Tu 6/20

Parametric surfaces and their areas

Parametric surfaces :

recall: for a space curve, we use the parametrization

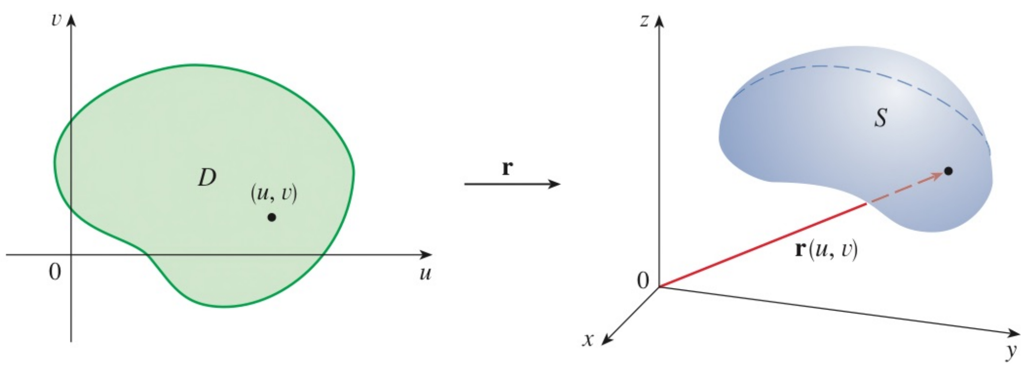
$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}\quad,\quad t \in [a,b]$$

we only need one parameter t !

For a surface, we certainly need 2 parameters u & v, then we can write

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k} \qquad (u,v) \in D$$

this describes a surface! parametrized by D.



Example:

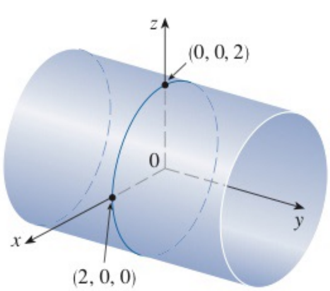


FIGURE 2

EXAMPLE 1 Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = 2\cos u\,\mathbf{i} + v\,\mathbf{j} + 2\sin u\,\mathbf{k}$$

SOLUTION The parametric equations for this surface are

$$x = 2\cos u \qquad y = v \qquad z = 2\sin u$$

So for any point (x, y, z) on the surface, we have

$$x^2 + z^2 = 4\cos^2 u + 4\sin^2 u = 4$$

This means that vertical cross-sections parallel to the xz-plane (that is, with y constant) are all circles with radius 2. Since $y = v$ and no restriction is placed on v, the surface is a circular cylinder with radius 2 whose axis is the y-axis (see Figure 2). ■

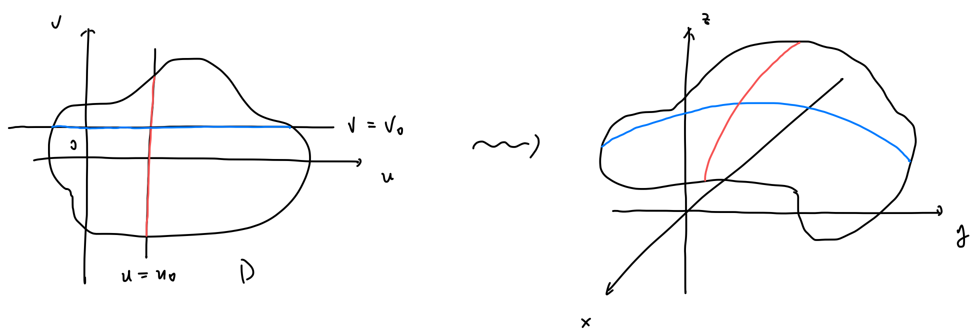
grid curves : Given parametric equations

$$\vec{r}(u, v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

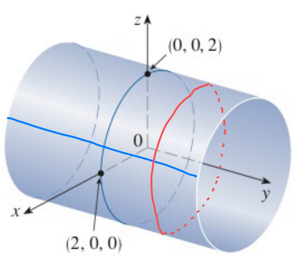
fix $u = u_0$, we get a space curve whose parametric equation is given by

$$\vec{r}(u_0, v) = x(u_0,v)\,\vec{i} + y(u_0,v)\,\vec{j} + z(u_0,v)\,\vec{k} \qquad v \text{ ranges}$$

$\left.\begin{array}{l}\text{for different } u_0\text{, we get a family of curves} \\ \text{fix } v = v_0\text{, we get another family of curves}\end{array}\right\} \Rightarrow$ grid curves



In previous example



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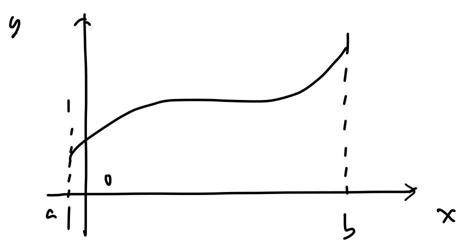
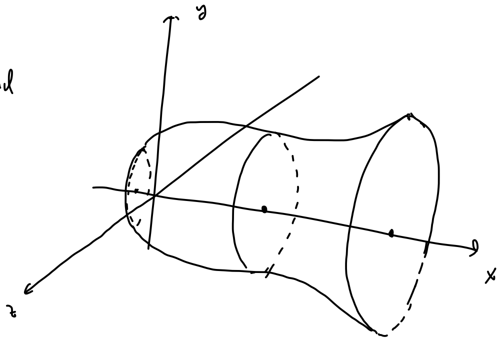
grid curves make up the whole surface

Surfaces of revolution

Given a function f, consider the graph of $y = f(x)$, $a \le x \le b$



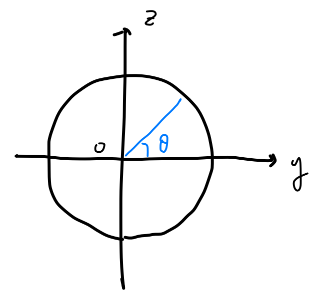
revolve around x-axis $\longrightarrow$



then we can give a parametrization of the surface by:

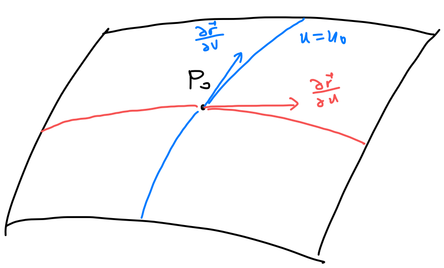
$$\underbrace{(x, \theta)}_{\substack{x \in [a, b],\ \theta \in [0, 2\pi] \\ \downarrow \\ \text{this is } D}} \longrightarrow \left\{ \begin{array}{l} x = x \\ y = f(x) \cos\theta \\ z = f(x) \sin\theta \end{array} \right.$$

$\Big\}\Rightarrow$ take a section of x



Tangent planes:

Consider the point $P_0 = \vec{r}(u_0, v_0)$, we are going to study the tangent space of the surface at P_0



tangent space of S at P_0 is the space spanned by $\frac{\partial \vec{r}}{\partial v}(u_0, v_0)$ & $\frac{\partial \vec{r}}{\partial u}(u_0, v_0)$

Consider the grid curve $u = u_0$

$$\vec{r}(u_0, v) = \left(x(u_0, v),\ y(u_0, v),\ z(u_0, v) \right)$$

it's tangent vector at P_0 is

$$\frac{\partial \vec{r}}{\partial v}(u_0, v_0) = \left(\frac{\partial x}{\partial v}(u_0, v_0),\ \frac{\partial y}{\partial v}(u_0, v_0),\ \frac{\partial z}{\partial v}(u_0, v_0) \right)$$

Consider the grid curve $v = v_0$

$$\vec{r}(u, v_0) = \left(x(u, v_0),\ y(u, v_0),\ z(u, v_0) \right)$$

it's tangent vector at P_0 is

$$\frac{\partial \vec{r}}{\partial u}(u_0, v_0) = \left(\frac{\partial x}{\partial u}(u_0, v_0),\ \frac{\partial y}{\partial u}(u_0, v_0),\ \frac{\partial z}{\partial u}(u_0, v_0) \right)$$

Smoothness: we say a parametric surface $S: \vec{r}(u,v)$ is smooth if

$$\vec{r}_u \times \vec{r}_v \neq 0 \quad \text{for } \forall P \in S$$

Note, if & only if $\vec{r}_u$ & $\vec{r}_v$ are not proportional to each other

$\Leftrightarrow$ the tangent space is 2-dim'l

then $\vec{r}_u \times \vec{r}_v$ is the normal vector of the tangent space!

Example

EXAMPLE 9 Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point $(1, 1, 3)$.

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u} = (2u, 0, 1), \quad \vec{r}_v = \frac{\partial \vec{r}}{\partial v} = (0, 2v, 2)$$

then at the point $(1,1,3)$,

$$\vec{r}_u = (2, 0, 1), \quad \vec{r}_v = (0, 2, 2), \text{ hence}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ 0 & 2 & 2 \end{vmatrix} = (-2, -4, 4)$$

then the tangent space is: $\quad x-1+2(y-1)-2(z-3)$

$$-2(x-1) - 4(y-1) + 4(z-3) = 0 \iff x + 2y - 2z = 1+2-6 = -3$$

the smooth locus: where $\vec{r}_u \times \vec{r}_v \neq 0$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = (-2v, -4u, 4uv)$$

hence S is smooth $\Leftrightarrow$ u, v are not both 0

EXAMPLE 3 Find a vector function that represents the plane that passes through the point P_0 with position vector $\mathbf{r}_0$ and that contains two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$.

SOLUTION If P is any point in the plane, we can get from P_0 to P by moving a certain distance in the direction of $\mathbf{a}$ and another distance in the direction of $\mathbf{b}$. So there are scalars u and v such that $\overrightarrow{P_0P} = u\mathbf{a} + v\mathbf{b}$. (Figure 6 illustrates how this works, by means of the Parallelogram Law, for the case where u and v are positive. See also Exercise 12.2.46.) If $\mathbf{r}$ is the position vector of P, then

$$\mathbf{r} = \overrightarrow{OP_0} + \overrightarrow{P_0P} = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

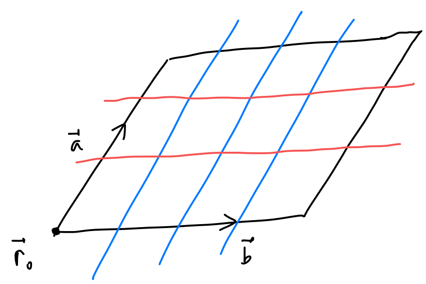
So the vector equation of the plane can be written as

$$\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

where u and v are real numbers.

If we write $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then we can write the parametric equations of the plane through the point (x_0, y_0, z_0) as follows:

$$x = x_0 + ua_1 + vb_1 \qquad y = y_0 + ua_2 + vb_2 \qquad z = z_0 + ua_3 + vb_3$$ ■



EXAMPLE 4 Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = a^2$$

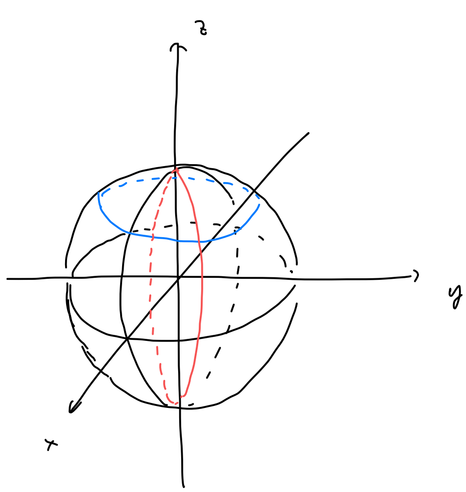
$x = a \sin\phi \cos\theta$

$y = a \sin\phi \sin\theta$

$z = a \cos\phi$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

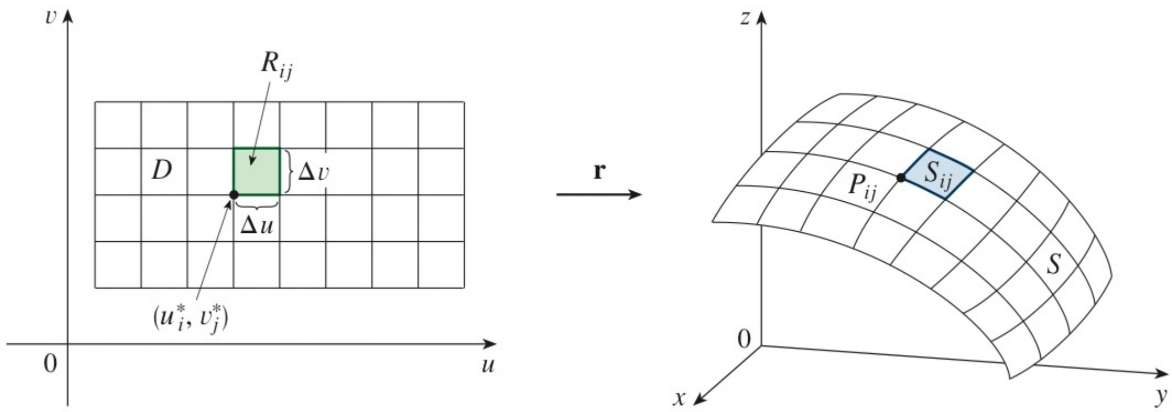


Surface Area

Now suppose we are given a parametric surface

$$S: \quad \vec{r} = \vec{r}(u,v), \quad \text{for } (u,v) \in D$$

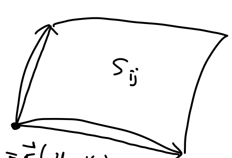
How to compute the area of the surface S ?



Step 1: divide the region D into sub-rectangles R_{ij}, S_{ij} = image of R_{ij} under $\vec{r}$

then $\quad \text{area}(S) = \sum_{i,j} \text{area}(S_{ij})$

Step 2: Approximate S_{ij}



$P_{ij} = \vec{r}(u_i, v_j)$

$$\text{area}(S_{ij}) \approx \left| \left(\vec{r}(u_i+\Delta u, v_j) - \vec{r}(u_i, v_j) \right) \times \left(\vec{r}(u_i, v_j+\Delta v) - \vec{r}(u_i, v_j) \right) \right|$$

$$= \left| \vec{r}_u(u_i^*) \cdot \Delta u \times \vec{r}_v(v_j^*) \cdot \Delta v \right|$$

$$= \left| \vec{r}_u(u_i^*) \times \vec{r}_v(v_j^*) \right| \cdot \Delta u \, \Delta v$$

$$\Rightarrow \quad \text{area}(S) = \sum_{i,j} |\vec{r}_u \times \vec{r}_v| \cdot \Delta u \, \Delta v$$

6 Definition If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k} \qquad (u,v) \in D$$

and S is covered just once as (u,v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

where $\quad \mathbf{r}_u = \dfrac{\partial x}{\partial u}\mathbf{i} + \dfrac{\partial y}{\partial u}\mathbf{j} + \dfrac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \dfrac{\partial x}{\partial v}\mathbf{i} + \dfrac{\partial y}{\partial v}\mathbf{j} + \dfrac{\partial z}{\partial v}\mathbf{k}$

EXAMPLE 10 Find the surface area of a sphere of radius a.

SOLUTION In Example 4 we found the parametric representation

$$x = a \sin\phi \cos\theta \qquad y = a \sin\phi \sin\theta \qquad z = a\cos\phi$$

where the parameter domain is

$$D = \{(\phi, \theta) \mid 0 \leqslant \phi \leqslant \pi,\ 0 \leqslant \theta \leqslant 2\pi\}$$

We first compute the cross product of the tangent vectors:

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial \phi} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial z}{\partial \phi} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin^2\phi\cos\theta\,\mathbf{i} + a^2\sin^2\phi\sin\theta\,\mathbf{j} + a^2\sin\phi\cos\phi\,\mathbf{k}$$

Thus

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{a^4\sin^4\phi + a^4\sin^2\phi\cos^2\phi} = a^2\sqrt{\sin^2\phi} = a^2\sin\phi$$

since $\sin\phi \geqslant 0$ for $0 \leqslant \phi \leqslant \pi$. Therefore, by Definition 6, the area of the sphere is

$$A = \iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta|\,dA = \int_0^{2\pi}\int_0^{\pi} a^2\sin\phi\,d\phi\,d\theta$$

$$= a^2\int_0^{2\pi} d\theta \int_0^{\pi}\sin\phi\,d\phi = a^2(2\pi)2 = 4\pi a^2$$

Review: surface area of the graph of a function

Suppose we have a function $z = f(x,y)$, $(x,y) \in D$

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + f(x,y)\vec{k} = (x, y, f(x,y))$$

then $\vec{r}_x = \left(1, 0, \dfrac{\partial f}{\partial x}\right)$, $\vec{r}_y = \left(0, 1, \dfrac{\partial f}{\partial y}\right)$, hence

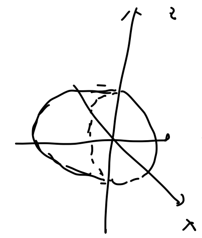
$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \dfrac{\partial f}{\partial x} \\ 0 & 1 & \dfrac{\partial f}{\partial y} \end{vmatrix} = \left(-\dfrac{\partial f}{\partial x}, -\dfrac{\partial f}{\partial y}, 1\right)$$

then area $= \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,dA$

Exercises :

22. The part of the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ that lies to the left of the xz-plane

$$\left.\begin{aligned} x &= \sin\phi \cos\theta \\ y &= \frac{1}{\sqrt{2}} \sin\phi \sin\theta \\ z &= \frac{1}{\sqrt{3}} \cos\phi \end{aligned}\right\} \Rightarrow \begin{aligned} \pi \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{aligned}$$



36. $\mathbf{r}(u, v) = \sin u\,\mathbf{i} + \cos u \sin v\,\mathbf{j} + \sin v\,\mathbf{k}$; $u = \pi/6,\ v = \pi/6$

$\vec{r}_u = (\cos u, -\sin u \sin v, 0)$

$\vec{r}_v = (0, \cos u \cos v, \cos v)$

at $u = \frac{\pi}{6}$ & $v = \frac{\pi}{6}$, $\vec{r}_u = \left(\frac{\sqrt{3}}{2}, -\frac{1}{4}, 0\right)$, $\vec{r}_v = \left(0, \frac{3}{4}, \frac{\sqrt{3}}{2}\right)$

then $r\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right)$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\sqrt{3}}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{3}{4} & \frac{\sqrt{3}}{2} \end{vmatrix} = \left(-\frac{\sqrt{3}}{8}, -\frac{3}{4}, \frac{3\sqrt{3}}{8}\right)$$

$$-\frac{\sqrt{3}}{8}\left(x - \frac{1}{2}\right) - \frac{3}{4}\left(y - \frac{\sqrt{3}}{4}\right) + \frac{3\sqrt{3}}{8}\left(z - \frac{1}{2}\right) = 0$$

Surface Integrals

Goal: analogue of line integrals in the case of surface

Parametric surface: The surface S is given by:

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

for $(u,v) \in D$

and given a function f on the surface (density function)

we want to define & evaluate

$$\iint_S f(x,y,z)\,dS$$

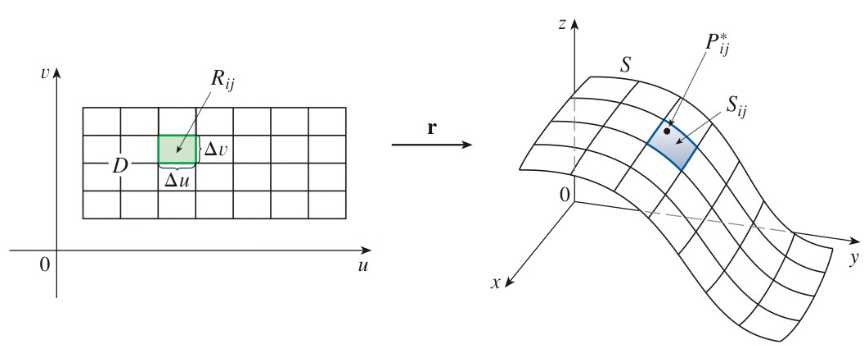
Step 1: divide D into sub-rectangles R_{ij}, image of R_{ij} is S_{ij}

Step 2: Choose $P_{ij}^* \in S_{ij}$, then form the Riemann sum:

$$\sum_{i,j} f(P_{ij}^*)\,\Delta S_{ij}$$

Step 3: define

$$\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{i,j} f(P_{ij}^*)\,\Delta S_{ij}$$



Recall that $\Delta S_{ij} \approx |r_u \times r_v|\,\Delta u\,\Delta v$, we have the following evaluation formula

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,dA}$$

Example:

EXAMPLE 1 Compute the surface integral $\iint_S x^2\,dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

$x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi$

where $0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi$

then $\vec{r}_\phi = (\cos\phi\cos\theta, \ \cos\phi\sin\theta, \ -\sin\phi)$

$\vec{r}_\theta = (-\sin\phi\sin\theta, \ \sin\phi\cos\theta, \ 0)$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} = \left(-\sin^2\phi\cos\theta, \ \sin^2\phi\sin\theta, \ \cos\phi\sin\phi\right)$$

hence $|\vec{r}_\phi \times \vec{r}_\theta| = \sin\phi$

then $$\iint_S x^2\,dS = \int_0^\pi \int_0^{2\pi} \sin^2\phi\cos^2\theta\sin\phi\,d\theta\,d\phi$$

Therefore, by Formula 2,

$$\iint_S x^2\,dS = \iint_D (\sin\phi\cos\theta)^2\,|\mathbf{r}_\phi \times \mathbf{r}_\theta|\,dA$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^2\phi\cos^2\theta\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\cos^2\theta\,d\theta \int_0^{\pi}\sin^3\phi\,d\phi$$

$$= \int_0^{2\pi} \tfrac{1}{2}(1 + \cos 2\theta)\,d\theta \int_0^{\pi} (\sin\phi - \sin\phi\cos^2\phi)\,d\phi$$

$$= \tfrac{1}{2}\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{2\pi}\Big[-\cos\phi + \tfrac{1}{3}\cos^3\phi\Big]_0^{\pi} = \frac{4\pi}{3}$$

■

Graph of functions

Suppose the surface S is given by the graph of some functions, then

$$x = x, \quad y = y, \quad z = g(x,y), \quad (x,y) \in D$$

and so we have $\qquad \mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}$

Thus

$$\boxed{3} \qquad \mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

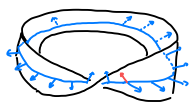
Therefore, in this case, Formula 2 becomes

$$\boxed{4} \qquad \iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Oriented surfaces

Let S be a surface, we say S is orientable if we can choose a normal vector $\vec{n}$ at every point of the surface S, such that $\vec{n}$ varies continuously
unit
the choice of $\vec{n}$ at every point is called an orientation on S,
Note: every orientable surface has exactly 2 orientations

non-Example: Mobiüs strip



Example. Graph of a function

For a surface $z = g(x, y)$ given as the graph of g, we use Equation 3 to associate with the surface a natural orientation given by the unit normal vector

$$\boxed{5} \qquad \mathbf{n} = \frac{-\dfrac{\partial g}{\partial x}\mathbf{i} - \dfrac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\dfrac{\partial g}{\partial x}\right)^2 + \left(\dfrac{\partial g}{\partial y}\right)^2}}$$

Since the $\mathbf{k}$-component is positive, this gives the *upward* orientation of the surface.

Example: smooth parametric surface

If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector

$$\boxed{6} \qquad \mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

Example: sphere surface

and the opposite orientation is given by $-\mathbf{n}$. For instance, in Example 16.6.4 we found the parametric representation

$$\mathbf{r}(\phi, \theta) = a \sin\phi \, \cos\theta \, \mathbf{i} + a \sin\phi \, \sin\theta \, \mathbf{j} + a \cos\phi \, \mathbf{k}$$

for the sphere $x^2 + y^2 + z^2 = a^2$. Then in Example 16.6.10 we found that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2 \sin^2\phi \, \cos\theta \, \mathbf{i} + a^2 \sin^2\phi \, \sin\theta \, \mathbf{j} + a^2 \sin\phi \, \cos\phi \, \mathbf{k}$$

and
$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin\phi$$

So the orientation induced by $\mathbf{r}(\phi, \theta)$ is defined by the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} = \sin\phi \, \cos\theta \, \mathbf{i} + \sin\phi \, \sin\theta \, \mathbf{j} + \cos\phi \, \mathbf{k} = \frac{1}{a}\mathbf{r}(\phi, \theta)$$

Observe that $\mathbf{n}$ points in the same direction as the position vector, that is, outward from the sphere (see Figure 8). The opposite (inward) orientation would have been obtained (see Figure 9) if we had reversed the order of the parameters because $\mathbf{r}_\theta \times \mathbf{r}_\phi = -\mathbf{r}_\phi \times \mathbf{r}_\theta$.

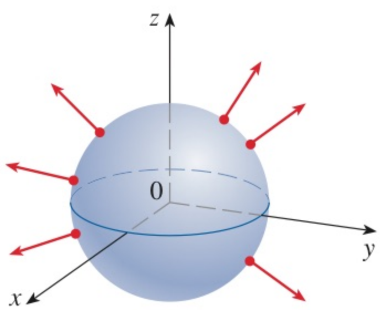


FIGURE 8
Positive orientation

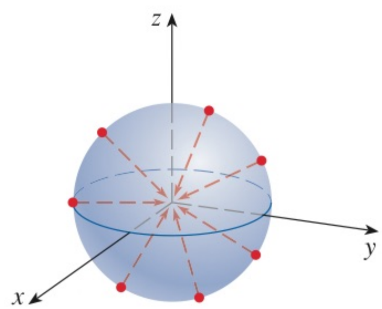


FIGURE 9
Negative orientation

Surface Integral of vector fields & flux

Let's now come to integral of vector fields: suppose S is a surface, $\vec{F}$ is a vector field on $\mathbb{R}^3$

think of $\vec{F}$ as a fluid

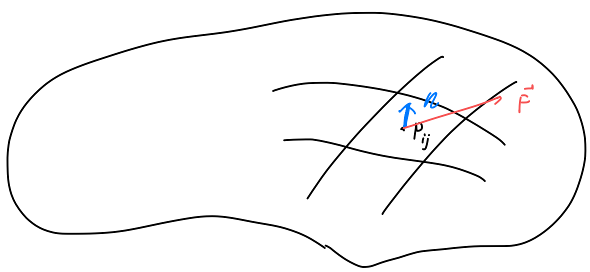
has velocity : $\vec{v}(x,y,z)$

has density : $\rho(x,y,z)$

$\Rightarrow \vec{F} = \rho \cdot \vec{v}$

Goal: find the mass of fluid per unit time crossing S

Step 1: divide S into sub-surfaces S_{ij}.



choose point $P_{ij} \in S_{ij}$, $\vec{n}_{ij}$ = the unit normal vector at P_{ij}

Step 2: the mass of fluid per unit time crossing S_{ij} is:

$$\rho \vec{v} \cdot \vec{n}_{ij} \cdot \Delta S_{ij}$$

Step 3: taking limits

$$m = \lim_{m,n\to\infty} \sum_{ij} \rho \vec{v} \cdot \vec{n}_{ij} \cdot \Delta S_{ij}$$

A surface integral of this form occurs frequently in physics, even when **F** is not ρ**v**, and is called the *surface integral* (or *flux integral*) of **F** over S.

8 Definition If **F** is a continuous vector field defined on an oriented surface S with unit normal vector **n**, then the **surface integral of F over S** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

This integral is also called the **flux** of **F** across S.

Evaluation formula

If S is given by a vector function $\mathbf{r}(u, v)$, then $\mathbf{n}$ is given by Equation 6, and from Definition 8 and Equation 2 we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\, dS$$

$$= \iint_D \left[\mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right] |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

where D is the parameter domain. Thus we have

9
$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA}$$

Formula 9 assumes the orientation of S induced by $\mathbf{r}_u \times \mathbf{r}_v$, as in Equation 6. For the opposite orientation, we multiply by -1.

Example:

EXAMPLE 4 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Figure 12 shows the vector field **F** in Example 4 at points on the unit sphere.

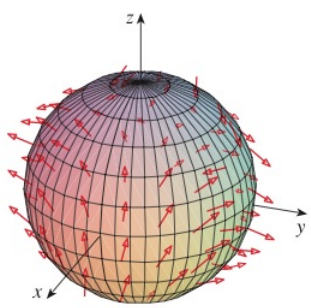


FIGURE 12

SOLUTION As in Example 1, we use the parametric representation

$$\mathbf{r}(\phi, \theta) = \sin\phi\,\cos\theta\,\mathbf{i} + \sin\phi\,\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k} \qquad 0 \leqslant \phi \leqslant \pi \qquad 0 \leqslant \theta \leqslant 2\pi$$

Then
$$\mathbf{F}(\mathbf{r}(\phi, \theta)) = \cos\phi\,\mathbf{i} + \sin\phi\,\sin\theta\,\mathbf{j} + \sin\phi\,\cos\theta\,\mathbf{k}$$

and, from Example 16.6.10,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2\phi\,\cos\theta\,\mathbf{i} + \sin^2\phi\,\sin\theta\,\mathbf{j} + \sin\phi\,\cos\phi\,\mathbf{k}$$

(You can check that these vectors correspond to the outward orientation of the sphere.) Therefore

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \cos\phi\,\sin^2\phi\,\cos\theta + \sin^3\phi\,\sin^2\theta + \sin^2\phi\,\cos\phi\,\cos\theta$$

and, by Formula 9, the flux is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, dA$$

$$= \int_0^{2\pi} \int_0^{\pi} (2\sin^2\phi\,\cos\phi\,\cos\theta + \sin^3\phi\,\sin^2\theta)\, d\phi\, d\theta$$

$$= 2\int_0^{\pi} \sin^2\phi\,\cos\phi\, d\phi \int_0^{2\pi} \cos\theta\, d\theta + \int_0^{\pi} \sin^3\phi\, d\phi \int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= 0 + \int_0^{\pi} \sin^3\phi\, d\phi \int_0^{2\pi} \sin^2\theta\, d\theta \qquad \left(\text{since } \int_0^{2\pi} \cos\theta\, d\theta = 0\right)$$

$$= \frac{4\pi}{3}$$

Special case: S is the graph of some functions:

$$x = x, \quad y = y, \quad z = g(x, y), \quad (x, y) \in D$$

then $\vec{r}_x = \left(1, 0, \frac{\partial g}{\partial x}\right)$, $\vec{r}_y = \left(0, 1, \frac{\partial g}{\partial y}\right)$

$$\vec{r}_x \times \vec{r}_y = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right)$$

$\vec{F} = (P, Q, R)$, then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R\right) dA$$

Example:

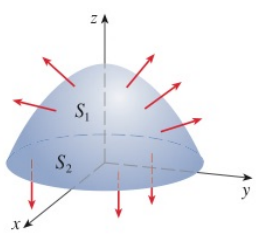


FIGURE 13

EXAMPLE 5 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + z\,\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

SOLUTION S consists of a parabolic top surface S_1 and a circular bottom surface S_2. (See Figure 13.) Since S is a closed surface, we use the convention of positive (outward) orientation. This means that S_1 is oriented upward and we can use Equation 10 with D being the projection of S_1 onto the xy-plane, namely, the disk $x^2 + y^2 \leqslant 1$. Since

$$P(x, y, z) = y \qquad Q(x, y, z) = x \qquad R(x, y, z) = z = 1 - x^2 - y^2$$

on S_1 and

$$\frac{\partial g}{\partial x} = -2x \qquad \frac{\partial g}{\partial y} = -2y$$

we have

$$\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R\right) dA \\
&= \iint_D [-y(-2x) - x(-2y) + 1 - x^2 - y^2]\, dA \\
&= \iint_D (1 + 4xy - x^2 - y^2)\, dA \\
&= \int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos\theta \sin\theta - r^2)\, r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 (r - r^3 + 4r^3 \cos\theta \sin\theta)\, dr\, d\theta \\
&= \int_0^{2\pi} \left(\tfrac{1}{4} + \cos\theta \sin\theta\right) d\theta = \tfrac{1}{4}(2\pi) + 0 = \frac{\pi}{2}
\end{aligned}$$

The disk S_2 is oriented downward, so its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and we have

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot (-\mathbf{k})\, dS = \iint_D (-z)\, dA = \iint_D 0\, dA = 0$$

since $z = 0$ on S_2. Finally, we compute, by definition, $\iint_S \mathbf{F} \cdot d\mathbf{S}$ as the sum of the surface integrals of $\mathbf{F}$ over the pieces S_1 and S_2:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$ ■

Exercise:

8. $\iint_S (x^2 + y^2)\, dS$,
S is the surface with vector equation
$\mathbf{r}(u, v) = \langle 2uv, u^2 - v^2, u^2 + v^2 \rangle$, $u^2 + v^2 \leqslant 1$

$\vec{r}_u = (2v, 2u, 2u)$, $\vec{r}_v = (2u, -2v, 2v)$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 2v & 2u & 2u \\ 2u & -2v & 2v \end{vmatrix} = \left(8uv,\ 4(u^2-v^2),\ -4(u^2+v^2) \right)$$

$$\Rightarrow |\vec{r}_u \times \vec{r}_v| = 4 \cdot \sqrt{(2uv)^2 + (u^2-v^2)^2 + (u^2+v^2)^2} = 4\sqrt{2}\,(u^2+v^2)$$

$$\iint_S (x^2+y^2)\, dS = \iint_D (2uv)^2 + (u^2-v^2)^2 \cdot 4\sqrt{2}\,(u^2+v^2)\, du\, dv$$

$$= \iint_D 4\sqrt{2} \cdot (u^2+v^2)^3\, du\, dv$$

$$= \int_0^{2\pi} \int_0^1 4\sqrt{2}\, r^6\, r\, dr\, d\theta$$

28. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$, S is the surface $z = x \sin y$, $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant \pi$, with upward orientation

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{[0,2]\times[0,\pi]} \left(-\sin y \cdot yx \sin y + x \cos y \cdot x^2 \sin y + xy \right) dA$$