sqrt[n]{y}$

This definition says that if f takes x to y, then f^{-1} takes y back to x. (If f were not one-to-one, then f^{-1} would not be defined uniquely.) The arrow diagram in Figure 6 indicates that f^{-1} reverses the effect of f. From the definition we have

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

Note: $f^{-1}(x)$ doesn't mean $\frac{1}{f(x)}$

The letter x is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on f, we usually reverse the roles of x and y in Definition 2 and write

3

$$f^{-1}(x) = y \iff f(y) = x$$

By substituting for y in Definition 2 and substituting for x in (3), we get the following **cancellation equations**:

4

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for every } x \text{ in } A \\ f(f^{-1}(x)) &= x \quad \text{for every } x \text{ in } B \end{aligned}$$

Example: logarithmic functions

$f(x) = a^x \quad (a>0,\ a \neq 1)$

is one-to-one: $\mathbb{R} \longrightarrow \mathbb{R}_{>0}$

so that we can define its inverse

$f^{-1}(x) =: \log_a x : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$

When we apply the Inverse Function Property described on page 222 to $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, we get

$$\log_a(a^x) = x \qquad x \in \mathbb{R}$$

$$a^{\log_a x} = x \qquad x > 0$$

We list these and other properties of logarithms discussed in this section.

PROPERTIES OF LOGARITHMS

Property	Reason
1. $\log_a 1 = 0$	We must raise a to the power 0 to get 1.
2. $\log_a a = 1$	We must raise a to the power 1 to get a.
3. $\log_a a^x = x$	We must raise a to the power x to get a^x.
4. $a^{\log_a x} = x$	$\log_a x$ is the power to which a must be raised to get x.

Laws of Logarithms If x and y are positive numbers, then

1. $\log_b(xy) = \log_b x + \log_b y$

2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

3. $\log_b(x^r) = r\log_b x \qquad$ (where r is any real number)

Graphs

Recall that if a one-to-one function f has domain A and range B, then its inverse function f^{-1} has domain B and range A. Since the exponential function $f(x) = a^x$ with $a \neq 1$ has domain $\mathbb{R}$ and range $(0, \infty)$, we conclude that its inverse function, $f^{-1}(x) = \log_a x$, has domain $(0, \infty)$ and range $\mathbb{R}$.

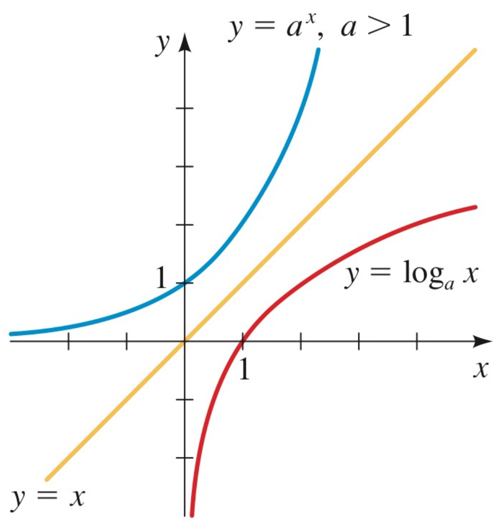
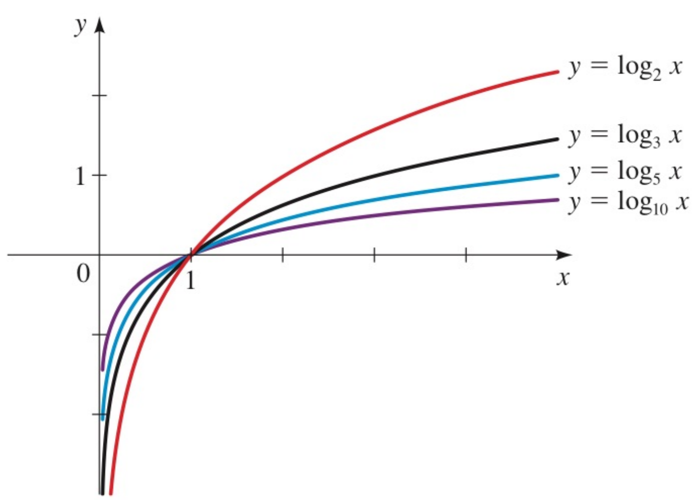


FIGURE 2 Graph of the logarithmic function $f(x) = \log_a x$

Figure 4 shows the graphs of the family of logarithmic functions with bases 2, 3, 5, and 10. These graphs are drawn by reflecting the graphs of $y = 2^x$, $y = 3^x$, $y = 5^x$, and $y = 10^x$ (see Figure 2 in Section 4.1) in the line $y = x$. We can also plot points as an aid to sketching these graphs, as illustrated in Example 4.



Special one: <u>Natural logarithmic</u>

$a = e$:

$$\ln x = \log_e x$$

Example: Inverse trigonometric function

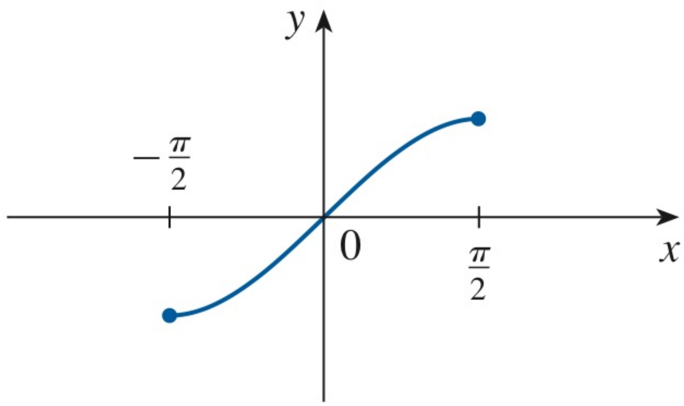


FIGURE 18
$y = \sin x, -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

Sin is 1-1 on this interval,

hence we can define its inverse

$\sin^{-1}: [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\boxed{\sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}}$$

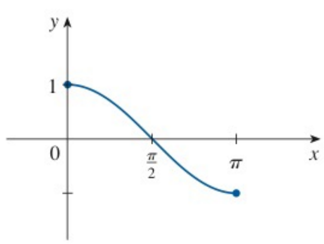


FIGURE 21
$y = \cos x, 0 \leqslant x \leqslant \pi$

The **inverse cosine function** is handled similarly. The restricted cosine function $f(x) = \cos x$, $0 \leqslant x \leqslant \pi$, is one-to-one (see Figure 21) and so it has an inverse function denoted by $\cos^{-1}$ or arccos.

$$\boxed{\cos^{-1}x = y \iff \cos y = x \quad \text{and} \quad 0 \leqslant y \leqslant \pi}$$

The cancellation equations are

$$\begin{aligned} \cos^{-1}(\cos x) &= x \quad \text{for } 0 \leqslant x \leqslant \pi \\ \cos(\cos^{-1}x) &= x \quad \text{for } -1 \leqslant x \leqslant 1 \end{aligned}$$

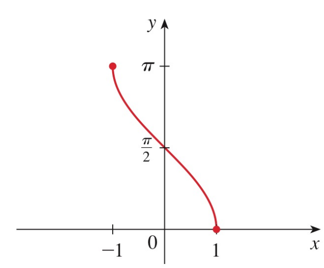


FIGURE 22
$y = \cos^{-1}x = \arccos x$

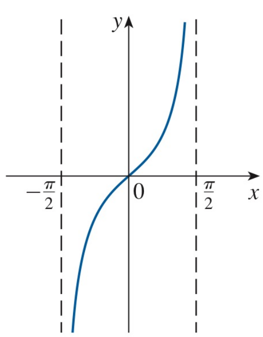


FIGURE 23
$y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

The tangent function can be made one-to-one by restricting it to the interval $(-\pi/2, \pi/2)$. Thus the **inverse tangent function** is defined as the inverse of the function $f(x) = \tan x$, $-\pi/2 < x < \pi/2$. (See Figure 23.) It is denoted by $\tan^{-1}$ or arctan.

$$\boxed{\tan^{-1}x = y \iff \tan y = x \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}}$$

The inverse tangent function, $\tan^{-1} = \arctan$, has domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$. Its graph is shown in Figure 25.