Tu 6/13

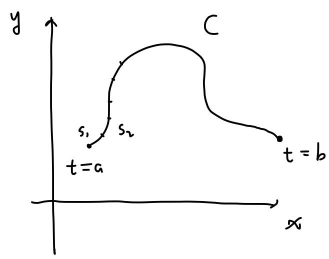
Line integrals

Previously, we knew how to define and evaluate the integral over an interval $[a, b]$

$[a, b]$ can be viewed as a curve, today we will define integral over a curve C

Line integrals in the plane

We first consider the case that the curve C is contained in the plane



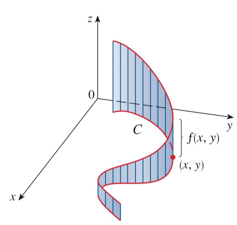
given by the parametric equations

$$x = x(t) \qquad a \le t \le b$$
$$y = y(t)$$

or given by $\vec{r}(t) = (x(t), y(t))$

we assume $\vec{r}\,'(t) = (x'(t), y'(t))$ is continuous

suppose that there is a function $f(x, y)$ on the plane, we want to define "the integral of f along the curve C"



Goal: Calculate the area of this "surface"

Step 1: We divide $[a, b]$ into n sub intervals $[t_{i-1}, t_i]_{i=1,2,\dots,n}$ with $t_0 = a$, $t_n = b$
then $[t_{i-1}, t_i]$ is mapped to a sub-arc s_i of length Δs_i of the curve C

Step 2: Choose $(x_i^*, y_i^*) \in s_i$, then form the sum:

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

Step 3: taking limits $n \to +\infty$

$$\int_C f(x, y)\, ds := \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

Question: How to evaluate the integral?

Recall the length formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

then

$$\Delta s_i = \int_{t_{i-1}}^{t_i} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \approx \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}(t_i^*) \cdot \Delta t_i$$

hence

$$\sum_{i=1}^n f(x_i^*, y_i^*)\, \Delta s_i \approx \sum_{i=1}^n f(x(t_i^*), y(t_i^*)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}(t_i^*) \cdot \Delta t_i$$

then

3 $$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Examples:

EXAMPLE 1 Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

consider the parametrization $x = \cos t$, $y = \sin t$, $0 \le t \le \pi$, then

$$\int_C (2 + x^2 y)\, ds = \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{(\sin t)^2 + (\cos t)^2}\, dt$$

$$= \int_0^\pi 2 + \cos^2 t \sin t\, dt = 2t - \frac{\cos^3 t}{3} \Big|_0^\pi = 2\pi + \frac{2}{3}$$

Example: A physical interpretation of line integrals

interpretation of the function f. Suppose that $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve C (see Example 3.7.2). Then the mass of the part of the wire from P_{i-1} to P_i in Figure 1 is approximately $\rho(x_i^*, y_i^*)\, \Delta s_i$ and so the total mass of the wire is approximately $\Sigma\, \rho(x_i^*, y_i^*)\, \Delta s_i$. By taking more and more points on the curve, we obtain the **mass** m of the wire as the limiting value of these approximations:

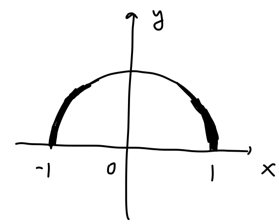
$$m = \lim_{n \to \infty} \sum_{i=1}^n \rho(x_i^*, y_i^*)\, \Delta s_i = \int_C \rho(x, y)\, ds$$

[For example, if $f(x, y) = 2 + x^2 y$ represents the density of a semicircular wire, then the integral in Example 1 would represent the mass of the wire.] The **center of mass** of the wire with density function ρ is located at the point $(\bar{x}, \bar{y})$, where

4 $$\bar{x} = \frac{1}{m} \int_C x \rho(x, y)\, ds \qquad \bar{y} = \frac{1}{m} \int_C y \rho(x, y)\, ds$$

EXAMPLE 3 A wire takes the shape of the semicircle $x^2 + y^2 = 1$, $y \geqslant 0$, and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

parametrization



SOLUTION As in Example 1 we use the parametrization $x = \cos t$, $y = \sin t$, $0 \leqslant t \leqslant \pi$, and find that $ds = dt$. The linear density is

$$\rho(x, y) = k(1 - y)$$

we first compute its mass

$$m = \int_C k(1 - y)\, ds = \int_0^\pi k(1 - \sin t)\, dt = k\Big[t + \cos t\Big]_0^\pi = k(\pi - 2)$$

then the center of mass:

$$\bar{y} = \frac{1}{m}\int_C y\rho(x, y)\, ds = \frac{1}{k(\pi - 2)}\int_C y\, k(1 - y)\, ds$$

$$= \frac{1}{\pi - 2}\int_0^\pi (\sin t - \sin^2 t)\, dt = \frac{1}{\pi - 2}\Big[-\cos t - \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t\Big]_0^\pi$$

$$= \frac{4 - \pi}{2(\pi - 2)}$$

$\bar{x} = 0$

Line integrals with respect to x or y

We can replace Δs_i by Δx_i or Δy_i.

Two other types of line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ in Definition 2. They are called the **line integrals of f along C with respect to x and y**:

5 $$\int_C f(x, y)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta x_i$$

6 $$\int_C f(x, y)\,dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta y_i$$

the computation formulas are

The following formulas say that line integrals with respect to x and y can also be evaluated by expressing everything in terms of t: $x = x(t)$, $y = y(t)$, $dx = x'(t)\,dt$, $dy = y'(t)\,dt$.

7 $$\int_C f(x, y)\,dx = \int_a^b f(x(t), y(t))\,x'(t)\,dt$$

$$\int_C f(x, y)\,dy = \int_a^b f(x(t), y(t))\,y'(t)\,dt$$

Example:

EXAMPLE 4 Evaluate $\int_C y^2\,dx + x\,dy$ for two different paths C.

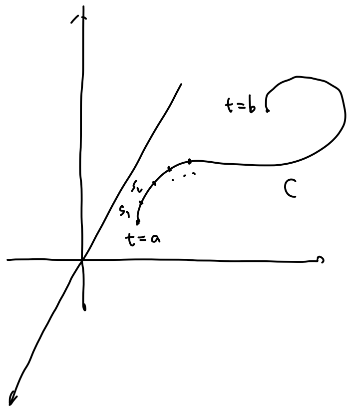
(a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$.
(b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$.
(See Figure 7.)

key: parametrization of a line segment: Suppose the line starts from $\vec{r}_0$ to $\vec{r}_1$ then an equation can be obtained by $\vec{r}(t) = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0) = (1-t)\vec{r}_0 + t\vec{r}_1$, $0 \le t \le 1$

C_1: $\vec{r}(t) = (1-t)(-5,-3) + t(0,2) = (5t-1,\ 5t-3)$ $\quad 0 \le t \le 1$

Line integrals in Space

Now we consider a curve in the 3-dim'l space $\mathbb{R}^3$, suppose that the curve C is given by the following equations:



$$C: \quad x = x(t), \quad y = y(t), \quad z = z(t)$$
$$a \le t \le b$$

Suppose we have a function $f(x, y, z)$, now we want to define the integral of f along the curve C

the definition is similar to the plane case, then

$$\int_C f(x, y, z)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta s_i$$

evaluation formula:

9 $$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

with respect to x, y, z

Line integrals along C with respect to x, y, and z can also be defined. For example,

$$\int_C f(x, y, z)\, dz = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\, \Delta z_i$$
$$= \int_a^b f(x(t), y(t), z(t))\, z'(t)\, dt$$

Therefore, as with line integrals in the plane, we evaluate integrals of the form

10 $$\int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$$

by expressing everything (x, y, z, dx, dy, dz) in terms of the parameter t.

Examples:

EXAMPLE 5 Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant 2\pi$. (See Figure 9.)

SOLUTION Formula 9 gives

$$\int_C y \sin z \, ds = \int_0^{2\pi} (\sin t) \sin t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

$$= \int_0^{2\pi} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1} \, dt = \sqrt{2} \int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2t) \, dt$$

$$= \frac{\sqrt{2}}{2} \Big[t - \tfrac{1}{2} \sin 2t \Big]_0^{2\pi} = \sqrt{2}\, \pi$$

■

EXAMPLE 6 Evaluate $\int_C y \, dx + z \, dy + x \, dz$, where C consists of the line segment C_1 from $(2, 0, 0)$ to $(3, 4, 5)$, followed by the vertical line segment C_2 from $(3, 4, 5)$ to $(3, 4, 0)$.

SOLUTION The curve C is shown in Figure 10. Using Equation 8, we write C_1 as

line segment equation $\rightarrow$

$$\mathbf{r}(t) = (1 - t)\langle 2, 0, 0 \rangle + t\langle 3, 4, 5 \rangle = \langle 2 + t, 4t, 5t \rangle$$

or, in parametric form, as

$$x = 2 + t \qquad y = 4t \qquad z = 5t \qquad 0 \leqslant t \leqslant 1$$

Thus

$$\int_{C_1} y \, dx + z \, dy + x \, dz = \int_0^1 (4t) \, dt + (5t)4 \, dt + (2 + t)5 \, dt$$

$$= \int_0^1 (10 + 29t) \, dt = 10t + 29 \frac{t^2}{2} \Big]_0^1 = 24.5$$

Likewise, C_2 can be written in the form

$$\mathbf{r}(t) = (1 - t)\langle 3, 4, 5 \rangle + t\langle 3, 4, 0 \rangle = \langle 3, 4, 5 - 5t \rangle$$

or

$$x = 3 \qquad y = 4 \qquad z = 5 - 5t \qquad 0 \leqslant t \leqslant 1$$

Then $dx = 0 = dy$, so

$$\int_{C_2} y \, dx + z \, dy + x \, dz = \int_0^1 3(-5) \, dt = -15$$

Adding the values of these integrals, we obtain

$$\int_C y \, dx + z \, dy + x \, dz = 24.5 - 15 = 9.5$$

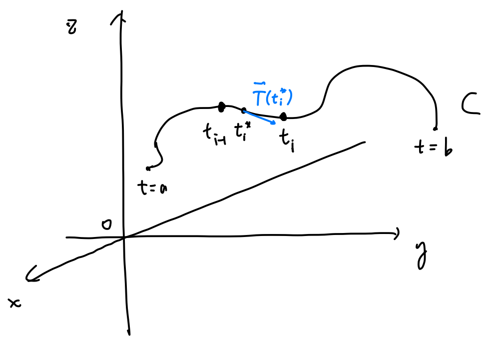
■

Line integrals of vector fields

Now we generalize the line integral of a scalar function to vector fields
the original idea: compute the work done by a force moving along a curve

Let's denote the curve C:

$x = x(t), \quad y = y(t), \quad z = z(t), \quad a \leq t \leq b$



$\vec{F} = (P, Q, R) = P\vec{i} + Q\vec{j} + R\vec{k}$ is a force field

Step 1: We divide the $[a,b]$ into n sub intervals $[t_{i-1}, t_i]$
every $[t_{i-1}, t_i]$ maps to a subarc S_i, with length ΔS_i

Step 2: choose $t_i^* \in [t_{i-1}, t_i]$, let $P_i^* = (x(t_i^*), y(t_i^*), z(t_i^*))$
$\vec{T}(t_i^*)$: the unit tangent vector of C at P_i^*

then $\vec{F}(x(t_i^*), y(t_i^*), z(t_i^*)) \cdot \vec{T}(t_i^*) \cdot \Delta S_i$
is an approximation of the work done by $\vec{F}$ along S_i

Step 3: Summing over all i & taking limits

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} \vec{F}(x(t_i^*), y(t_i^*), z(t_i^*)) \cdot \vec{T}(t_i^*) \cdot \Delta S_i$$

If the curve C is given by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, then $\mathbf{T}(t) = \mathbf{r}'(t)/|\,\mathbf{r}'(t)\,|$, so using Equation 9 we can rewrite Equation 12 in the form

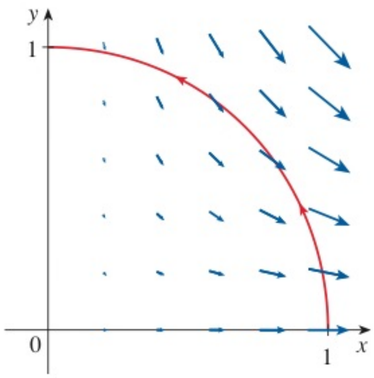
$$W = \int_a^b \left[\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\,\mathbf{r}'(t)\,|} \right] |\,\mathbf{r}'(t)\,|\, dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

This integral is often abbreviated as $\int_C \mathbf{F} \cdot d\mathbf{r}$ and occurs in other areas of physics as well. Therefore we make the following definition for the line integral of *any* continuous vector field.

13 Definition Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then the **line integral of F along *C*** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

Figure 13 shows the force field and the curve in Example 7. The work done is negative because the field impedes movement along the curve.



EXAMPLE 7 Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} - xy\,\mathbf{j}$ in moving a particle along the quarter-circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $0 \leqslant t \leqslant \pi/2$.

SOLUTION Since $x = \cos t$ and $y = \sin t$, we have

$$\mathbf{F}(\mathbf{r}(t)) = \cos^2 t\,\mathbf{i} - \cos t\,\sin t\,\mathbf{j}$$

and

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

Therefore the work done is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{\pi/2} (-\cos^2 t\,\sin t - \cos^2 t\,\sin t)\,dt$$

$$= \int_0^{\pi/2} (-2\cos^2 t\,\sin t)\,dt = 2\,\frac{\cos^3 t}{3}\Bigg]_0^{\pi/2} = -\frac{2}{3}$$

■

Exercise :

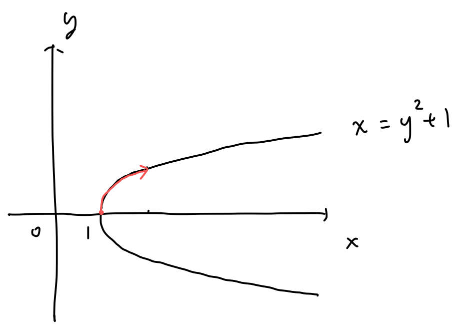
24. $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + z^3\,\mathbf{j} + y\,\mathbf{k}$,
$\mathbf{r}(t) = e^t\,\mathbf{i} + e^{2t}\,\mathbf{j} + e^{-t}\,\mathbf{k}, \quad -1 \leqslant t \leqslant 1$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{-1}^{1} F(e^t, e^{2t}, e^{-t}) \cdot (e^t, 2e^{2t}, -e^{-t})\, dt$$

$$= \int_{-1}^{1} (1, e^{-3t}, e^{2t}) \cdot (e^t, 2e^{2t}, -e^{-t})\, dt$$

$$= \int_{-1}^{1} e^t + 2e^{-t} - e^t\, dt = -2e^{-t}\Big|_{-1}^{1} = 2(e - e^{-1})$$

42. Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} + ye^x\,\mathbf{j}$ on a particle that moves along the parabola $x = y^2 + 1$ from $(1, 0)$ to $(2, 1)$.



$\vec{r}(t) = (t^2+1,\ t), \quad 0 \le t \le 1$

then

$$\int_C \vec{F}\, d\vec{r} = \int_0^1 \vec{F}(t^2+1, t) \cdot (2t, 1)\, dt$$

$$= \int_0^1 (t^4 + 2t^2 + 1,\ te^{t^2+1}) \cdot (2t, 1)\, dt$$

$$= \int_0^1 2t^5 + 4t^3 + 2t + te^{t^2+1}\, dt$$

$$= \frac{t^6}{3} + t^4 + t^2 + \frac{1}{2}e^{t^2+1}\Big|_0^1$$

$$= \frac{1}{3} + 2 + \frac{1}{2}e^2 - \frac{1}{2}e = \frac{7}{3} + \frac{e^2 - e}{2}$$

The Fundamental theorem for line integrals

Recall: the fundamental theorem for calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

The main theorem

If we think of the gradient vector ∇f of a function f of two or three variables as a sort of derivative of f, then the following theorem can be regarded as a version of the Fundamental Theorem for line integrals.

2 Theorem Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

PROOF OF THEOREM 2 Using Definition 16.2.13, we have

$$\begin{aligned}
\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt \\
&= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\,dt \qquad \text{(by the Chain Rule)} \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a))
\end{aligned}$$

The last step follows from the Fundamental Theorem of Calculus (Equation 1). ■

Example:

EXAMPLE 1 Find the work done by the gravitational field

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}$$

in moving a particle with mass m from the point $(3, 4, 12)$ to the point $(2, 2, 0)$ along a piecewise-smooth curve C. (See Example 16.1.4.)

SOLUTION From Section 16.1 we know that $\mathbf{F}$ is a conservative vector field and, in fact, $\mathbf{F} = \nabla f$, where

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

Therefore, by Theorem 2, the work done is

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} \\ &= f(2, 2, 0) - f(3, 4, 12) \\ &= \frac{mMG}{\sqrt{2^2 + 2^2}} - \frac{mMG}{\sqrt{3^2 + 4^2 + 12^2}} = mMG\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right) \end{aligned}$$

■

Independence of path

Let $\vec{F}$ be a continuous vector field. with domain D. We say that the line integral

$$\int_C \vec{F}\cdot d\vec{r}$$

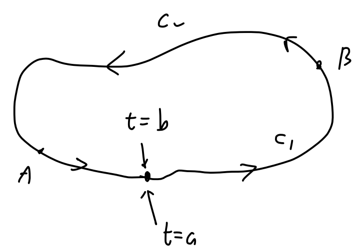
is independent of the path, if for any two paths C_1 & C_2 in D with the same initial points & terminal points, we have

$$\int_{C_1} \vec{F}\, d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$

Example: conservative vector field, i.e. $\vec{F} = \nabla f$ for some smooth function f on D

then $\int_C \vec{F}\, d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(\text{terminal point}) - f(\text{initial point})$

Closed curve: terminal point = initial point



if $\vec{F}$ is independent of path, then

$$\int_C \vec{F}\, d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\, d\vec{r}$$

$$= \underbrace{\int_{C_1} \vec{F}\, d\vec{r} - \int_{-C_2} \vec{F}\cdot d\vec{r}}_{\text{Curves with the same initial points \& terminal points}} = 0$$

A B

if for any closed curve C, we have

$\int_C \vec{F}\, d\vec{r} = 0$, then for C_1 & C_2 with the same initial points & terminal points (A, B)

$C_1 + (-C_2)$ is a closed curve, then

$$\int_{C_1} \vec{F}\, d\vec{r} + \int_{-C_2} \vec{F}\, d\vec{r} = 0 \Rightarrow \int_{C_1} \vec{F}\, d\vec{r} = \int_{C_2} \vec{F}\, d\vec{r}$$

3 Theorem $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed path C in D.

space curves. We assume that D is **open**, which means that for every point P in D there is a disk with center P that lies entirely in D. (So D doesn't contain any of its boundary points.) In addition, we assume that D is **connected**: this means that any two points in D can be joined by a path that lies in D.

$\boxed{4}$ **Theorem** Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Conservative Vector fields and potential functions

5 Theorem If $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

pf: $\vec{F}(x,y) = \nabla f$, i.e. $P(x,y) = \frac{\partial f}{\partial x}(x,y)$, $Q(x,y) = \frac{\partial f}{\partial y}(x,y)$, then

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

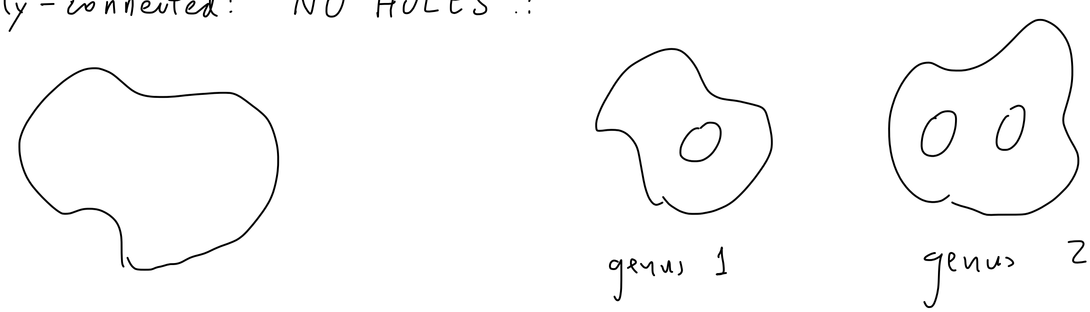
the converse version:

6 Theorem Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

simply-connected: NO HOLES !!



Example:

EXAMPLE 2 Determine whether or not the given vector field is conservative.

(a) $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + (x - 2)\,\mathbf{j}$

(b) $\mathbf{F}(x, y) = (3 + 2xy)\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$

(a) $\frac{\partial P}{\partial y} = -1, \quad \frac{\partial Q}{\partial x} = 1$

(b) $\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x} \Rightarrow$ Since $\vec{F}$ is defined on $\mathbb{R}^2$ (simply-connected)

hence it is a conservative vector field

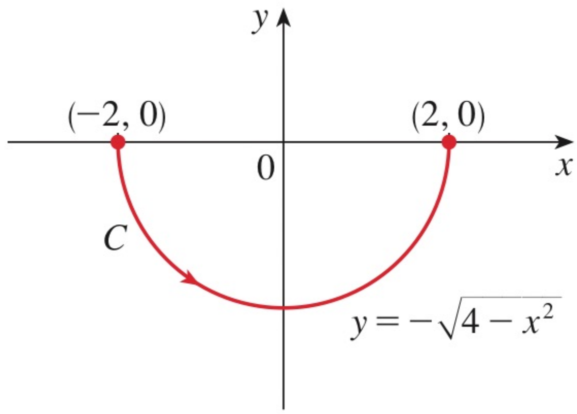
$\frac{\partial f}{\partial x} = 3 + 2xy, \quad \frac{\partial f}{\partial y} = x^2 - 3y^2$

$\Rightarrow f = 3x + x^2y + g(y) \Rightarrow \frac{\partial f}{\partial y} = x^2 + \underbrace{g'(y)}_{= -3y^2} \Rightarrow g(y) = -y^3 + C$

$\Rightarrow f = 3x + x^2y - y^3 + C$

Exercise:

13. Let $\mathbf{F}(x, y) = (3x^2 + y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$ and let C be the curve shown.



(a) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ directly.

(b) Show that $\mathbf{F}$ is conservative and find a function f such that $\mathbf{F} = \nabla f$.

(c) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ using Theorem 2.

(d) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ by first replacing C by a simpler curve that has the same initial and terminal points.

(a) $\vec{r}(t) = (2\cos t,\ 2\sin t)$, $\pi \le t \le 2\pi$

$$\int_C \vec{F}\cdot d\vec{r} = \int_\pi^{2\pi} \vec{F}(2\cos t, 2\sin t)\cdot(-2\sin t, 2\cos t)\,dt$$

$$= \int_\pi^{2\pi} \left(4(3\cos^2 t + \sin^2 t),\ 8\sin t\cos t\right)\left(-2\sin t,\ 2\cos t\right)dt$$

$$= 8\int_\pi^{2\pi} (2\cos^2 t + 1)(-\sin t) + (2\sin t\cos t)\cdot\cos t\ dt$$

$$= 8\int_\pi^{2\pi} -(2+\cos 2t)\sin t + (\sin 2t)\cos t\ dt$$

(b) $\frac{\partial P}{\partial y} = 2y = \frac{\partial Q}{\partial x}$ since $\mathbb{R}^2$ is simply-connected, $\vec{F}$ is conservative

$\vec{F} = \nabla f$, $\frac{\partial f}{\partial x} = 3x^2 + y^2 \Rightarrow f = x^3 + xy^2 + g(y) \Rightarrow \frac{\partial f}{\partial y} = 2xy + \underbrace{g'(y)}_{0} \Rightarrow g(y) = C$

hence $f = x^3 + xy^2 + C$

(c) $\int_C \vec{F}\cdot d\vec{r} = x^3 + xy^2 + C \Big|_{(-2,0)}^{(2,0)} = 8-(-8) = 16$

(d) $\vec{r}(t) = (t, 0), \quad -2 \le t \le 2$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{-2}^{2} \vec{F}(t,0)\cdot(1,0)\,dt = \int_{-2}^{2} (3t^2, 0)\cdot(1,0)\,dt = \int_{-2}^{2} 3t^2\,dt = 8-(-8) = 16$$

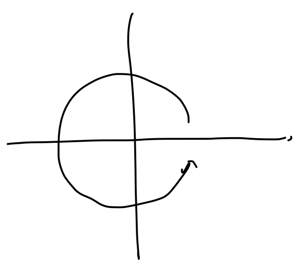
41. Let $\mathbf{F}(x, y) = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$.

(a) Show that $\partial P/\partial y = \partial Q/\partial x$.

(b) Show that $\int_C \mathbf{F}\cdot d\mathbf{r}$ is not independent of path. [*Hint:* Compute $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$, where C_1 and C_2 are the upper and lower halves of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$.] Does this contradict Theorem 6?

(a) $\dfrac{\partial P}{\partial y} = \dfrac{-(x^2+y^2) - 2y(-y)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}, \quad \dfrac{\partial Q}{\partial x} = \dfrac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$

(b)



$\vec{r}(t) = (\cos t, \sin t)$, then

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (-\sin t, \cos t)\cdot(-\sin t, \cos t)\,dt$$

$$= 2\pi \neq 0 \;!$$

No contradiction, since $\vec{F}$ is defined on $\mathbb{R}^2 - \{(0,0)\}$, which is not simply-connected