Tu 6/6

Change of variables in multiple integrals

- Case of double integral

Comparison of two formulas:

In 1-dim'l case:

1 $$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

where $x = g(u)$ and $a = g(c)$, $b = g(d)$. Another way of writing Formula 1 is as follows:

2 $$\int_a^b f(x)\,dx = \int_c^d f(x(u))\boxed{\frac{dx}{du}}\,du$$

changing factors

In 2-dim'l case: polar coordinate

We have already seen an example of a change of variables for double integrals: conversion to polar coordinates. The new variables r and θ are related to the old variables x and y by the equations

$$x = r\cos\theta \qquad y = r\sin\theta$$

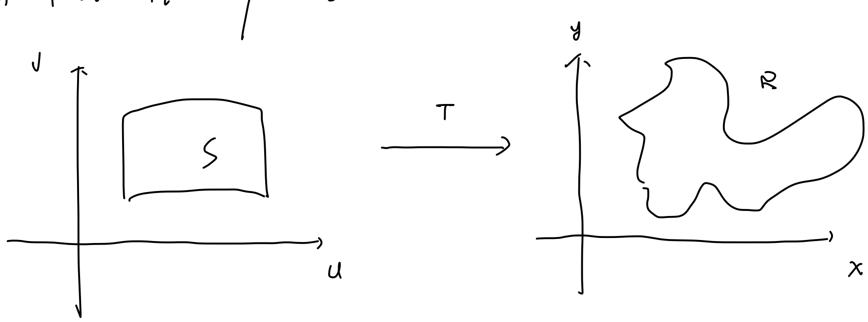
and the change of variables formula (15.3.2) can be written as

$$\iint_R f(x, y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\,\boxed{r}\,dr\,d\theta$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane.

Change of variables makes the computation much more easier!!

In general, $R \subset \mathbb{R}^2$ is a region, $S \subset \mathbb{R}^2$ is another region. Suppose T is a map such that it maps S to R



$x = x(u,v)$
$y = y(u,v)$

and S is a "good shape", then we want to integrate over S, not R, so

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \cdot \underbrace{\boxed{}}_{\text{some factor!}} \cdot dA' \qquad dA = \Box\, dA'$$

EXAMPLE 1 A transformation is defined by the equations

$$x = u^2 - v^2 \qquad y = 2uv$$

Find the image of the square $S = \{(u, v) \mid 0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1\}$.

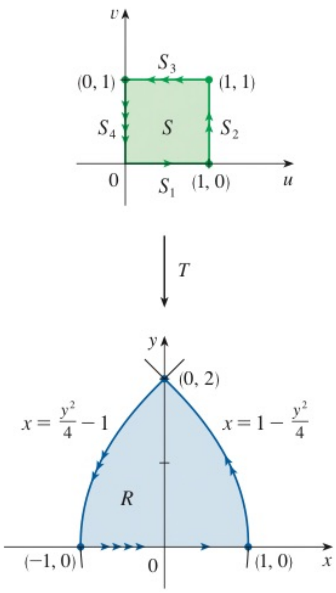


FIGURE 2

SOLUTION The transformation maps the boundary of S into the boundary of the image. So we begin by finding the images of the sides of S. The first side, S_1, is given by $v = 0$ $(0 \leqslant u \leqslant 1)$. (See Figure 2.) From the given equations we have $x = u^2$, $y = 0$, and so $0 \leqslant x \leqslant 1$. Thus S_1 is mapped onto the line segment from (0, 0) to (1, 0) in the xy-plane. The second side, S_2, is $u = 1$ $(0 \leqslant v \leqslant 1)$ and, putting $u = 1$ in the given equations, we get

$$x = 1 - v^2 \qquad y = 2v$$

Eliminating v, we obtain

$$\boxed{4} \qquad x = 1 - \frac{y^2}{4} \qquad 0 \leqslant x \leqslant 1$$

which is part of a parabola. Similarly, S_3 is given by $v = 1$ $(0 \leqslant u \leqslant 1)$, whose image is the parabolic arc

$$\boxed{5} \qquad x = \frac{y^2}{4} - 1 \qquad -1 \leqslant x \leqslant 0$$

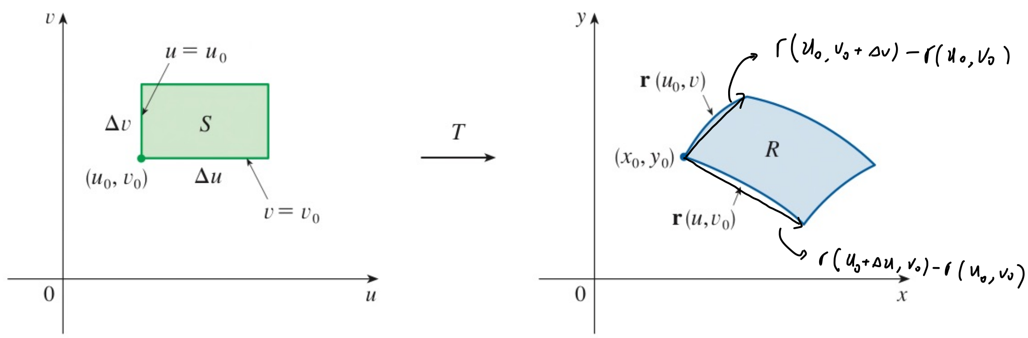
Finally, S_4 is given by $u = 0$ $(0 \leqslant v \leqslant 1)$ whose image is $x = -v^2$, $y = 0$, that is, $-1 \leqslant x \leqslant 0$. (Notice that as we move around the square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.) The image of S is the region R (shown in Figure 2) bounded by the x-axis and the parabolas given by Equations 4 and 5. ■

key: find the image of the boundary

How to find this factor?

Let's look at the local behavior of the transformation map T

Now let's see how a change of variables affects a double integral. We start with a small rectangle S in the uv-plane whose lower left corner is the point (u_0, v_0) and whose dimensions are Δu and Δv. (See Figure 3.)



The image of S is a region R in the xy-plane, one of whose boundary points is $(x_0, y_0) = T(u_0, v_0)$. The vector

$$\mathbf{r}(u, v) = g(u, v)\,\mathbf{i} + h(u, v)\,\mathbf{j}$$

$\Delta A' =$ area of S, $\Delta A =$ area of R

$= \Delta u \cdot \Delta v$

Let's try to approximate ΔA

Rough idea: approximate R be parallelgram formed by tangent vectors

$$\Delta A \approx \left| \Big(r(u_0+\Delta u, v_0) - r(u_0, v_0) \Big) \times \Big(r(u_0, v_0+\Delta v) - r(u_0, v_0) \Big) \right|$$

$$\approx \left| r_u'(u_0, v_0) \cdot \Delta u \times r_v'(u_0, v_0) \cdot \Delta v \right|$$

$$= \left| r_u'(u_0, v_0) \times r_v'(u_0, v_0) \right| \cdot \Delta u \, \Delta v$$

because by definition:

$$r_u'(u_0, v_0) = \lim_{\Delta u \to 0} \frac{r(u_0+\Delta u, v_0) - r(u_0, v_0)}{\Delta u}$$

$$r_v'(u_0, v_0) = \lim_{\Delta v \to 0} \frac{r(u_0, v_0+\Delta v) - r(u_0, v_0)}{\Delta v}$$

both are vectors!

Since $r(u,v) = \left(x(u,v),\ y(u,v) \right)$, we got

$$r_u'(u_0, v_0) = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right), \qquad r_v'(u_0, v_0) = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right)$$

then

$$r_u'(u_0, v_0) \times r_v'(u_0, v_0) = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right)(u_0, v_0) \cdot \vec{k}$$

and then

$$\left| r_u'(u_0, v_0) \times r_v'(u_0, v_0) \right| = \left| \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \right| (u_0, v_0)$$

7 Definition The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

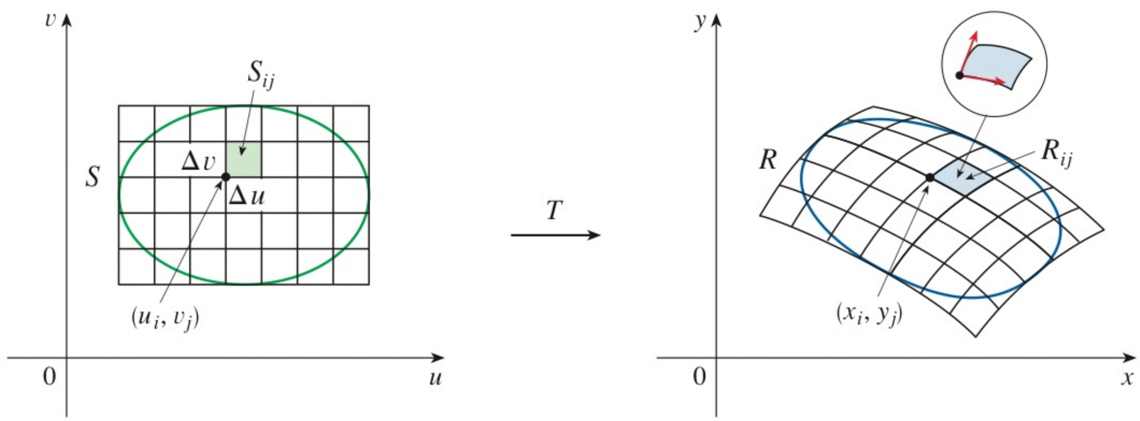
$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

With this notation we can use Equation 6 to give an approximation to the area ΔA of R:

8
$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \, \Delta v$$

where the Jacobian is evaluated at (u_0, v_0).

Next we divide a region S in the uv-plane into rectangles S_{ij} and call their images in the xy-plane R_{ij}. (See Figure 6.)



Applying the approximation (8) to each R_{ij}, we approximate the double integral of f over R as follows:

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

$$\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(g(u_i, v_j), h(u_i, v_j)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u\, \Delta v$$

where the Jacobian is evaluated at (u_i, v_j). Notice that this double sum is a Riemann sum for the integral

$$\iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

The foregoing argument suggests that the following theorem is true. (A full proof is given in books on advanced calculus.)

9 Change of Variables in a Double Integral Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Review of Polar coordinate

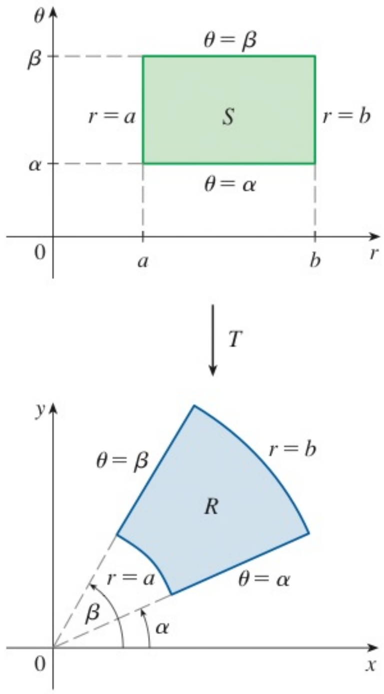


FIGURE 7

As a first illustration of Theorem 9, we show that the formula for integration in polar coordinates is just a special case. Here the transformation T from the $r\theta$-plane to the xy-plane is given by

$$x = g(r, \theta) = r\cos\theta \qquad y = h(r, \theta) = r\sin\theta$$

and the geometry of the transformation is shown in Figure 7: T maps an ordinary rectangle in the $r\theta$-plane to a polar rectangle in the xy-plane. The Jacobian of T is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r > 0$$

Thus Theorem 9 gives

$$\iint_R f(x, y)\, dx\, dy = \iint_S f(r\cos\theta, r\sin\theta) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr\, d\theta$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

which is the same as Formula 15.3.2.

Examples:

which is the same as Formula 15.3.2.

FIGURE 7
The polar coordinate transformation

EXAMPLE 2 Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\, dA$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$ and $y^2 = 4 + 4x$, $y \geqslant 0$.

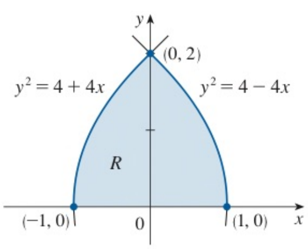


FIGURE 8

SOLUTION The region R is pictured in Figure 8. It is the region from Example 1 (see Figure 2); in that example we discovered that $T(S) = R$, where S is the square $[0, 1] \times [0, 1]$. Indeed, the reason for making the change of variables to evaluate the integral is that S is a much simpler region than R. First we need to compute the Jacobian:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$$

Therefore, by Theorem 9,

$$\iint_R y\, dA = \iint_S 2uv \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA = \int_0^1 \int_0^1 (2uv)4(u^2 + v^2)\, du\, dv$$

$$= 8 \int_0^1 \int_0^1 (u^3 v + uv^3)\, du\, dv = 8 \int_0^1 \Big[\tfrac{1}{4}u^4 v + \tfrac{1}{2}u^2 v^3 \Big]_{u=0}^{u=1} dv$$

$$= \int_0^1 (2v + 4v^3)\, dv = \Big[v^2 + v^4 \Big]_0^1 = 2$$

■

EXAMPLE 3 Evaluate the integral $\iint_R e^{(x+y)/(x-y)}\,dA$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$.

SOLUTION Since it isn't easy to integrate $e^{(x+y)/(x-y)}$, we make a change of variables suggested by the form of this function:

$$\boxed{10} \qquad u = x + y \qquad v = x - y$$

These equations define a transformation T^{-1} from the xy-plane to the uv-plane. Theorem 9 talks about a transformation T from the uv-plane to the xy-plane. It is obtained by solving Equations 10 for x and y:

$$\boxed{11} \qquad x = \tfrac{1}{2}(u + v) \qquad y = \tfrac{1}{2}(u - v)$$

The Jacobian of T is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\tfrac{1}{2}$$

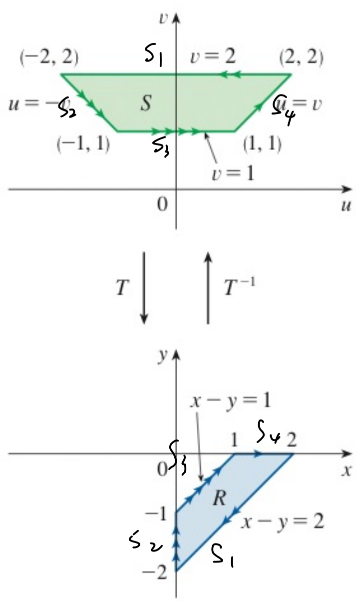


FIGURE 9

To find the region S in the uv-plane corresponding to R, we note that the sides of R lie on the lines

$$y = 0 \qquad x - y = 2 \qquad x = 0 \qquad x - y = 1$$

and, from either Equations 10 or Equations 11, the image lines in the uv-plane are

$$u = v \qquad v = 2 \qquad u = -v \qquad v = 1$$

Thus the region S is the trapezoidal region with vertices $(1, 1)$, $(2, 2)$, $(-2, 2)$, and $(-1, 1)$ shown in Figure 9. Since

$$S = \{(u, v) \mid 1 \leqslant v \leqslant 2,\ -v \leqslant u \leqslant v\}$$

Theorem 9 gives

$$\iint_R e^{(x+y)/(x-y)}\,dA = \iint_S e^{u/v} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\,dv$$

$$= \int_1^2 \int_{-v}^{v} e^{u/v}\left(\tfrac{1}{2}\right) du\,dv = \tfrac{1}{2}\int_1^2 \Big[ve^{u/v}\Big]_{u=-v}^{u=v} dv$$

$$= \tfrac{1}{2}\int_1^2 (e - e^{-1})v\,dv = \tfrac{3}{4}(e - e^{-1})$$

S_1 : $v = 2 \Rightarrow u = x + y,\ 2 = x - y \Rightarrow y = x - 2$ & $x = \frac{u}{2} + 1 \in [0, 2]$
$u \in [-2, 2]$

S_2 : $u + v = 0,\ u \in [-2, -1] \Rightarrow u + v = 2x = 0,\ x = 0$
$y = \frac{u - v}{2} = u \in [-2, -1]$

Change of variables in triple integrals

There is a similar change of variables formula for triple integrals. Let T be a one-to-one transformation that maps a region S in uvw-space onto a region R in xyz-space by means of the equations

$$x = g(u, v, w) \qquad y = h(u, v, w) \qquad z = k(u, v, w)$$

The **Jacobian** of T is the following 3×3 determinant:

12
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Under hypotheses similar to those in Theorem 9, we have the following formula for triple integrals:

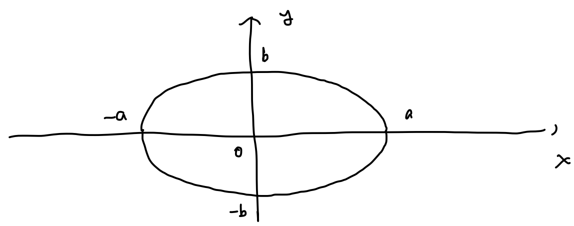
13
$$\iiint_R f(x, y, z)\, dV = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

Exercises:

6. S is the disk given by $u^2 + v^2 \leqslant 1$; $x = au,\ y = bv$

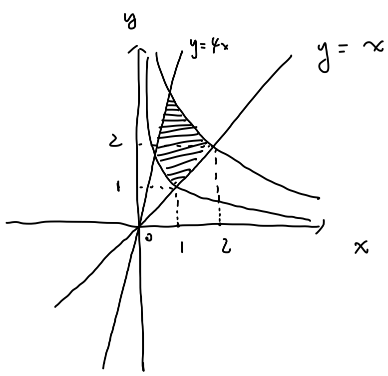
$u = \frac{x}{a}$, $v = \frac{y}{b}$, then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$



10. R is bounded by the hyperbolas $y = 1/x$, $y = 4/x$ and the lines $y = x$, $y = 4x$ in the first quadrant

Sketch R:



$xy = 1$ & $\frac{y}{x} = 1$

$xy = 4$ & $\frac{y}{x} = 4$

hence let $u = xy$, $v = \frac{y}{x}$ $\Rightarrow$ $x = \sqrt{\frac{u}{v}}$, $y = \sqrt{uv}$

$1 \leq u \leq 4$ & $1 \leq v \leq 4$

Review of Ch. 15

	Double integrals	Triple Integrals
Definition :	over rectangles $\Downarrow$ over general region: type I, type II	over rectangular boxes $\Downarrow$ over general region: type 1, type 2, type 3
Changing of variables	Polar coordinate $dA = r\,dr\,d\theta$	cylindrical coordinate $dV = r\,dz\,dr\,d\theta$ spherical coordinate $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

Exercise

7. $\int_0^{\pi} \int_0^1 \int_0^{\sqrt{1-y^2}} y \sin x \, dz\, dy\, dx$

$$= \int_0^{\pi} \sin x \, dx \cdot \int_0^1 y\sqrt{1-y^2}\, dy$$

$$= 2\int_0^1 y\sqrt{1-y^2}\, dy \underset{=}{\overset{u=y^2}{=}} \int_0^1 \sqrt{1-u}\, du = -\frac{2}{3}(1-u)^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}$$

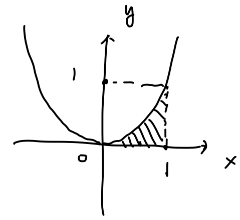
$$\int_0^1 \int_0^x \cos(x^2)\, dy\, dx \quad \text{or try} \quad \int_0^1 \int_y^1 \cos(x^2)\, dx\, dy$$

change variable

$$= \int_0^1 x\cos(x^2)\, dx$$

$$\underset{=}{\overset{u=x^2}{=}} \frac{1}{2}\int_0^1 \cos(u)\, du = \frac{1}{2}\sin(u)\Big|_0^1 = \frac{1}{2}\sin(1)$$

20. $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\, dx\, dy$



$\sqrt{y} \le x \le 1 \Rightarrow y \le x^2 \le 1$

$$= \int_0^1 \int_0^{x^2} \frac{ye^{x^2}}{x^3}\, dy\, dx$$

$$= \int_0^1 \frac{1}{2} x e^{x^2} dx = \frac{1}{4} \int_0^1 e^{x^2} d(x^2)$$

$$= \frac{1}{4} \int_0^1 e^u du = \frac{1}{4}(e-1)$$

14. Identify the surfaces whose equations are given.
(a) $\theta = \pi/4$ (b) $\phi = \pi/4$

(a) $\rho \sin\phi \cos\theta = \rho \sin\phi \sin\theta$

$\boxed{x = y}$

& $x \ge 0$

(b) $\boxed{z = \sqrt{x^2 + y^2}}$

$\rho \cos\phi = \rho \sin\phi$

17. Describe the region whose area is given by the integral

$$\int_0^{\pi/2} \int_0^{\sin 2\theta} r\, dr\, d\theta$$

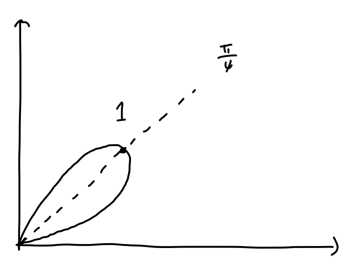
$0 \le r \le \sin 2\theta, \quad 0 \le \theta \le \frac{\pi}{2}$

$r = \sin 2\theta$

$\sqrt{x^2 + y^2} = 2 \sin\theta \cos\theta$

$= \dfrac{2xy}{x^2 + y^2}$

$\Rightarrow (x^2 + y^2)^{\frac{3}{2}} = 2xy$



$\theta = 0 \quad r \le 0$

$\theta = \frac{\pi}{8} \quad r \le \frac{\sqrt{2}}{2}$

$\theta = \frac{\pi}{6}, \quad r = \frac{1}{2}$

$\theta = \frac{\pi}{4}, \quad r \le 1$

$\theta = \frac{3\pi}{8}, \quad r \le \frac{\sqrt{2}}{2}$

58. (a) Evaluate

$$\iint_D \frac{1}{(x^2 + y^2)^{n/2}}\, dA$$

where n is an integer and D is the region bounded by the circles with center the origin and radii r and R, $0 < r < R$.

(b) For what values of n does the integral in part (a) have a limit as $r \to 0^+$?

(c) Find

$$\iiint_E \frac{1}{(x^2 + y^2 + z^2)^{n/2}}\, dV$$

where E is the region bounded by the spheres with center the origin and radii r and R, $0 < r < R$.

(d) For what values of n does the integral in part (c) have a limit as $r \to 0^+$?

(a) $$\iint_D \frac{1}{(x^2+y^2)^{n/2}}\, dA$$

$$= \int_0^{2\pi} \int_r^R \frac{1}{r^n} \cdot r\, dr\, d\theta$$

$$= 2\pi \cdot \frac{r^{2-n}}{2-n} \Big|_r^R$$

$$= \frac{2\pi}{n-2} \left(\frac{1}{r^{n-2}} - \frac{1}{R^{n-2}} \right)$$

(b) $n-2 \leq 0 \Leftrightarrow n \leq 2$

(c) $$\iiint_E \frac{1}{(x^2+y^2+z^2)^{n/2}}\, dV$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_r^R \frac{1}{\rho^n} \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 2\pi \cdot 2 \cdot \int_r^R \frac{1}{\rho^{n-2}}\, d\rho$$

$$= \frac{4\pi}{n-3} \left(\frac{1}{r^{n-3}} - \frac{1}{R^{n-3}} \right)$$

(d) $n-3 \leq 0 \Leftrightarrow n \leq 3$

5. The double integral $\int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy$ is an improper integral and could be defined as the limit of double integrals over the rectangle $[0, t] \times [0, t]$ as $t \to 1^-$. But if we expand th integrand as a geometric series, we can express the integral as the sum of an infinite series. Show that

$$\int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

6. Leonhard Euler was able to find the exact sum of the series in Problem 5. In 1736 he proved that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

In this problem we ask you to prove this fact by evaluating the double integral in Problem 5. Start by making the change of variables

$$x = \frac{u-v}{\sqrt{2}} \qquad y = \frac{u+v}{\sqrt{2}}$$

This gives a rotation about the origin through the angle $\pi/4$. You will need to sketch the corresponding region in the uv-plane.

[*Hint:* If, in evaluating the integral, you encounter either of the expressions $(1 - \sin\theta)/\cos\theta$ or $(\cos\theta)/(1 + \sin\theta)$, you might like to use the identity $\cos\theta = \sin((\pi/2) - \theta)$ and the corresponding identity for $\sin\theta$.]

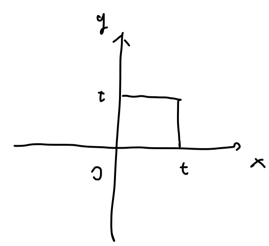
5. $\int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy = \int_0^1 -\frac{1}{y}\ln(1-xy)\Big|_0^1 dy = \int_0^1 -\frac{1}{y}\ln(1-y)\,dy$

$= \int_0^1 \frac{1}{y}\sum_{n=1}^{\infty}\frac{y^n}{n}\,dy$

$= \int_0^1 \sum_{n=1}^{\infty}\frac{y^{n-1}}{n}\,dy = \sum_{n=1}^{\infty}\int_0^1 \frac{y^{n-1}}{n}\,dy = \sum_{n=1}^{\infty}\frac{1}{n^2}$

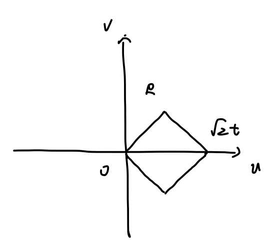
or $= \int_0^1\int_0^1 \sum_{n=0}^{\infty}(xy)^n\,dx\,dy$

$= \int_0^1 \sum_{n=0}^{\infty} y^n \cdot \frac{1}{n+1}\,dy = \sum_{n=1}^{\infty}\frac{1}{n^2}$



6. $\iint_{[0,t]\times[0,t]} \frac{1}{1-xy}\,dx\,dy = \iint_R \frac{1}{1-\frac{u^2-v^2}{2}}\,du\,dv$

$= 4\iint_{R^+} \frac{1}{2-u^2+v^2}\,dv\,du$



$= 4\int_0^{\frac{1}{\sqrt2}}\int_0^u \frac{1}{v^2+2-u^2}\,dv\,du + 4\int_{\frac{1}{\sqrt2}}^{\sqrt2}\int_0^{\sqrt2-u}\frac{1}{v^2+2-u^2}\,dv\,du$

$= 4\int_0^{\frac{1}{\sqrt2}} \frac{1}{\sqrt{2-u^2}}\arctan\frac{u}{\sqrt{2-u^2}}\,du + 4\int_{\frac{1}{\sqrt2}}^{\sqrt2}\frac{1}{\sqrt{2-u^2}}\arctan\frac{\sqrt2-u}{\sqrt{2-u^2}}\,du$

$\| \; u=\sqrt2\sin\theta$ $\qquad \| \; u=\sqrt2\cos\theta$

$\frac{\sqrt2(1-\cos\theta)}{\sqrt2\sin\theta} = \frac{2\sqrt2\cdot\sin^2\frac{\theta}{2}}{2\sqrt2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$

$= 4\int_0^{\frac{\pi}{6}} \frac{\theta}{\sqrt2\cos\theta}\sqrt2\cos\theta\,d\theta + 4\int_0^{\frac{\pi}{3}}\frac{1}{\sqrt2\sin\theta}\cdot\frac{\theta}{2}\sqrt2\sin\theta\,d\theta = 4\times\frac{\pi^2}{72} + 4\times\frac14\times\frac{\pi^2}{9} = \frac{\pi^2}{18}+\frac{\pi^2}{9} = \frac{\pi^2}{6}$