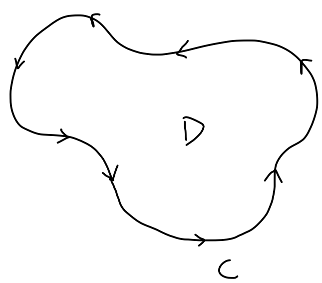
Th 6/15

Green's Theorem

orientation: Let C be a simple closed curve (a closed curve without self-intersection)
D = the region bounded by C

the positive orientation of C is a counterclockwise traversal of C, i.e. the region D is always on the left of C



Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\| $$

$$\int_{\partial D} P\,dx + Q\,dy$$

Rmk: Counterpart of Fundamental theorem of Calculus

FTC: $\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$, a, b can be viewed as the boundary of $[a, b]$

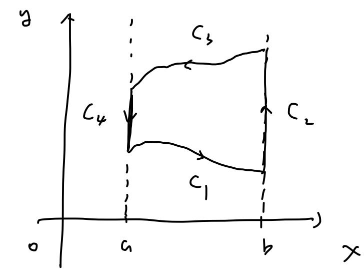
Goal for today's class:

- prove Green's theorem in some simple cases
- Applications

proof when D is simple: D is both type I & type II

Strategy: prove that $\underbrace{\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA}_{①}$, $\underbrace{\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA}_{②}$

<u>Claim</u>: ① is true when D is of type I



$D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$, then

$$\int_C P\,dx = \int_{C_1} P\,dx + \int_{C_2} P\,dx + \int_{C_3} P\,dx + \int_{C_4} P\,dx$$

$$= \int_a^b P(x, g_1(x))\,dx + \int_b^a P(x, g_2(x))\,dx$$

$$= -\int_a^b \Big(P(x, g_2(x)) - P(x, g_1(x))\Big)\,dx$$

$$\iint_D \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x,y)\,dy\,dx$$

$$= \int_a^b \Big(P(x, g_2(x)) - P(x, g_1(x))\Big)\,dx$$

therefore $\displaystyle\int_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$

hence the Claim is true

<u>Claim</u>: ② is true when D is of type II : proof is similiar to the previous one

Examples of simple region

EXAMPLE 1 Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

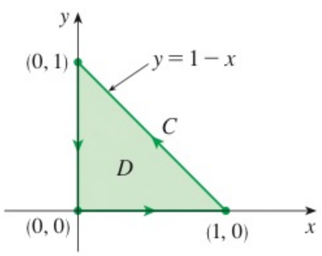


FIGURE 4

SOLUTION Although the given line integral could be evaluated as usual by the methods of Section 16.2, that would involve setting up three separate integrals along the three sides of the triangle, so let's use Green's Theorem instead. Notice that the region D enclosed by C is simple and C has positive orientation (see Figure 4). If we let $P(x, y) = x^4$ and $Q(x, y) = xy$, then we have

$$\int_C x^4\,dx + xy\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^{1-x} (y - 0)\,dy\,dx$$

$$= \int_0^1 \left[\tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx = \tfrac{1}{2}\int_0^1 (1 - x)^2\,dx$$

$$= -\tfrac{1}{6}(1 - x)^3\Big]_0^1 = \tfrac{1}{6}$$

EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4 + 1}\right) dy$, where C is the circle $x^2 + y^2 = 9$.

SOLUTION The region D bounded by C is the disk $x^2 + y^2 \leqslant 9$, so let's change to polar coordinates after applying Green's Theorem:

$$\oint_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4 + 1}\right) dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}\left(7x + \sqrt{y^4 + 1}\right) - \frac{\partial}{\partial y}(3y - e^{\sin x})\right] dA$$

$$= \int_0^{2\pi} \int_0^3 (7 - 3)\,r\,dr\,d\theta = 4\int_0^{2\pi} d\theta \int_0^3 r\,dr = 36\pi$$

Instead of using polar coordinates, we could simply use the fact that D is a disk of radius 3 and write

$$\iint_D 4\,dA = 4 \cdot \pi(3)^2 = 36\pi$$

Finding Areas

idea: $\iint_D 1\, dA$ = area of D = $\int_C P\,dx + Q\,dy$

we choose suitable P & Q:

There are several possibilities:

$$P(x, y) = 0 \qquad P(x, y) = -y \qquad P(x, y) = -\tfrac{1}{2}y$$
$$Q(x, y) = x \qquad Q(x, y) = 0 \qquad Q(x, y) = \tfrac{1}{2}x$$

Then Green's Theorem gives the following formulas for the area of D:

5 $$\boxed{A = \oint_C x\,dy = -\oint_C y\,dx = \tfrac{1}{2}\oint_C x\,dy - y\,dx}$$

Example

EXAMPLE 3 Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

SOLUTION The ellipse has parametric equations $x = a\cos t$ and $y = b\sin t$, where $0 \leqslant t \leqslant 2\pi$. Using the third formula in Equation 5, we have

$$A = \tfrac{1}{2}\int_C x\,dy - y\,dx$$
$$= \tfrac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)\,dt - (b\sin t)(-a\sin t)\,dt$$
$$= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab$$

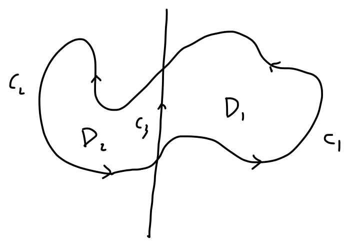
In general. $\iint_D 1\, dA = \int_{-a}^{a} \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy\, dx$

$$= \int_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}\, dx \overset{x=at}{=\!=} 2ab\int_{-1}^{1}\sqrt{1-t^2}\, dt$$

$$\overset{t=\sin\theta}{=\!=} 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^2\, d\theta = \pi a b$$

Extended versions: D is a finite union of simple regions (itself may not be simple)

proof of Green's theorem: only need to prove $D = D_1 \cup D_2$



$$\int_{C_1 + (-C_3)} P\,dx + Q\,dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\int_{C_3 + C_2} P\,dx + Q\,dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\Downarrow$$

$$\int_{C} P\,dx + Q\,dy = \int_{C_1 + C_2} P\,dx + Q\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Examples

FIGURE 7

EXAMPLE 4 Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

SOLUTION Notice that although D is not simple, the y-axis divides it into two simple regions (see Figure 8). In polar coordinates we can write

$$D = \{(r, \theta) \mid 1 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant \pi\}$$

Therefore Green's Theorem gives

$$\oint_C y^2\,dx + 3xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2) \right] dA$$

$$= \iint_D y\,dA = \int_0^{\pi} \int_1^2 (r \sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi} \sin\theta\,d\theta \int_1^2 r^2\,dr = \Big[-\cos\theta\Big]_0^{\pi} \Big[\tfrac{1}{3}r^3\Big]_1^2 = \tfrac{14}{3}$$

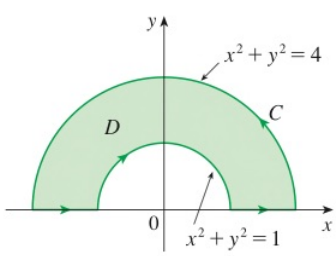


FIGURE 8

D has holes:

the orientation on C make the region D is always on the left of the curve

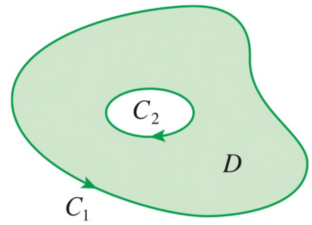


FIGURE 9

$$D': \quad \int_{\partial D'} P\,dx + Q\,dy = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$D'': \quad \int_{\partial D''} P\,dx + Q\,dy = \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

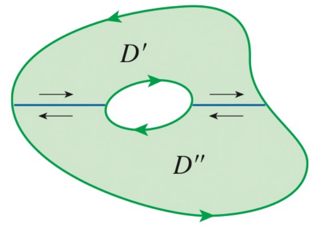
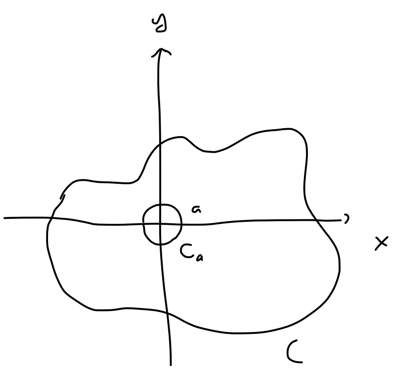


FIGURE 10

$\partial D' + \partial D'' = \partial D$ boundaries with reverse orientation cancel

$$\int_{\partial D} P\,dx + Q\,dy = \iint_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y\,\mathbf{i} + x\,\mathbf{j})/(x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

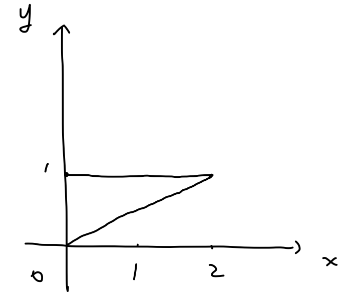


$$\int_C \vec{F}\,d\vec{r} = \int_{C_a} \vec{F} \cdot d\vec{r} = 2\pi$$

$$\int_{C + (-C_a)} \vec{F}\,d\vec{r} = \iint_{D'} 0\,dA = 0$$

Exercises:

8. $\int_C (x^2 + y^2)\, dx + (x^2 - y^2)\, dy$,
C is the triangle with vertices $(0, 0)$, $(2, 1)$, and $(0, 1)$



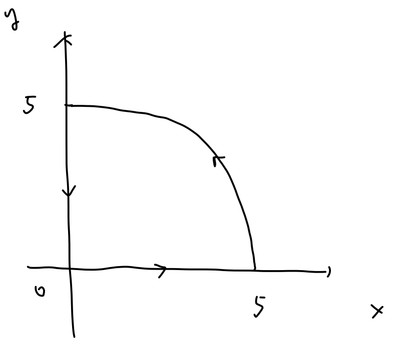
$$\int_C (x^2+y^2)\,dx + (x^2-y^2)\,dy$$

$$= \iint_D (2x - 2y)\, dA$$

$$= \int_0^2 \int_{\frac{1}{2}x}^{1} 2(x-y)\, dy\, dx$$

$$= \int_0^2 (2xy - y^2)\Big|_{\frac{1}{2}x}^{1}\, dx$$

22. A particle starts at the origin, moves along the x-axis to $(5, 0)$, then along the quarter-circle $x^2 + y^2 = 25$, $x \geqslant 0$, $y \geqslant 0$ to the point $(0, 5)$, and then down the y-axis back to the origin. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = \langle \sin x, \sin y + xy^2 + \frac{1}{3}x^3 \rangle$.



$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C (\sin x)\, dx + \left(\sin y + xy^2 + \frac{1}{3}x^3\right) dy$$

$$= \iint_D (y^2 + x^2)\, dA$$

$$= \int_0^5 \int_0^{\frac{\pi}{2}} r^2 \cdot r\, d\theta\, dr = \frac{\pi}{2} \times \frac{5^4}{4}$$

Curl and divergence

Curl If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\boxed{1} \qquad \operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Another expression (easier to remember)
Consider the following differential operator:

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

It has meaning when it operates on a scalar function to produce the gradient of f:

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

If we think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, we can also consider the formal cross product of ∇ with the vector field $\mathbf{F}$ as follows:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$= \operatorname{curl} \mathbf{F}$$

So the easiest way to remember Definition 1 is by means of the symbolic expression

$$\boxed{2} \qquad \boxed{\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}}$$

Example:

EXAMPLE 1 If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find curl $\mathbf{F}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = \vec{i}\left(-2y - xy\right) - \vec{j}\left(0 - x\right) + \vec{k}\left(yz - 0\right)$$

$$= \left(-2y - xy,\ x,\ yz\right)$$

3 Theorem If f is a function of three variables that has continuous second-order partial derivatives, then

$$\operatorname{curl}(\nabla f) = \mathbf{0}$$

$\rightarrow$ two dim'l analogue of $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

pf: $\operatorname{curl}(\nabla f) = \nabla \times (\nabla f)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \vec{i}\left(f_{yz} - f_{zy}\right) - \vec{j}\left(f_{xz} - f_{zx}\right) + \vec{k}\left(f_{xy} - f_{yx}\right)$$

$$= 0$$

EXAMPLE 2 Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ is not conservative.

SOLUTION In Example 1 we showed that

$$\operatorname{curl} \mathbf{F} = -y(2 + x)\,\mathbf{i} + x\,\mathbf{j} + yz\,\mathbf{k}$$

This shows that curl $\mathbf{F} \neq \mathbf{0}$ and so, by the remarks preceding this example, $\mathbf{F}$ is not conservative. ■

the reverse direction:

4 Theorem If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

EXAMPLE 3

(a) Show that

$$\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$$

is a conservative vector field.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

Divergence

Def:

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of F** is the function of three variables defined by

9
$$\boxed{\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

(If $\mathbf{F}$ is a vector field on $\mathbb{R}^2$, then div $\mathbf{F}$ is a function of two variables defined similarly to the three-variable case.) Observe that curl $\mathbf{F}$ is a vector field but div $\mathbf{F}$ is a scalar field. In terms of the gradient operator $\nabla = (\partial/\partial x)\,\mathbf{i} + (\partial/\partial y)\,\mathbf{j} + (\partial/\partial z)\,\mathbf{k}$, the divergence of $\mathbf{F}$ can be written symbolically as the dot product of ∇ and $\mathbf{F}$:

10
$$\boxed{\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F}}$$

Example:

EXAMPLE 4 If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find div $\mathbf{F}$.

SOLUTION By the definition of divergence (Equation 9 or 10), we have

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz$$ ■

Thm: $\operatorname{div} \circ \operatorname{curl} = 0$

11 Theorem If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\operatorname{div}\operatorname{curl}\mathbf{F} = 0$$

EXAMPLE 5 Show that the vector field $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$ can't be written as the curl of another vector field, that is, $\mathbf{F} \neq \operatorname{curl}\mathbf{G}$ for any vector field $\mathbf{G}$.

SOLUTION In Example 4 we showed that

$$\operatorname{div}\mathbf{F} = z + xz$$

and therefore div $\mathbf{F} \neq 0$. If it were true that $\mathbf{F} = \operatorname{curl}\mathbf{G}$, then Theorem 11 would give

$$\operatorname{div}\mathbf{F} = \operatorname{div}\operatorname{curl}\mathbf{G} = 0$$

which contradicts div $\mathbf{F} \neq 0$. Therefore $\mathbf{F}$ is not the curl of another vector field. ■

Exercise

15–20 Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

18. $\mathbf{F}(x, y, z) = yz \sin xy\,\mathbf{i} + xz \sin xy\,\mathbf{j} - \cos xy\,\mathbf{k}$

18. $$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz\sin(xy) & xz\sin(xy) & -\cos(xy) \end{vmatrix}$$

$$= \vec{i}\left(\sin(xy)\cdot x - x\sin(xy)\right) - \vec{j}\left(\sin(xy)\cdot y - y\sin(xy)\right)$$

$$+ \vec{k}\left(z\sin(xy) + xzy\cos(xy) - z\sin(xy) - xyz\sin(xy)\right)$$

$$= \vec{0}$$

$\Rightarrow$ conservative, hence $\exists f$, s.t. $\nabla f = \vec{F}$

$f_x = yz\sin(xy) \Rightarrow f = -z\cos(xy) + g(y, z)$

$f_y = xz\sin(xy) + \frac{\partial g}{\partial y} = xz\sin(xy) \Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g(y, z) = g(z)$

$f_z = -\cos(xy) + g'(z) = -\cos(xy) \Rightarrow g'(z) = 0 \Rightarrow$

$$f = -z\cos(xy) + C$$

4. $\mathbf{F}(x, y, z) = \sin yz\,\mathbf{i} + \sin zx\,\mathbf{j} + \sin xy\,\mathbf{k}$

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(yz) & \sin(zx) & \sin(xy) \end{vmatrix}$$

$$= \vec{i}\left(x\cos(xy) - x\cos(xz) \right) - \vec{j}\left(y\cos(xy) - y\cos(yz) \right) + \vec{k}\left(z\cos(xz) - z\cos(yz) \right)$$

$$\text{div } \vec{F} = 0 + 0 + 0 = 0$$