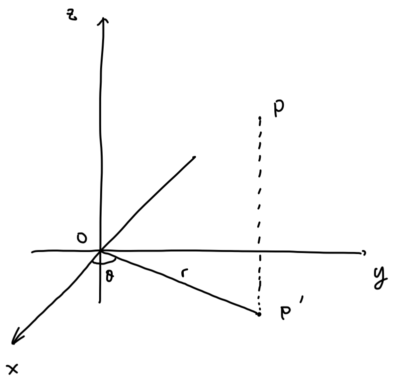
Th 6/1

Triple integrals in Cylindrical Coordinates

Cylindrical coordinate

Given a point $P = (x, y, z) \in \mathbb{R}^3$, we first project it to the xy-plane to get a point (x, y)



then from (x, y), we get a point (r, θ) by the usual polar coordinate on $\mathbb{R}^2$, i.e.

$$r = \sqrt{x^2 + y^2}$$

and $x = r\cos\theta$, $y = r\sin\theta \Rightarrow \tan\theta = \frac{y}{x}$

then we get a point (r, θ, z)

Examples

(a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.
(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

(a): $\left(2, \frac{2\pi}{3}, 1\right) \rightsquigarrow \left(2\cos\frac{2\pi}{3}, 2\sin\frac{2\pi}{3}, 1\right) = (-1, \sqrt{3}, 1)$

(b) $(3, -3, -7) \rightsquigarrow r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$, $\cos\theta = \frac{x}{3\sqrt{2}} = \frac{\sqrt{2}}{2}$, $\sin\theta = \frac{y}{3\sqrt{2}} = -\frac{\sqrt{2}}{2}$, then

$$\theta = \frac{7\pi}{4}$$

then $\left(3\sqrt{2}, \frac{7\pi}{4}, -7\right)$

Cylindrical coordinates are useful in problems that involve symmetry about an axis, and the z-axis is chosen to coincide with this axis of symmetry. For instance, the axis of

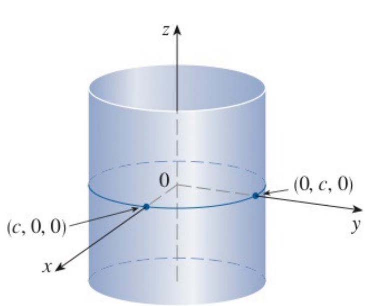


FIGURE 4
$r = c$, a cylinder

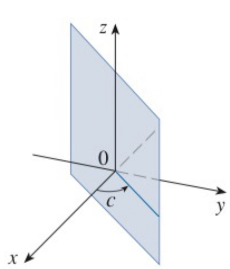


FIGURE 5
$\theta = c$, a vertical plane

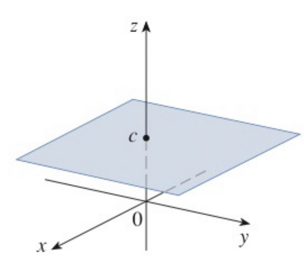


FIGURE 6
$z = c$, a horizontal plane

Triple integrals in Cylindrical coordinates

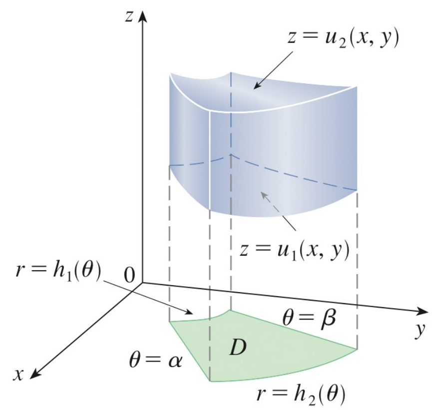
Let's consider a region E such that

- E is of type 1, i.e

$$E = \{ (x, y, z) \mid (x, y) \in D,\quad u_1(x,y) \le z \le u_2(x,y) \}$$

- D is a type I region in Polar coordinate, i.e.

$$D = \{ (r, \theta) \mid \alpha \le \theta \le \beta,\quad h_1(\theta) \le r \le h_2(\theta) \}$$



Let f be a continuous function on E, then

$$\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$

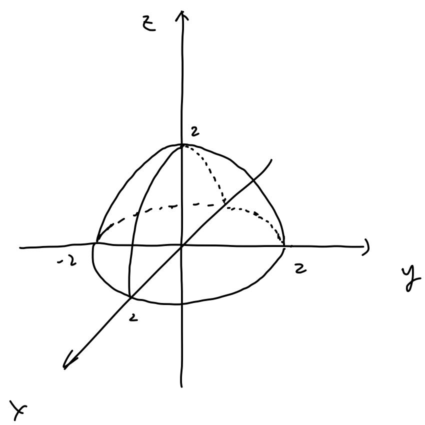
$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,dz \cdot r\,dr\,d\theta$$

$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

Example:

EXAMPLE 3 Evaluate $\iiint_E x^2\,dV$, where E is the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the xy-plane (see Figure 10).

Sketch the region E



$$E = \{(x,y,z) \mid \underbrace{x^2+y^2 \le 4}_{D},\ 0 \le z \le 4-(x^2+y^2)\}$$

$$D = \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 2\}$$, then

$$\iiint_E x^2\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} r^2\cos^2\theta\ r\,dz\,dr\,d\theta$$

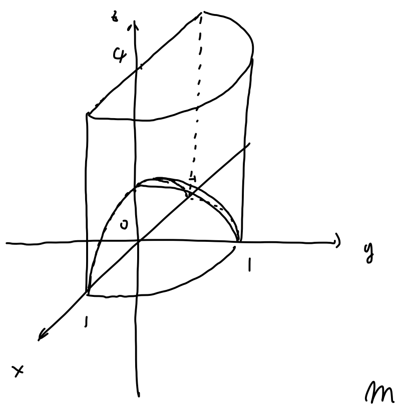
$$= \int_0^{2\pi}\int_0^2 (4-r^2)r^3\cos^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\cos^2\theta\,d\theta \cdot \int_0^2 (4r^3 - r^5)\,dr$$

$$= \int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta \cdot \left. r^4 - \frac{1}{6}r^6\right|_0^2$$

$$= \pi \times \left(16 - \frac{64}{6}\right) = \frac{16\pi}{3}$$

EXAMPLE 4 A solid E lies within the cylinder $x^2 + y^2 = 1$ to the right of the xz-plane, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. (See Figure 12.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.



$$E = \{ (x, y, z) \mid \underbrace{x^2 + y^2 \le 1}_{\downarrow}, \ 1 - x^2 - y^2 \le z \le 4 \} \quad y \ge 0$$

$$D = \{ (r, \theta) \mid 0 \le \theta \le \pi, \ 0 \le r \le 1 \}$$

then $\rho(x, y, z) = K \cdot \sqrt{x^2 + y^2}$

$$m = K \cdot \iiint_E \sqrt{x^2+y^2} \, dV$$

$$= K \iint_D \left(\int_{1-x^2-y^2}^{4} \sqrt{x^2+y^2} \, dz \right) dA$$

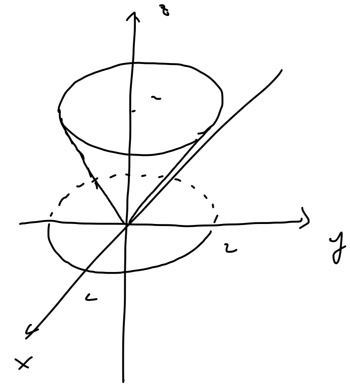
$$= K \int_0^{\pi} \int_0^1 \int_{1-r^2}^{4} r \cdot r \, dz \, dr \, d\theta$$

$$= K \cdot \pi \cdot \int_0^1 (r^2+3) r^2 \, dr = \pi K \left(\frac{1}{5} + 1 \right) = \frac{6\pi K}{5}$$

EXAMPLE 5 Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx.$

$$E = \left\{ (x,y,z) \;\middle|\; \underbrace{-2 \le x \le 2,\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}}_{①},\ \sqrt{x^2+y^2} \le z \le 2 \right\}$$

Sketch:



$(x,y) \in$ circle with radius 2, centered at O

$$= \int_0^{2\pi}\int_0^2\int_r^2 r^2\cdot r\,dz\,dr\,d\theta$$

$$= 2\pi\cdot\int_0^2 r^3(2-r)\,dr$$

$$= 2\pi\cdot\ \frac{1}{2}r^4 - \frac{1}{5}r^5\Big|_0^2$$

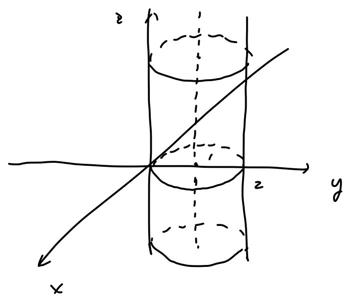
$$= 2\pi\cdot\left(8-\frac{32}{5}\right) = \frac{16\pi}{5}$$

Exercise

8. $r = 2\sin\theta \Rightarrow r^2 = 2r\sin\theta$

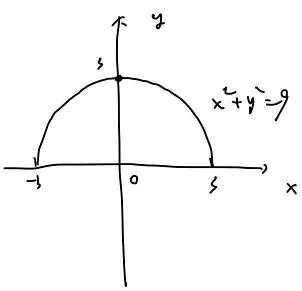
$$\Downarrow \qquad \Downarrow$$

$$x^2 + y^2 = 2y \Rightarrow x^2 + (y-1)^2 = 1$$



32. $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$

$-3 \le x \le 3$ & $0 \le y \le \sqrt{9-x^2} \rightarrow (r,\theta)$: $0 \le \theta \le \pi$, $0 \le r \le 3$, then



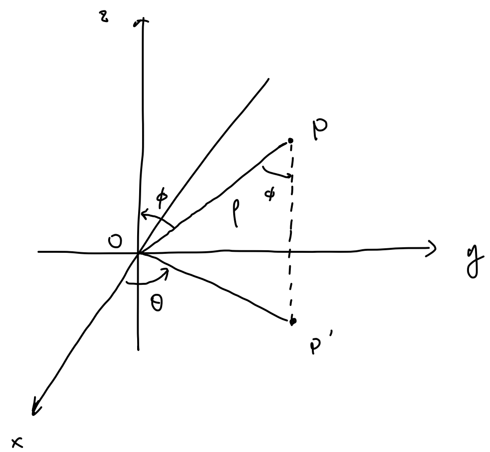
$$= \int_0^{\pi}\int_0^{3}\int_0^{9-r^2} r \cdot r \, dz\, dr\, d\theta$$

$$= \pi \int_0^{3} r^2(9-r^2)\, dr$$

Triples integrals in spherical coordinates

Spherical coordinate

Given a point $P = (x, y, z) \in \mathbb{R}^3$, project it to xy-plane, we get $P' = (x, y)$



connect O & P', O & P

ϕ = angle between z-axis & OP $\in [0, \pi]$

θ = angle between OP' & x-axis $\in [0, 2\pi]$

$\rho = |OP| = \sqrt{x^2 + y^2 + z^2} \geq 0$

therefore $(x, y, z) \rightsquigarrow (\rho, \phi, \theta)$

How to recover the original point from (ρ, ϕ, θ) ?

By definition of ϕ, $|OP'| = |OP| \cdot \sin\phi = \rho \sin\phi$, $z = \rho\cos\phi$

By definition of θ, $OP' = (|OP'|\cos\theta, |OP'|\sin\theta)$

$$= (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta) = (x, y)$$

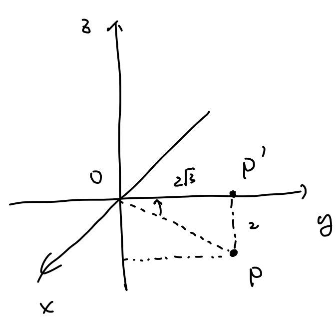
then $(x, y, z) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$

Examples:

EXAMPLE 1 The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

$x = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = \frac{\sqrt{6}}{2}$, $y = 2\sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{6}}{2}$, $z = 2\cos\frac{\pi}{3} = 1$

EXAMPLE 2 The point $(0, 2\sqrt{3}, -2)$ is given in rectangular coordinates. Find spherical coordinates for this point.

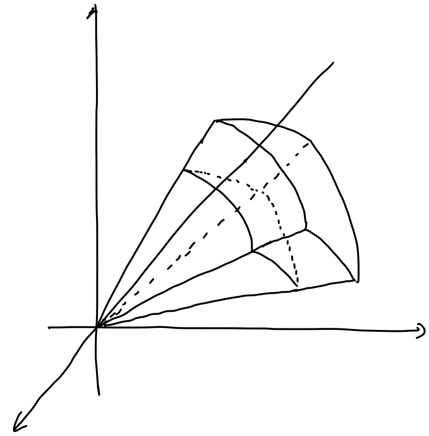


$$\left.\begin{aligned} &P' = (0, 2\sqrt{3}) \Rightarrow \theta = \frac{\pi}{2}, \\ &\rho = \sqrt{12 + 4} = 4 \\ &\phi = \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3} \end{aligned}\right\} \Rightarrow \left(4, \frac{2\pi}{3}, \frac{\pi}{2}\right)$$

Triple integrals in Spherical coordinates

Rectangular Box in Spherical coordinate:

$$a \le \rho \le b, \quad \alpha \le \theta \le \beta, \quad c \le \phi \le d$$



it is not a rectangular box in $\mathbb{R}^3$! we denote the region by E

Let $f: E \to \mathbb{R}$ be a continuous function, you may think that

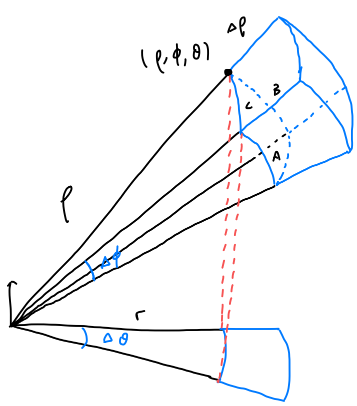
$$\iiint_E f(x,y,z)\, dV = \int_a^b \int_\alpha^\beta \int_c^d f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, d\rho\, d\phi\, d\theta$$

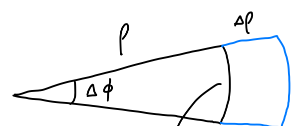
but this is not true!. the problem arises because

$$dV \neq d\rho\, d\phi\, d\theta$$

Let's study the relation between $\underset{dx\,dy\,dz}{\overset{\shortparallel}{dV}}$ & $d\rho\, d\phi\, d\theta$:

If we are given a small "box" with $\Delta\rho$, $\Delta\phi$, $\Delta\theta$, then





$= \rho \cdot \Delta\phi = A$

$C = r \cdot \Delta\theta$

$= \rho \sin\phi \cdot \Delta\theta$

$B = \Delta\rho$

$$\Rightarrow \quad \Delta V \approx \Delta\rho \cdot \rho\Delta\phi \cdot \rho\sin\phi\,\Delta\theta = \rho^2 \sin\phi\, \Delta\rho\, \Delta\phi\, \Delta\theta \quad \Rightarrow \quad dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

3 $\displaystyle\iiint_E f(x, y, z)\, dV$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$

Examples:

EXAMPLE 3 Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\,dV$, where B is the unit ball:

$$B = \left\{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\right\}$$

in spherical coordinate: $0 \le \rho \le 1$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$

$$\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^1 e^{\rho^3}\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi \cdot -\cos\phi\Big|_0^{\pi} \cdot \frac{1}{3}e^{\rho^3}\Big|_0^1$$

$$= 2\pi \cdot 2 \cdot \frac{1}{3}(e-1) = \frac{4\pi(e-1)}{3}$$

NOTE It would have been extremely awkward to evaluate the integral in Example 3 without spherical coordinates. In rectangular coordinates the iterated integral would have been

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\,dz\,dy\,dx$$

EXAMPLE 4 Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. (See Figure 9.)

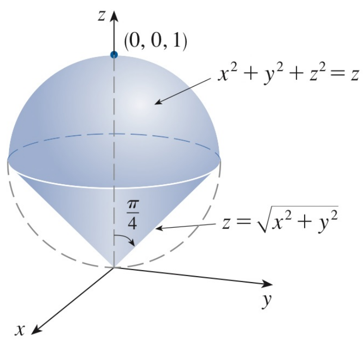


FIGURE 9

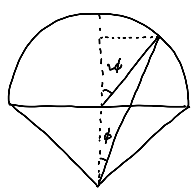
first sketch the solid

$x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4} \Rightarrow$ centered at $(0, 0, \frac{1}{2})$, radius $\frac{1}{2}$

how to express it in spherical coordinate?

$$z = \sqrt{x^2 + y^2}$$

$\rho\cos\phi \qquad \rho\sin\phi \Rightarrow \sin\phi = \cos\phi \rightsquigarrow \phi = \frac{\pi}{4}$



for given ϕ such that $0 \le \phi \le \frac{\pi}{4}$

ρ ranges from 0 to $\dfrac{\frac{1}{2} + \frac{1}{2}\cos 2\phi}{\cos\phi} = \cos\phi$

$\theta \in [0, 2\pi]$, hence

$$\text{Vol} = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \frac{(\cos\phi)^3}{3} \sin\phi \, d\phi \overset{u = \cos\phi}{=\!=} \frac{2\pi}{3} \int_{\frac{\sqrt{2}}{2}}^{1} u^3 \, du$$

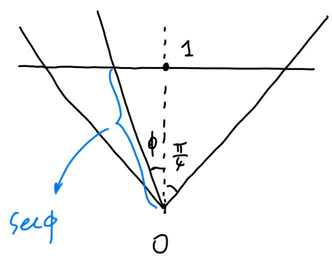
$$= \frac{2\pi}{3} \times \frac{1}{4} u^4 \Big|_{\frac{\sqrt{2}}{2}}^{1}$$

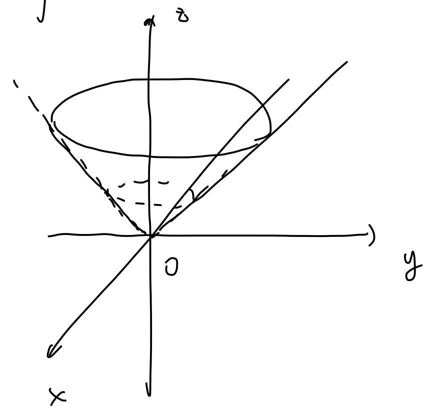
$$= \frac{\pi}{8}$$

Exercise

18. $\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

Sketch the solid: take a section with fixed θ, we get





$$\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \frac{(\sec\phi)^3}{3} \sin\phi \, d\phi$$

$$\underline{\underline{u=\cos\phi}} \quad \frac{2\pi}{3} \int_{\frac{\sqrt{2}}{2}}^{1} \frac{1}{3u^3} \, du$$

26. Evaluate $\iiint_E y^2\,dV$, where E is the solid hemisphere $x^2 + y^2 + z^2 \leqslant 9$, $y \geqslant 0$.

$0 \le \rho \le 3, \quad 0 \le \phi \le \frac{\pi}{2}, \quad 0 \le \theta \le 2\pi$

$$\iiint_E y^2\,dV = \int_0^3 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \rho^2 \sin^2\phi \sin^2\theta \, \rho^2 \sin\phi \, d\theta\, d\phi\, d\rho$$

$$= \int_0^3 \rho^4\,d\rho \cdot \int_0^{\frac{\pi}{2}} \sin^3\phi\,d\phi \cdot \int_0^{2\pi} \sin^2\theta\,d\theta$$

$$= \frac{3^5}{5} \cdot \frac{2}{3} \cdot \pi$$

$$\int_0^{\frac{\pi}{2}} \sin^3\phi\,d\phi = -\int_0^{\frac{\pi}{2}} \sin^2\phi\,d(\cos\phi) \overset{u=\cos\phi}{=\!=} \int_0^1 (1-u^2)\,du = \frac{2}{3}$$

$$\int_0^{2\pi} \sin^2\theta\,d\theta = \int_0^{2\pi} \frac{1-\cos 2\theta}{2}\,d\theta = \pi$$