Syzygies of Unimodular Lawrence Ideals

Dave Bayer Sorin Popescu Bernd Sturmfels

Abstract: *Infinite hyperplane arrangements whose vertices form a lattice are studied from the point of view of commutative algebra. The quotient of such an arrangement modulo the lattice action represents the minimal free resolution of the associated binomial ideal, which defines a toric subvariety in a product of projective lines. Connections to graphic arrangements and to Beilinson's spectral sequence are explored.*

1 Introduction

We are interested in the defining ideals of toric subvarieties in a product of projective lines $\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. Writing $(x_i : y_i)$ for the homogeneous coordinates of the i-th factor $\mathbb{P}^1$, these are the binomial ideals in $2n$ variables of the following form:

$$J_L \quad = \quad \langle\, \mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}} - \mathbf{x}^{\mathbf{b}}\mathbf{y}^{\mathbf{a}} \mid \mathbf{a}-\mathbf{b} \in L \,\rangle \quad \subset \quad S \,=\, k[x_1,\ldots,x_n,\, y_1,\ldots,y_n],$$

where L is a sublattice of $\mathbb{Z}^n$ and k is a field. Here $\mathbf{x}^{\mathbf{a}} = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$ for $\mathbf{a} = (a_1,\ldots,a_n) \in \mathbb{N}^n$. Binomial ideals of the form J_L are called *Lawrence ideals.* They provide the algebraic analogue to the Lawrence construction for convex polytopes [Zi, §6.6]. Lawrence polytopes enjoy remarkable rigidity properties, such as [Zi, Theorem 6.27]. On the algebraic side, rigidity of Lawrence ideals manifests itself in the following result, which appears in [Stu, Theorem 7.1]. Recall for part (d) that the *Graver basis* consists of all binomials $\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}} - \mathbf{x}^{\mathbf{b}}\mathbf{y}^{\mathbf{a}}$ in J_L such that the only vector $\mathbf{a}'-\mathbf{b}' \in L\backslash\{\mathbf{0}\}$ with $\mathbf{0} \le \mathbf{a}' \le \mathbf{a}$ and $\mathbf{0} \le \mathbf{b}' \le \mathbf{b}$ is $\mathbf{a}-\mathbf{b}$ itself.

Proposition 1.1 *The following sets of binomials in a Lawrence ideal J_L coincide:*
(a) *Any minimal set of binomial generators of J_L.*
(b) *Any reduced Gröbner basis for J_L.*
(c) *The universal Gröbner basis for J_L (the union of all reduced Gröbner bases).*
(d) *The Graver basis for J_L.*

Cellular resolutions, as defined in [BS], provide a natural geometric framework for studying homological, algorithmic and combinatorial properties of monomial and binomial ideals. One instance is the *Scarf complex*, which gives the minimal resolution for generic monomial ideals [BPS] and generic lattice ideals [PS]. The *hull resolution* of [BS] generalizes the Scarf complex and provides a cellular resolution for arbitrary co-Artinian monomial modules; however, it need not be minimal. For lattice modules, the hull resolution is compatible with the lattice action and determines a cellular resolution of the corresponding lattice ideal [BS, Theorem 3.9].

In this paper we present a minimal cellular resolution, which happens to also coincide with the hull resolution, for a remarkable class of nongeneric lattice ideals. These are the unimodular Lawrence ideals J_L, which are characterized as follows:

Theorem 1.2 *For a sublattice L of $\mathbb{Z}^n$ the following conditions are equivalent:*

(a) *The Lawrence ideal J_L possesses an initial monomial ideal which is radical.*
(b) *Every initial monomial ideal of the Lawrence ideal J_L is a radical ideal.*
(c) *Every minimal generator of J_L is a difference of two squarefree monomials.*
(d) *The lattice L is the image of an integer matrix B with linearly independent columns, such that all maximal minors of B lie in the set $\{0, 1, -1\}$.*
(e) *The lattice L is the kernel of an integer matrix A with linearly independent rows, such that all maximal minors of A lie in $\{0, m, -m\}$ for some integer m.*
(f) *The quotient ring S/J_L is a normal domain.*

Theorem 1.2 is proved in Section 2. If any (and thus all) of these six equivalent conditions for L holds, then we say that the Lawrence ideal J_L is *unimodular*. A first example is the ideal of 2×2-minors of a $2 \times n$-matrix of indeterminates:

$$J_L \quad = \quad I_2 \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ y_1 & y_2 & y_3 & \cdots & y_n \end{pmatrix} \tag{1.1}$$

Here L is the kernel of $A = (\, 1\ 1\ 1\ \cdots 1\,)$, or the image of the matrix B whose rows are $\mathbf{e}_i - \mathbf{e}_{i+1}$, $i \in \{1, \ldots, n-1\}$, the differences of consecutive unit vectors in $\mathbb{R}^n$. The minimal resolution of (1.1) is an Eagon-Northcott complex, whose polyhedral model is the hypersimplicial complex of Gel'fand and MacPherson [BS, Ex. 3.15].

In Section 2 we introduce an infinite periodic hyperplane arrangement $\mathcal{H}_L$ whose vertices are the elements of L. It is shown in Section 3 that this arrangement supports the minimal free resolution of J_L, and coincides with the hull complex of J_L. In Section 4 we prove that this resolution is universal in the sense that it is stable under all Gröbner deformations. In particular, all initial ideals of J_L have the same Betti numbers as J_L, and their minimal resolutions are also cellular. We also construct cellular resolutions for the monomial ideals defined by the fibers of L. In Section 5 we discuss Lawrence ideals associated with directed graphs and we present open combinatorial problems. In Section 6 we reinterpret Lawrence ideals in terms of the Audin-Cox homogeneous coordinate ring, and we generalize Beilinson's spectral sequence from projective space to unimodular toric varieties.

The authors would like to thank the Mathematical Sciences Research Institute in Berkeley for its support while part of this paper was being written. All three authors were partially supported by the NSF during the preparation of this work.

2 Unimodularity and an infinite hyperplane arrangement

In this section we establish some basic facts about unimodular lattices and their Lawrence ideals. We start out by proving the equivalences stated in Section 1.

Proof of Theorem 1.2. A monomial ideal is radical if and only if it is generated by squarefree monomials. Clearly, the first term of a binomial $\mathbf{x^a y^b} - \mathbf{x^b y^a}$ is squarefree if and only if the second term is squarefree. The equivalence of (a), (b) and (c) follows directly from Proposition 1.1.

In an exact sequence of free abelian groups,

$$0 \longrightarrow \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d, \tag{2.1}$$

each $(n-d) \times (n-d)$-minor of B is equal to the complementary $d \times d$-minor of A, up to a global constant m. Since the cokernel of B is torsion free, the maximal minors of B are integers with no common factor. This implies $m \in \mathbb{Z}$ and the equivalence of (d) and (e), for $L = \ker(A) = \operatorname{im}(B)$.

The condition (e) is precisely the defining condition given in [Stu, §8, page 70] for a matrix A to be unimodular. A matrix A is unimodular if and only if its Lawrence lifting $\Lambda(A)$ is unimodular. Following [Stu, §7, page 55], the *Lawrence lifting* of the $d \times n$-matrix A is obtained by appending the zero $d \times n$-matrix $\mathbf{0}_{d,n}$ and two copies of the $n \times n$-identity matrix $\mathbf{I}_n$ as follows:

$$\Lambda(A) \quad = \quad \begin{pmatrix} A & \mathbf{0}_{d,n} \\ \mathbf{I}_n & \mathbf{I}_n \end{pmatrix}.$$

Hence the equivalence of (b) and (e) is a reformulation of [Stu, Remark 8.10].

The conjunction of (b) and (e) implies property (f), namely, the lattice L is the kernel of an integer matrix if and only if J_L is a prime ideal, and (b) implies normality by [Stu, Proposition 13.15]. To complete the proof, it suffices to show that (f) implies (c). Suppose that (c) is false, i.e., the ideal J_L has a minimal generator $\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}} - \mathbf{x}^{\mathbf{b}}\mathbf{y}^{\mathbf{a}}$ whose terms are not squarefree. We may assume that this generator is a circuit, which means that it has minimal support. By setting all pairs of variables not appearing in this circuit to zero, we reduce to the case where J_L is a principal ideal. But an affine binomial hypersurface is normal if and only if at least one of its monomials is squarefree. This completes our proof of Theorem 1.2. ∎

Here is another, more invariant, formulation of the unimodularity condition.

Proposition 2.1 *A lattice L is unimodular if and only if, for every projection $\pi(L)$ of L to a coordinate sublattice $\mathbb{Z}^r \subset \mathbb{Z}^n$, the quotient $\mathbb{Z}^r/\pi(L)$ is torsion free.*

Proof. We will use the notation of the proof of Theorem 1.2. We first prove the "if" direction. Set $m = n - d$ and consider any nonsingular $m \times m$-submatrix C of B. The image of C in $\mathbb{Z}^m$ equals $\pi(L)$ for the corresponding coordinate projection $\pi : \mathbb{Z}^n \to \mathbb{Z}^m$. The finite abelian group $\mathbb{Z}^m/\pi(L)$ is torsion free if and only if it is zero. Hence $\pi(L) = \mathbb{Z}^m$ and we conclude that the determinant of the $m \times m$-submatrix of B under consideration is either 1 or -1.

For the "only-if" direction, suppose that L is unimodular. Every coordinate projection $\pi(L)$ of L is unimodular as well. This can be seen by choosing A (resp. B) to have appropriate unit vectors among its columns (resp. rows). Thus it suffices to show that the group $\mathbb{Z}^n/L$ is torsion free. But this follows from the exact sequence (2.1), which identifies $\mathbb{Z}^n/L$ with the image of the matrix A, showing $\mathbb{Z}^n/L \simeq \mathbb{Z}^d$. ∎

Let L be any m-dimensional sublattice of $\mathbb{Z}^n$, and write $\mathbb{R}L$ for the linear subspace of $\mathbb{R}^n$ spanned by L. We denote by $\mathcal{H}_L$ the *affine hyperplane arrangement* in $\mathbb{R}L$ obtained by intersecting $\mathbb{R}L$ with all lattice translates of the coordinate hyperplanes in $\mathbb{R}^n$. These are the hyperplanes $\{x_i = j\}$ for $1 \leq i \leq n$ and $j \in \mathbb{Z}$. Thus $\mathcal{H}_L$ is an infinite m-dimensional hyperplane arrangement in the vector space $\mathbb{R}L$.

It is convenient to embed $\mathcal{H}_L$ as a hyperplane arrangement in the Euclidean space $\mathbb{R}^m$ as follows. Let B be an integer $n \times m$-matrix such that $L = \mathrm{im}(B)$ as in part (d) of Theorem 1.2. Write $\mathbf{b}_i \in \mathbb{Z}^m$ for the i-th row vector of B. Then $\mathcal{H}_L$ is isomorphic to the infinite arrangement in $\mathbb{R}^m$ consisting of the hyperplanes

$$H_{ij} \;=\; \{\, \mathbf{x} \in \mathbb{R}^m \;\mid\; \mathbf{b}_i \cdot \mathbf{x} = j \,\} \qquad \text{for all } i \in \{1,2,\ldots,n\} \text{ and all } j \in \mathbb{Z}.$$

Proposition 2.2 *Each lattice point in L is a vertex of the affine hyperplane arrangement $\mathcal{H}_L$. There are no additional vertices in $\mathcal{H}_L$ if and only if L is unimodular.*

Proof. We identify L with $\mathbb{Z}^m$ via the matrix B and thus consider $\mathcal{H}_L$ as the hyperplane arrangement $\mathcal{H}_L = \{H_{ij}\}$ in $\mathbb{R}^m$. The intersection point of m linearly independent such hyperplanes,

$$H_{i_1 j_1} \cap H_{i_2 j_2} \cap \cdots \cap H_{i_m j_m} \quad = \quad \{\mathbf{x}\},$$

is defined by the linear equations $\mathbf{b}_{i_\nu} \cdot \mathbf{x} = j_\nu$ for $\nu = 1, \ldots, m$. The point $\mathbf{x} \in \mathbb{R}^m$ has integer coordinates for all $j_1, \ldots, j_m \in \mathbb{Z}$ if and only if $\det(\mathbf{b}_{i_1}, \ldots, \mathbf{b}_{i_m}) = \pm 1$. By part (d) in Theorem 1.2, this means that the lattice L is unimodular. The first assertion holds because each $\mathbf{x} \in \mathbb{Z}^m$ can be expressed as such an intersection. ∎

From now on we assume that L is a unimodular sublattice of $\mathbb{Z}^n$. Recall from [Stu, Proposition 8.11] that the Graver basis of the Lawrence ideal J_L is given precisely by the circuits of the lattice L. (The *circuits* of L are the primitive vectors in L whose supports are minimal with respect to inclusion.) We view $\mathcal{H}_L$ as an infinite regular cell complex, equipped with an action by the abelian group L.

Lemma 2.3 *Two vertices $\mathbf{a}$ and $\mathbf{b}$ of the arrangement $\mathcal{H}_L$ are connected by an edge if and only if their difference $\mathbf{a} - \mathbf{b}$ is a circuit of the unimodular lattice L.*

Proof. Since L acts transitively on the vertices of $\mathcal{H}_L$ we may assume that $\mathbf{b} = \mathbf{0}$. Our assertion states that $\mathbf{a}$ is a circuit if and only if $\{\mathbf{0}, \mathbf{a}\}$ forms an edge in $\mathcal{H}_L$. This holds because the circuits in the subspace $\mathbb{R}L$ of $\mathbb{R}^n$ are computed by the rule

$$H_{i_1 0} \cap H_{i_2 0} \cap \cdots \cap H_{i_{m-1} 0} \quad = \quad \mathbb{R}\mathbf{a}$$

for all possible increasing sequences of indices $1 \leq i_1 < i_2 < \cdots < i_{m-1} \leq n$. ∎

In this section we have introduced a family of hyperplane arrangements $\mathcal{H}_L$ whose vertices form a lattice L. Such arrangements appear in many parts of the mathematical literature; for example, see [BLSWZ]. The group L acts on the faces of the arrangement $\mathcal{H}_L$ with finitely many orbits, and we shall be interested in the quotient complex $\mathcal{H}_L/L$. Nontrivial examples will be presented in Section 5. First we reinterpret $\mathcal{H}_L$ and $\mathcal{H}_L/L$ as minimal free resolutions in the sense of commutative algebra. This will be done in the next two sections by labeling the faces of $\mathcal{H}_L$ with Laurent monomials, following the general recipes in [BS].

Our warmup example (1.1) is the case where L is spanned by the vectors $\mathbf{e}_i - \mathbf{e}_{i+1}$, $i \in \{1, \ldots, n-1\}$, the differences of the consecutive unit vectors in $\mathbb{R}^n$. Thus J_L is here the ideal of 2×2-minors, and $\mathcal{H}_L$ is the *hypersimplicial arrangement*. The quotient complex $\mathcal{H}_L/L$ has $n-1$ distinct maximal faces, called *hypersimplices*. The hypersimplicial arrangement for $n = 3$ is depicted in the following figure:

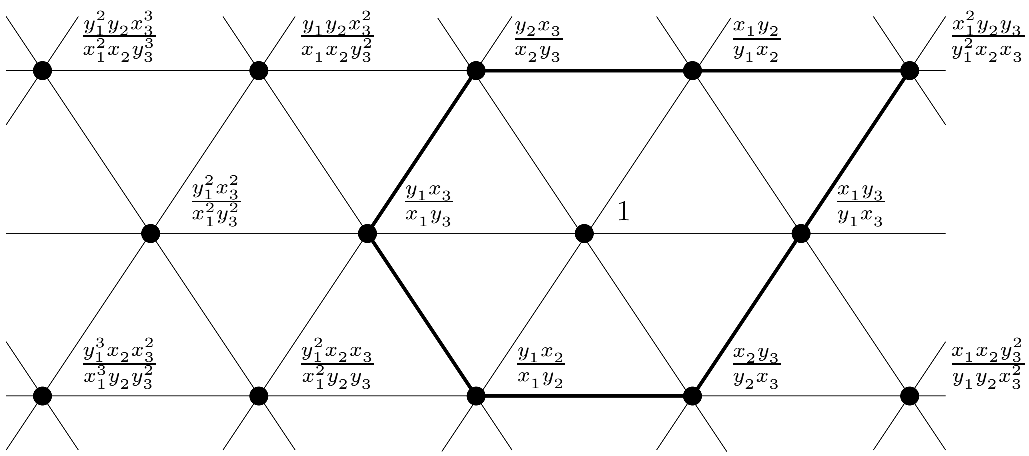

Figure 1: The hypersimplicial arrangement for $n = 3$.

Its hypersimplices are up-triangles and down-triangles. Each vertex is labeled by a Laurent monomial. The finite quotient complex $\mathcal{H}_L/L$ is a torus, subdivided by one vertex, three edges and two 2-cells, one up-triangle and one down-triangle.

3 From hyperplane arrangements to minimal free resolutions

We fix a unimodular sublattice L of dimension m in $\mathbb{Z}^n$. The Laurent polynomial ring $T = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$ is a module over the polynomial ring $S = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$. We consider the following monomial S-submodule of T:

$$M_L \quad := \quad S \cdot \{\, \mathbf{x}^{\mathbf{a}}\mathbf{y}^{-\mathbf{a}} \mid \mathbf{a} \in L \,\} \quad \subset \quad T.$$

Each lattice point $\mathbf{a} = (a_1, \ldots, a_n)$ in L is a vertex of the arrangement $\mathcal{H}_L$, and we label that vertex of $\mathcal{H}_L$ with the corresponding generator of M_L, namely,

$$\mathbf{x}^{\mathbf{a}}\mathbf{y}^{-\mathbf{a}} \quad := \quad x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}y_1^{-a_1}y_2^{-a_2}\cdots y_n^{-a_n}.$$

Each face F of $\mathcal{H}_L$ is then labeled by the least common multiple m_F of the labels of its vertices. (The *least common multiple* of a set of Laurent monomials is the Laurent monomial whose exponents are the coordinatewise maxima of the given exponents.) The labeled cell complex $\mathcal{H}_L$ defines a complex of free $\mathbb{Z}^{2n}$-graded S-modules

$$\mathbf{F}_{\mathcal{H}_L} \;=\; \bigoplus_{\substack{F\in\mathcal{H}_L\\ F\neq\emptyset}} S(-m_F),$$

where the summand $S(-m_F)$ has homological degree $\dim(F)$. The differential of the complex $\mathbf{F}_{\mathcal{H}_L}$ is the homogenized differential of the cell complex $\mathcal{H}_L$, defined by

$$\partial(F) \quad = \sum_{\substack{F'\subset F\\ \mathrm{cod}(F',F)=1}} \epsilon(F,F')\cdot\frac{m_F}{m_{F'}}\cdot F', \qquad \text{for faces} \quad F \text{ of } \mathcal{H}_L.$$

Here $\epsilon(F,F')$ is either $+1$ or -1, indicating the orientation of F' in the boundary of F. See [BS, §1] for details on this construction. The complex $(\mathbf{F}_{\mathcal{H}_L},\partial)$ is not S-finite, but has finite length $m=\mathrm{rank}(L)$.

As an example consider the complex of free $k[x_1,x_2,x_3,y_1,y_2,y_3]$-modules defined by the hypersimplicial arrangement in Figure 1. The edge E connecting the module generators $m_1=\frac{x_2y_3}{y_2x_3}$ and $m_2=\frac{x_1y_3}{y_1x_3}$ is labeled by their least common multiple, which is the Laurent monomial $m_E=\frac{x_1x_2y_3}{x_3}$. This edge E represents the first syzygy $(x_1y_2)\cdot m_1-(y_1x_2)\cdot m_2$ of the monomial submodule M_L.

Theorem 3.1 *The complex $(\mathbf{F}_{\mathcal{H}_L},\partial)$ is a minimal $\mathbb{Z}^{2n}$-graded free S-resolution of the lattice module M_L.*

Proof. The complex $(\mathbf{F}_{\mathcal{H}_L},\partial)$ consists of free S-modules and is clearly $\mathbb{Z}^{2n}$-graded. To show that it is a resolution, we apply the exactness criterion in [BS, Proposition 1.2] to the labeled cell complex $X=\mathcal{H}_L$. For any $(\mathbf{a},\mathbf{b})\in\mathbb{Z}^{2n}$ we consider the subcomplex $X_{\le(\mathbf{a},\mathbf{b})}$ consisting of all faces F of the arrangement X whose label $m_F=\mathbf{x}^{\mathbf{c}}\mathbf{y}^{\mathbf{d}}$ satisfies the coordinatewise inequalities $\mathbf{c}\le\mathbf{a}$ and $\mathbf{d}\le\mathbf{b}$. We shall prove that $X_{\le(\mathbf{a},\mathbf{b})}$ is contractible, by identifying this subcomplex with a convex polytope in $\mathbb{R}L$. For instance, the marked pentagon in Figure 1 is $X_{\le(2,1,1,1,1,1)}$.

Our labeled hyperplane arrangement can be described as follows. For $\mathbf{u}\in\mathbb{R}^n$ we write $(\lceil\mathbf{u}\rceil,\lfloor\mathbf{u}\rfloor)$ for the vector in $\mathbb{Z}^{2n}$ obtained by rounding up and down each coordinate of $\mathbf{u}$. For instance, if $\mathbf{u}=(-2/5,-1,7/5)$ then $(\lceil\mathbf{u}\rceil,\lfloor\mathbf{u}\rfloor)=(0,-1,2,-1,-1,1)$. Two vectors $\mathbf{u}$ and $\mathbf{v}$ in the subspace $\mathbb{R}L$ of $\mathbb{R}^n$ are called *equivalent* if $(\lceil\mathbf{u}\rceil,\lfloor\mathbf{u}\rfloor)=(\lceil\mathbf{v}\rceil,\lfloor\mathbf{v}\rfloor)$. The resulting equivalence classes on $\mathbb{R}L$ are the (relatively open) faces of $X=\mathcal{H}_L$. The face F containing $\mathbf{u}\in\mathbb{R}L$ is labeled by the vector $(\lceil\mathbf{u}\rceil,-\lfloor\mathbf{u}\rfloor)$, or by the corresponding Laurent monomial $m_F=\mathbf{x}^{\lceil\mathbf{u}\rceil}\mathbf{y}^{-\lfloor\mathbf{u}\rfloor}$.

For $\mathbf{a},\mathbf{b}\in\mathbb{Z}^n$ we consider the following subset of the subspace $\mathbb{R}L$ in $\mathbb{R}^n$:

$$\{\,\mathbf{u}\in\mathbb{R}L \mid \lceil\mathbf{u}\rceil\le\mathbf{a} \text{ and } -\lfloor\mathbf{u}\rfloor\le\mathbf{b}\,\} \quad = \quad \{\,\mathbf{u}\in\mathbb{R}L \mid -\mathbf{b}\le\mathbf{u}\le\mathbf{a}\,\}. \quad \text{(3.1a)}$$

This is a convex polytope with facet hyperplanes taken from $\mathcal{H}_L$. By the construction in the previous paragraph, the complex $X_{\le(\mathbf{a},\mathbf{b})}$ is a polyhedral subdivision of the convex polytope identified in (3.1a). Therefore $X_{\leq(\mathbf{a},\mathbf{b})}$ is contractible, and, by [BS, Proposition 1.2], the complex $(\mathbf{F}_{\mathcal{H}_L},\partial)$ is exact over S.

It follows from the description of the labeled complex $X = \mathcal{H}_L$ in the second-to-last paragraph that distinct faces F and F' of X have distinct labels $m_F \neq m_{F'}$. This shows that the resolution $(\mathbf{F}_{\mathcal{H}_L},\partial)$ is minimal (compare [BS, Remark 1.4]). ∎

Corollary 3.2 *The $\mathbb{Z}^{2n}$-graded Betti numbers of M_L are 0 or 1.* ∎

We retain the following identification for the rest of the paper:

$$X_{\leq(\mathbf{a},\mathbf{b})} \quad = \quad \{\, \mathbf{u} \in \mathbb{R}L \mid -\mathbf{b} \leq \mathbf{u} \leq \mathbf{a} \,\} \qquad \text{for} \quad \mathbf{a},\mathbf{b} \in \mathbb{Z}^n. \tag{3.1b}$$

In the next section we shall further identify these polytopes with the *fibers* of the Lawrence ideal J_L, i.e., with the congruence classes modulo J_L of monomials in S.

Note that the polytope $X_{\leq(\mathbf{a},\mathbf{b})}$ is itself a (closed) face of the arrangement $X = \mathcal{H}_L$ if and only if there exists a vector $\mathbf{u} \in \mathbb{R}L$ such that $(\mathbf{a},\mathbf{b}) = (\lceil \mathbf{u} \rceil, -\lfloor \mathbf{u} \rfloor)$. Suppose that this holds. Then each coordinate of $\mathbf{a}+\mathbf{b}$ is either 0 or 1, and, taking any vertex of $X_{\leq(\mathbf{a},\mathbf{b})}$, we get a lattice point $\mathbf{v} \in L$ with $-\mathbf{b} \leq \mathbf{v} \leq \mathbf{a}$. By translating our face with the lattice vector $\mathbf{v}$, we obtain now the following conclusion:

Proposition 3.3 *Modulo the action by the unimodular lattice L, each face of $\mathcal{H}_L$ has the form $X_{\leq(\mathbf{a},\mathbf{b})}$, where $\mathbf{a} \in \{0,1\}^n$ and $\mathbf{b} \in \{0,1\}^n$ have disjoint support.* ∎

We next consider the quotient complex $\mathcal{H}_L/L$, which is formally defined as the face poset of $\mathcal{H}_L$ modulo the action by the lattice L. Proposition 3.3 implies:

Corollary 3.4 *The number of faces of the quotient complex $\mathcal{H}_L/L$ is finite.* ∎

We apply the algebraic quotient construction in [BS, Section 3] to this quotient complex. The labeling of the $\mathbb{Z}^{2n}$-graded cell complex $X = \mathcal{H}_L$ is consistent with the action by the following lattice which is canonically isomorphic to L,

$$\Lambda(L) \quad := \quad \{\, (\mathbf{u},-\mathbf{u}) \in \mathbb{Z}^{2n} \mid \mathbf{u} \in L \,\}. \tag{3.2}$$

Following [BS, Lemma 3.5], the complex $(\mathbf{F}_{\mathcal{H}_L},\partial)$ has the structure of a complex of $\mathbb{Z}^{2n}$-graded free modules over the group algebra $S[L]$. The rank of $\mathbf{F}_{\mathcal{H}_L}$ over $S[L]$ equals the number of faces of $\mathcal{H}/L$, which is finite by Corollary 3.4. Now the functor in [BS, Theorem 3.2] defines an equivalence of categories between the category of $\mathbb{Z}^{2n}$-graded $S[L]$-modules and the category of $\mathbb{Z}^{2n}/\Lambda(L)$-graded S-modules. Applying this functor to the $S[L]$-complex $(\mathbf{F}_{\mathcal{H}_L},\partial)$ we obtain the cellular quotient complex $(\mathbf{F}_{\mathcal{H}_L/L},\partial)$ which is a complex of $\mathbb{Z}^{2n}/\Lambda(L)$-graded free S-modules. From Theorem 3.1 and [BS, Corollary 3.7] we conclude that $(\mathbf{F}_{\mathcal{H}_L/L},\partial)$ is the minimal free resolution of S/J_L over S. This concludes the proof of the following theorem.

Theorem 3.5 *The quotient complex $\mathcal{H}_L/L$ of the hyperplane arrangement modulo L supports the minimal S-free resolution of the unimodular Lawrence ideal J_L.* ∎

Corollary 3.6 *The minimal free resolution of the unimodular Lawrence ideal J_L is independent of the characteristic of the base field k. The number of minimal i-th syzygies of S/J_L equals the number of i-dimensional faces of the quotient complex $\mathcal{H}_L/L$, and the Betti numbers of S/J_L in the $\mathbb{Z}^{2n}/\Lambda(L)$-grading are all 0 or 1.* ∎

To write down the matrices in the minimal cellular resolution $(\mathbf{F}_{\mathcal{H}_L/L}, \partial)$ of a unimodular Lawrence ideal J_L, one must select a fundamental domain of $\mathcal{H}_L$ modulo L and identify the cover relation in the poset of faces of $\mathcal{H}_L/L$. In higher dimensions it is convenient to use the monomial ideals in Section 4 for that purpose, but in dimensions 2 and 3 we can do the identifications directly on the picture. For instance, if $L = \ker(1\ 1\ 1)$ and thus J_L is the ideal of 2×2-minors of a generic 2×3-matrix, then the resolution $(\mathbf{F}_{\mathcal{H}_L/L}, \partial)$ is supported by the hypersimplicial arrangement in $\mathbb{R}^2$ from Figure 1. Here the quotient complex $\mathcal{H}_L/L$ consists of one vertex, three edges and two triangles, which are glued to form a torus. Hence S/J_L has one generator, three first syzygies and two second syzygies:

$$0 \longrightarrow S^2 \xrightarrow{\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}} S^3 \xrightarrow{(\, x_2y_3 - x_3y_2 \ \ x_3y_1 - x_1y_3 \ \ x_1y_2 - x_2y_1 \,)} S \longrightarrow S/J_L \longrightarrow 0$$

We now show that the minimal cellular resolution of S/J_L equals the hull resolution introduced in [BS, Section 3]. For $\mathbf{a}$, $\mathbf{b} \in \mathbb{Z}^n$ and $t > 0$, we write

$$(t^{\mathbf{a}}, t^{\mathbf{b}}) \quad = \quad (t^{a_1}, \ldots, t^{a_n}, t^{b_1}, \ldots, t^{b_n}) \in \mathbb{R}^{2n}$$

Recall that $\mathrm{hull}(M_L)$ is the complex of bounded faces of the polyhedron P_t for large t, where P_t is the convex hull of $\{\ (t^{\mathbf{a}}, t^{\mathbf{b}}) \mid \mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}} \in M_L\ \} \subset \mathbb{R}^{2n}$. The vertices of P_t correspond to the minimal generators of M_L, and the *hull resolution* of S/J_L is the cellular resolution supported on $\mathrm{hull}(M_L)/L$. We shall use the following lemma.

Lemma 3.7 *Let $\{a, b\}$ and $\{c, d\}$ be pairs of integers so $a + b = c + d$, and let $t > 0$, $t \neq 1$. If $|a - b| = |c - d|$, then $t^a + t^b = t^c + t^d$. If $|a - b| > |c - d|$, then $t^a + t^b > t^c + t^d$.*

Proof. If $|a - b| = |c - d|$, then $\{a, b\} = \{c, d\}$ as multisets. If $|a - b| > |c - d|$, suppose to be definite that $a > c \geq d > b$. In view of $a - c = d - b$, we have $t^a - t^c - t^d + t^b \ = \ t^c(t^{a-c} - 1) - t^b(t^{d-b} - 1) \ = \ t^b(t^{c-b} - 1)(t^{a-c} - 1) \ > \ 0.$ ∎

Theorem 3.8 *The hull resolution of S/J_L agrees with the minimal free resolution.*

Proof. We show that $\mathrm{hull}(M_L)$ and $\mathcal{H}_L$ agree as cell complexes, using the fact that $\mathrm{hull}(M_L)$ consists of those faces of P_t supported by strictly positive inner normals.

Let F be a face of $\mathcal{H}_L$. Then $F = X_{\leq (\lceil \mathbf{u} \rceil, -\lfloor \mathbf{u} \rfloor)}$ for some $\mathbf{u} \in \mathbb{R}L$. In other words, the vertices of F are precisely those elements $\mathbf{a} \in L$ so $\lfloor u_i \rfloor \leq a_i \leq \lceil u_i \rceil$ for each i.

Let $\mathbf{v} = (t^{-\lceil \mathbf{u} \rceil}, t^{\lfloor \mathbf{u} \rfloor})$. If $\mathbf{a}$ is a vertex of F, then $\mathbf{v} \cdot (t^{\mathbf{a}}, t^{-\mathbf{a}}) = n + m + (n-m)t^{-1}$, where m is the number of coordinates in which $\mathbf{u}$ is an integer. Suppose that $\mathbf{b} \in L$ is not a vertex of F, and let $t > 1$. Write $\mathbf{v}_i = (t^{-\lceil u_i \rceil}, t^{\lfloor u_i \rfloor})$, so $\mathbf{v} \cdot (t^{\mathbf{a}}, t^{-\mathbf{a}}) = \sum_{i=1}^{n} \mathbf{v}_i \cdot (t^{a_i}, t^{-a_i})$. For each i we have $\mathbf{v}_i \cdot (t^{b_i}, t^{-b_i}) \geq \mathbf{v}_i \cdot (t^{a_i}, t^{-a_i})$, with equality if and only if $\lfloor u_i \rfloor \leq b_i \leq \lceil u_i \rceil$. This follows by applying Lemma 3.7 to the pairs $\{b_i - \lceil u_i \rceil, \lfloor u_i \rfloor - b_i\}$ and $\{a_i - \lceil u_i \rceil, \lfloor u_i \rfloor - a_i\}$. Thus $\mathbf{v} \cdot (t^{\mathbf{b}}, t^{-\mathbf{b}}) > \mathbf{v} \cdot (t^{\mathbf{a}}, t^{-\mathbf{a}})$, so $\mathbf{v}$ supports F as a face of $\mathrm{hull}(M_L)$.

Let $\mathbf{a}$ and $\mathbf{b}$ be two vertices of $\mathcal{H}_L$ which do not belong to a common face of $\mathcal{H}_L$, and let $F = X_{\leq(\lceil \mathbf{u} \rceil, -\lfloor \mathbf{u} \rfloor)}$ be the face of $\mathcal{H}_L$ determined by $\mathbf{u} = (\mathbf{a}+\mathbf{b})/2$. Then $|a_i - b_i| \geq 2$ for some i, for otherwise $\mathbf{a}$ and $\mathbf{b}$ would both belong to F by Proposition 3.3. Let $\mathbf{c}$ be any vertex of F, and let $\mathbf{d} = \mathbf{a}+\mathbf{b}-\mathbf{c}$. Note that for each j, $\{c_j, d_j\} = \{\lfloor u_j \rfloor, \lceil u_j \rceil\}$ as multisets, so $|c_j - d_j| \leq 1$. Then $\mathbf{a}+\mathbf{b} = \mathbf{c}+\mathbf{d}$, and $|a_j - b_j| \geq |c_j - d_j|$ for each j, with strict inequality for $j = i$. Let $t > 1$. Applying Lemma 3.7 to the pairs $\{a_j, b_j\}$ and $\{c_j, d_j\}$,

$$t^{a_j} + t^{b_j} \geq t^{c_j} + t^{d_j} \qquad \text{and} \qquad t^{-a_j} + t^{-b_j} \geq t^{-c_j} + t^{-d_j}$$

for each j, with strict inequalities for $j = i$. Let $\mathbf{p}$ be the midpoint of the line segment from $(t^{\mathbf{a}}, t^{-\mathbf{a}})$ to $(t^{\mathbf{b}}, t^{-\mathbf{b}})$, and let $\mathbf{q}$ be the midpoint of the line segment from $(t^{\mathbf{c}}, t^{-\mathbf{c}})$ to $(t^{\mathbf{d}}, t^{-\mathbf{d}})$. We have shown that $\mathbf{p} - \mathbf{q}$ is a nonzero, nonnegative vector. Therefore, the point $\mathbf{p}$ cannot lie on any face of $\mathrm{hull}(M_L)$, because $\mathbf{v} \cdot \mathbf{p} > \mathbf{v} \cdot \mathbf{q}$ for any strictly positive vector $\mathbf{v}$. Thus $\mathbf{a}$ and $\mathbf{b}$ cannot belong to a common face of $\mathrm{hull}(M_L)$. We conclude that the cell complexes $\mathrm{hull}(M_L)$ and $\mathcal{H}_L$ are equal. ∎

4 Fiber monomial ideals and initial monomial ideals

With any lattice ideal in a polynomial ring one can associate two families of monomial ideals. First, there are the *initial monomial ideals*, with respect to various term orders. Their minimal free resolutions can always be lifted, by Schreyer's construction in Gröbner basis theory, to a (possibly nonminimal) resolution of the lattice ideal. Second, we have the *fiber monomial ideals*, which are generated by the fibers, that is, the equivalence classes of monomials modulo the lattice ideal [PS, Section 2]. It follows from the results in [BS, Section 3] that the resolution of the lattice ideal is always determined by the resolution of a large enough fiber ideal.

In this section we make these results precise for the unimodular Lawrence case. Fix a unimodular sublattice $L \subset \mathbb{Z}^n$. Two monomials m and m' in $S = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ are considered *equivalent* if $m - m'$ lies in the unimodular Lawrence ideal J_L. The equivalence classes are finite and called the *fibers* of J_L. For given $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$, let $\mathrm{fib}(\mathbf{a}, \mathbf{b})$ denote the fiber of the monomial $\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}$. We shall identify the fibers with the lattice points in the polytopes of the form (3.1b):

Lemma 4.1 *Let $X = \mathcal{H}_L$ be the labeled cell complex introduced in Section 3. Then the map* $\phi : \mathrm{fib}(\mathbf{a}, \mathbf{b}) \to X_{\leq(\mathbf{a},\mathbf{b})} \cap L$, $\mathbf{x}^{\mathbf{c}}\mathbf{y}^{\mathbf{d}} \mapsto \mathbf{a} - \mathbf{c}$ *is a bijection.*

Proof. A monomial $\mathbf{x^c y^d}$ in S lies in $\mathrm{fib}(\mathbf{a},\mathbf{b})$ if and only if the vector $(\mathbf{a}-\mathbf{c},\mathbf{b}-\mathbf{d}) \in \mathbb{Z}^{2n}$ lies in the lattice $\Lambda(L)$ defined in (3.2). The latter condition means that $\mathbf{u}=\mathbf{a}-\mathbf{c}$ lies in L and $\mathbf{b}+\mathbf{u}=\mathbf{d}$. Thus $\mathbf{u}$ is a vertex of $X=\mathcal{H}_L$ and its label

$$\mathbf{x^u y^{-u}} \quad = \quad \mathbf{x^{a-c} y^{b-d}} \quad \text{divides} \quad \mathbf{x^a y^b},$$

which shows that $\mathbf{u}$ is actually a vertex of $X_{\leq(\mathbf{a},\mathbf{b})}$. Hence the map ϕ is well-defined. To see that it is a bijection we note that the inverse map is given by $\phi^{-1}(\mathbf{u}) = \mathbf{x^{a-u} y^{b+u}}$. This map is an analogue of the bijection in [PS, (2.1)]. ∎

Let $\langle \mathrm{fib}(\mathbf{a},\mathbf{b})\rangle$ denote the ideal in S generated by all monomials in the fiber of $\mathbf{x^a y^b}$. The vertices of the cell complex $X_{\leq(\mathbf{a},\mathbf{b})}$ (considered as a subcomplex of $\mathcal{H}_L$) are labeled with certain Laurent monomials of the form $\mathbf{x^u y^{-u}}$, with $\mathbf{u} \in L$. Consider their preimages under the bijection ϕ, and let $Y_{(\mathbf{a},\mathbf{b})}$ denote the same cell complex as $X_{\leq(\mathbf{a},\mathbf{b})}$ but with the vertices labeled by the monomials in $\mathrm{fib}(\mathbf{a},\mathbf{b})$. Thus the vertex with label $\mathbf{x^u y^{-u}}$ in $X_{\leq(\mathbf{a},\mathbf{b})}$ becomes the vertex with label $\mathbf{x^{a-u} y^{b+u}}$ in $Y_{(\mathbf{a},\mathbf{b})}$. The labeled cell complex $Y_{(\mathbf{a},\mathbf{b})}$ gives rise to a $\mathbb{Z}^{2n}$-graded complex of free S-modules as in [BS, Section 1]. This complex is always exact and minimal:

Theorem 4.2 *Let L be a unimodular sublattice of $\mathbb{Z}^n$ and $\mathbf{a},\mathbf{b} \in \mathbb{N}^n$. The labeled cell complex $Y_{(\mathbf{a},\mathbf{b})}$ defines a minimal free resolution of the monomial ideal* $\langle \mathrm{fib}(\mathbf{a},\mathbf{b})\rangle$.

Proof. For any $\mathbf{c},\mathbf{d} \in \mathbb{N}^n$ we can make the following identification of cell complexes

$$(Y_{(\mathbf{a},\mathbf{b})})_{\leq(\mathbf{c},\mathbf{d})} \quad = \quad X_{\leq(\min\{\mathbf{b},\mathbf{c}-\mathbf{a}\},\, \min\{\mathbf{a},\mathbf{d}-\mathbf{b}\})}, \tag{4.1}$$

where "min" refers to the coordinatewise minimum, and the labels are shifted appropriately. To see that (4.1) holds, one identifies both sides with the convex polytope

$$\{\ \mathbf{u} \in \mathbb{R}L \ \mid\ (\mathbf{0},\mathbf{0}) \leq (\mathbf{a}+\mathbf{u},\mathbf{b}-\mathbf{u}) \leq (\mathbf{c},\mathbf{d})\ \},$$

together with its natural subdivision. We conclude that the relevant subcomplexes for the exactness criterion in [BS, Proposition 1.2] are all contractible. As above, in the proof of Theorem 3.1, distinct faces of $Y_{(\mathbf{a},\mathbf{b})}$ have distinct labels. Therefore $Y_{(\mathbf{a},\mathbf{b})}$ supports a minimal cellular resolution of the fiber ideal $\langle \mathrm{fib}(\mathbf{a},\mathbf{b})\rangle$. ∎

Example 4.3 Let L be the corank 1 lattice of all vectors in $\mathbb{Z}^n$ with coordinate sum zero. As discussed at the end of Section 2, $X=\mathcal{H}_L$ is the hypersimplicial arrangement in $\mathbb{R}^{n-1}$, while J_L is the ideal of 2×2-minors of the matrix (1.1). Let M be the ideal generated by the monomials $x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n} y_1^{v_1} y_2^{v_2} \cdots y_n^{v_n}$ where

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{pmatrix} \tag{4.2}$$

runs over all nonnegative integer matrices having the same row sums and the same column sums. Then the minimal free resolution of M is cellular and supported by a suitably relabeled subcomplex $Y_{(\mathbf{a},\mathbf{b})}$ of the hypersimplicial arrangement $X=\mathcal{H}_L$.

For instance, let $n = 3$ and consider the following monomial ideal

$$M \quad = \quad \langle\, x_2^2x_3^2y_1^3,\; x_1x_2x_3^2y_1^2y_2,\; x_1^2x_3^2y_1y_2^2,\; x_1x_2^2x_3y_1^2y_3, \\ x_1^2x_2x_3y_1y_2y_3,\; x_1^3x_3y_2^2y_3,\; x_1^2x_2^2y_1y_3^2,\; x_1^3x_2y_2y_3^2 \,\rangle$$

Here the matrices (4.2) have row sums $4, 3$ and column sums $3, 2, 2$. The minimal free resolution of M is supported on the complex $Y_{(2,1,1,\,1,1,1)}$ which is a pentagon:

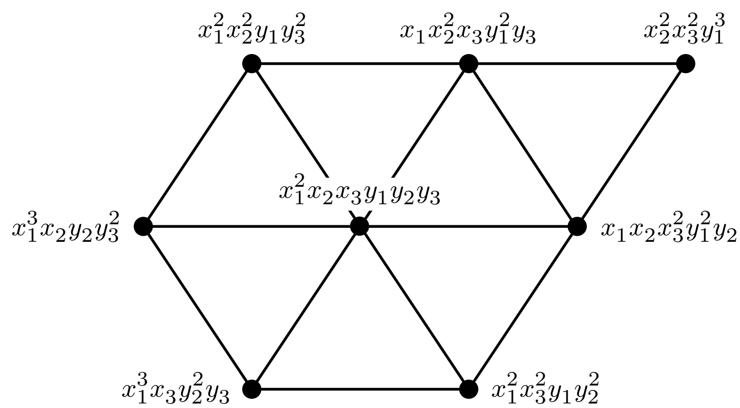

Figure 2: Minimal free resolution of a fiber in the hypersimplicial arrangement.

Let $\prec$ be any term order on the polynomial ring $S = k[x_1, \dots, x_n, y_1, \dots, y_n]$. Consider the initial monomial ideal $\text{in}_\prec(J_L)$ of the unimodular Lawrence ideal J_L. We know from Theorem 1.2 that $\text{in}_\prec(J_L)$ is a squarefree monomial ideal. We shall describe a minimal cellular free resolution of $\text{in}_\prec(J_L)$ and show that it has the same Betti numbers as the resolution $(\mathbf{F}_{\mathcal{H}_L/L}, \partial)$ of the Lawrence ideal J_L.

Let $\mathcal{H}_L^0$ be the set of all faces of the infinite hyperplane arrangement $\mathcal{H}_L$ which contain the origin $\mathbf{0} \in L$. This is a finite cell complex which we identify with the central hyperplane arrangement in $\mathbb{R}L$ given by the n coordinate hyperplanes $x_i = 0$. The faces of $\mathcal{H}_L^0$ are cones in $\mathbb{R}L$ with their apex at the origin. Under the embedding of $\mathcal{H}_L$ in $\mathbb{R}^m$ given prior to Proposition 2.2, the complex $\mathcal{H}_L^0$ becomes the central arrangement defined by the hyperplanes $H_{10}, H_{20}, \dots, H_{n0}$ in $\mathbb{R}^m$. For instance, in the example of Figure 1, $\mathcal{H}_L^0$ consists of one 0-face, six 1-faces and six 2-faces.

The term order $\prec$ extends uniquely to a total order on all Laurent monomials $\mathbf{x}^\mathbf{u}\mathbf{y}^{-\mathbf{u}}$ and hence on all vertices $\mathbf{u}$ of $\mathcal{H}_L$. A positive-dimensional cone F in the central arrangement $\mathcal{H}_L^0$ is called *$\prec$-positive* if all nonzero vertices $\mathbf{u} \in L$ of the face F satisfy the inequality $1 \prec \mathbf{x}^\mathbf{u}\mathbf{y}^{-\mathbf{u}}$. We can represent the term order $\prec$ by a generic hyperplane $H_\prec$ not containing the origin in $\mathbb{R}L$, such that the $\prec$-positive cones of $\mathcal{H}_L^0$ are precisely those cones which have bounded, nonempty intersection with $H_\prec$. If F is an i-dimensional $\prec$-positive cone in $\mathcal{H}_L^0$ then $F \cap H_\prec$ is an $(i{-}1)$-dimensional convex polytope. We write $\text{in}_\prec(\mathcal{H}_L)$ for the $(m{-}1)$-dimensional labeled cell complex consisting of those (bounded) polytopes $F \cap H_\prec$. Here $F \cap H_\prec$ inherits the label m_F from the face F of $\mathcal{H}_L$. Note that m_F is a monomial in S since $\mathbf{0} \in F$.

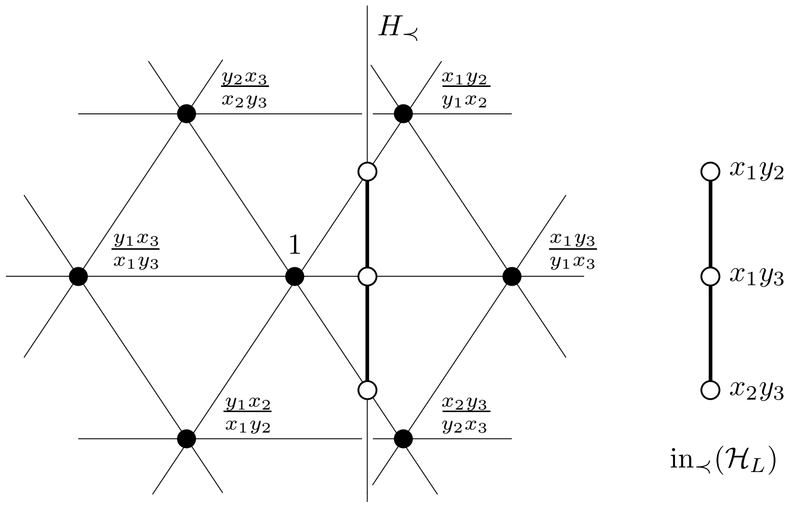

Figure 3: Initial monomial ideal of a unimodular Lawrence ideal.

Figure 3 shows such a hyperplane $H_\prec$ and the resulting complex $\text{in}_\prec(\mathcal{H}_L)$ arising from the hypersimplicial complex in Figure 1. The 1-dimensional complex $\text{in}_\prec(\mathcal{H}_L)$ represents the minimal S-free resolution of the monomial ideal $in_\prec(J_L)$:

$$0 \longrightarrow S^2 \xrightarrow{\begin{pmatrix} x_1 & 0 \\ -x_2 & -y_2 \\ 0 & y_3 \end{pmatrix}} S^3 \xrightarrow{(\,x_2y_3 \;\; x_1y_3 \;\; x_1y_2\,)} S \longrightarrow S/\text{in}_\prec(J_L) \longrightarrow 0$$

Theorem 4.4 *The minimal free resolution of the initial monomial ideal* $\text{in}_\prec(J_L)$ *of the unimodular Lawrence ideal* J_L *is given by the labeled cell complex* $\text{in}_\prec(\mathcal{H}_L)$.

Proof. The $\prec$-positive 1-dimensional faces C of $\mathcal{H}_L^0$ are the circuits of L, and their labels m_C are the minimal generators $\text{in}_\prec(J_L)$. Any higher-dimensional $\prec$-positive face F of $\mathcal{H}_L^0$ corresponds to a region in the central hyperplane arrangement in $\mathbb{R}L$ given by the coordinates. Its label m_F is a squarefree monomial by Proposition 3.3. We can identify m_F with the signed vector in the oriented matroid which represents the region F. Now, every vector in an oriented matroid is a conformal union of the circuits [BLSWZ, Proposition 3.7.2]. This implies that m_F is the least common multiple of the labels m_C of all circuits C with $C \subseteq F$. In other words, m_F is the least common multiple of those vertices of $\text{in}_\prec(J_L)$ which lie on $F \cap H_\prec$.

The labeled cell complex $\text{in}_\prec(\mathcal{H}_L)$ satisfies the exactness criterion in [BS, Proposition 1.2] because the subcomplex of faces whose label divides $\mathbf{x}^\mathbf{a}\mathbf{y}^\mathbf{b}$ is contractible. This follows from a result of Björner and Ziegler [BLSWZ, Theorem 4.5.7]. Minimality is inherited from $\mathcal{H}_L$ since different faces in $\mathcal{H}_L$ have different labels. ∎

Theorem 4.4 tells us that the minimal cellular resolution $(\mathbf{F}_{\mathcal{H}_L/L}, \partial)$ of a unimodular Lawrence ideal J_L is a *universal resolution* in the sense that it is stable with respect to any term order $\prec$. This generalizes the fact that the minimal generators of J_L are a *universal Gröbner basis* (Proposition 1.1). In other words, the universal Gröbner basis property extends from the generators of J_L to all the higher syzygies.

5 Lawrence ideals arising from graphs

Unimodular lattices arise naturally in the study of directed graphs and matroids. See Chapters 1 and 6 in [Whi1] for a first introduction. Detailed information on unimodularity appears in [Whi2, Chapter 3]. For instance, a famous theorem of Seymour [Whi2, Theorem 3.1.1(9)] states that every unimodular lattice can be built up in a simple way (by duality, 1-sums, 2-sums and 3-sums) from graphic lattices and a certain five-dimensional lattice in $\mathbb{Z}^{10}$. In this section we discuss the minimal free resolutions of cographic and graphic lattice ideals in combinatorial terms.

Let $G = (V, E)$ be a finite directed graph on d vertices, which we assume are labeled with numbers $1, 2, \ldots, d$. Suppose that E has n edges. The edge-node incidence matrix of G is the $n \times d$-matrix whose rows are $\mathbf{e}_i - \mathbf{e}_j \in \mathbb{Z}^d$ for $(i,j) \in E$. The image of this matrix in $\mathbb{Z}^n$ is denoted by L_G, and is called the *graphic lattice* of G. The orthogonal complement of L_G in $\mathbb{Z}^n$ is denoted by L_G^*, and is called the *cographic lattice* of G. In the language of matroid theory, the graphic lattice L_G is spanned by the *cocircuits* of G, and the cographic lattice L_G^* is spanned by the *circuits* of G. The following classical result appears in [Whi2, Theorem 1.5.3].

Proposition 5.1 *The graphic and cographic lattices L_G and L_G^* are unimodular.*

We write $J_G = J_{L_G}$ and $J_G^* = J_{L_G^*}$ for the two unimodular Lawrence ideals associated with a directed graph G. We call J_G the *graphic ideal* and J_G^* the *cographic ideal*. In fact, these ideals depend only on the undirected graph underlying G, and they can be described as follows. Replace each edge (i,j) of the directed graph $G = (V, E)$ by two directed edges, one from vertex i to vertex j and the other one from vertex j to vertex i. We associate the variables x_{ij} and x_{ji} with these directed edges. This gives a polynomial ring S with $2n$ variables over k. We interpret the variable x_{ji} as the homogenizing variable for x_{ij}. The pair of variables $\{x_{ij}, x_{ji}\}$ will play the same role as the pair $\{x_i, y_i\}$ in the previous sections.

We first discuss the graphic ideal of G. It can be computed as an ideal quotient:

$$J_G \quad = \quad \Big\langle \prod_{j:(i,j)\in E} x_{ij} \; - \prod_{j:(i,j)\in E} x_{ji} \;\; \Big| \;\; i = 1, 2, \ldots, d \Big\rangle : \langle \prod_{(r,s)\in E} x_{rs} x_{sr} \rangle^{\infty}.$$

The d binomials listed above correspond to a lattice basis of L_G. Hence J_G has codimension d. The minimal generators of J_G are the *cocircuits* of the graph G.

Example 5.2 Consider the complete graph on five nodes, $G = K_5$. The graphic ideal J_{K_5} is generated by five quartics such as $x_{12}x_{13}x_{14}x_{15} - x_{21}x_{31}x_{41}x_{51}$ and ten sextics such as $x_{13}x_{14}x_{15}x_{23}x_{24}x_{25} - x_{31}x_{32}x_{41}x_{42}x_{51}x_{52}$. They correspond to the

15 cocircuits of K_5. The minimal free resolution of J_{K_5} is given by a 4-dimensional simplicial complex with 24 facets, 60 tetrahedra, 50 triangles and 15 edges.

We generalize this example by describing the minimal free resolution of J_{K_d}, the graphic ideal of the complete graph on d nodes. Our construction is related to Lie algebra cohomology; a geometric version was found independently by Björner and Wachs [BW]. Let S be the polynomial ring over k in the $d(d-1)$ variables x_{ij}. Let F_{r-1} denote the free S-module whose basis elements correspond to ordered partitions $(A_1 \,|\, A_2 \,|\, \ldots \,|\, A_r)$ of the set $\{1, \ldots, d\}$, such that $1 \in A_1$, for all $1 \leq r \leq d-1$.

Theorem 5.3 *The minimal resolution of the graphic ideal J_{K_d} is the exact complex*

$$\mathbf{F}_\bullet : \quad 0 \to F_{d-2} \xrightarrow{\partial_{d-2}} F_{d-3} \xrightarrow{\partial_{d-3}} F_{d-4} \longrightarrow \ldots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to 0, \quad (5.1)$$

where the differential ∂_{r-1} acts on the basis elements of F_{r-1} by the "cyclic rule"

$$\partial_{r-1}(A_1 \mid A_2 \mid \cdots \mid A_r) \quad = \quad (-1)^{r+1} \prod_{i \in A_r} \prod_{j \in A_1} x_{ij} \cdot (A_r \cup A_1 \mid A_2 \mid \cdots \mid A_{r-1})$$

$$+ \sum_{s=2}^{r} (-1)^s \prod_{i \in A_{s-1}} \prod_{j \in A_s} x_{ij} \cdot (A_1 \mid A_2 \mid \cdots \mid A_{s-2} \mid A_{s-1} \cup A_s \mid A_{s+1} \mid \cdots \mid A_r).$$

Proof. The central hyperplane arrangement $\mathcal{H}^0_{L_{K_d}}$ can be identified with the familiar *braid arrangement* which consists of the hyperplanes $z_i = z_j$ in the $(d-1)$-dimensional vector space $\{ \, (z_1, z_2 \ldots, z_d) \in \mathbb{R}^d \mid z_1 + \cdots + z_d = 0 \, \}$. The $(r-2)$-faces of $\mathcal{H}^0_{L_{K_d}}$ are naturally labeled by the ordered partition $(A_1 \mid A_2 \mid \ldots \mid A_r)$ of $\{1, 2, \ldots, d\}$. More precisely, the unique face having a given point in its relative interior is determined by sorting the coordinates of that point.

We shall apply the initial ideal construction of Theorem 4.4. Let $H_\prec$ denote the hyperplane $\{z_1 = 1\}$ which represents a lexicographic order with the variables $x_{12}, \ldots, x_{1d}$ being highest. The faces of $\mathcal{H}^0_{L_{K_d}}$ which have bounded intersection with $H_\prec$ are indexed by those ordered partitions $(A_1 \mid A_2 \mid \ldots \mid A_r)$ which satisfy $1 \in A_1$. The simplicial complex $\mathrm{in}_\prec(\mathcal{H}_{L_{K_d}})$ is the first barycentric subdivision of the $(d-2)$-simplex, as shown for $d = 4$ in Figure 4. The vertices of $\mathrm{in}_\prec(\mathcal{H}_{L_{K_d}})$ are indexed by the monomials $\prod_{i \in A_1} \prod_{j \in A_2} x_{ij}$ which represent partitions (A_1, A_2) with $1 \in A_1$. Higher dimensional faces are labeled by ordered partitions with three or more parts. The cellular resolution given by the labeled simplicial complex $\mathrm{in}_\prec(\mathcal{H}_{L_{K_d}})$ has the format (5.1) with the differential mapping $(A_1 \mid A_2 \mid \cdots \mid A_r)$ to

$$\sum_{s=2}^{r} (-1)^s \prod_{i \in A_{s-1}} \prod_{j \in A_s} x_{ij} \cdot (A_1 \mid A_2 \mid \cdots \mid A_{s-2} \mid A_{s-1} \cup A_s \mid A_{s+1} \mid \cdots \mid A_r).$$

This complex is the minimal free resolution of the monomial ideal $\mathrm{in}_\prec(J_{K_d})$ as in Theorem 4.4. We lift this complex to the minimal free resolution of J_{K_d} by a construction as in [PS, Theorem 5.4]. Because $\mathcal{H}_{L_{K_d}}$ is simplicial, lifting amounts to adding one more term to each differential, and that term is exactly the remaining cyclic term, which is the first one listed in the statement of Theorem 5.3. ∎

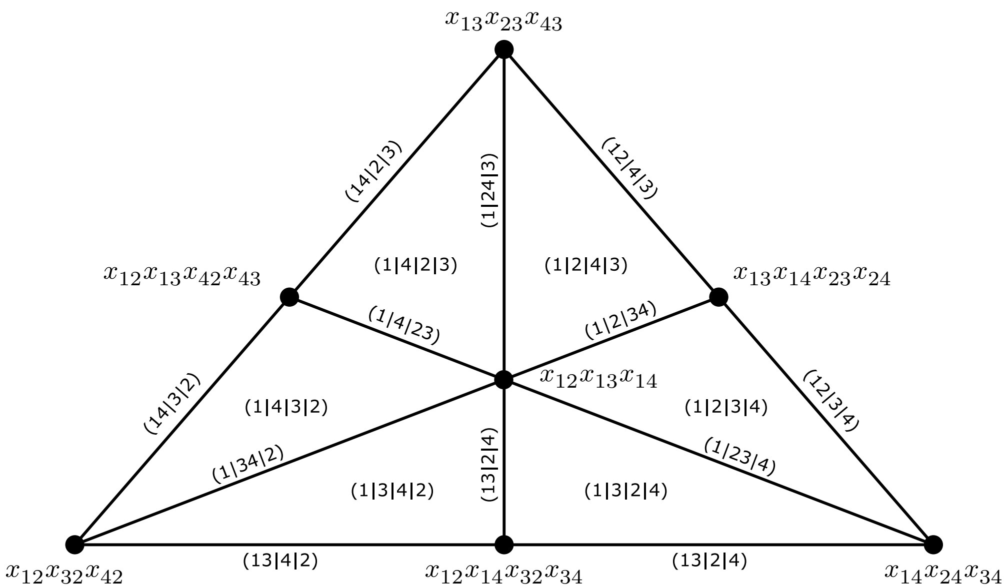

Figure 4: An initial ideal of the complete graphic Lawrence ideal.

We leave it to a future project to compute the Betti numbers of the graphic ideal J_G for arbitrary directed graphs G. We expect that there exist nice formulas in terms of the characteristic polynomials of graphic arrangements following [Za].

Another open combinatorial problem is to find a formula for the Betti numbers of the *cographic ideals* J_G^* which can be described as follows. Let $C = (C^+, C^-)$ be a signed circuit of the directed graph G; see [BLSWZ, §1.1]. This means that C^+ and C^- are disjoint subsets of the set of edges such that the edges in C^+ together with the reversals of the edges in C^- form a directed cycle which meets each vertex of G at most once. Every signed circuit $C = (C^+, C^-)$ is coded into a binomial:

$$\prod_{(i,j)\in C^+} \prod_{(k,l)\in C^-} x_{ij}x_{lk} \quad - \quad \prod_{(i,j)\in C^+} \prod_{(k,l)\in C^-} x_{ji}x_{kl}.$$

The cographic ideal J_G^* is minimally generated by these binomials where $C = (C^+, C^-)$ runs over all signed circuits of the directed graph G. For instance, if $G = K_4$ is the complete graph on $\{1,2,3,4\}$ with edges (i,j) for $i < j$, then there are seven circuits [BLSWZ, bottom of page 3] and our cographic ideal equals

$$\begin{aligned} J_{K_4}^* \;=\; \big\langle\, & x_{12}x_{23}x_{31} - x_{21}x_{32}x_{13}\,,\; x_{12}x_{24}x_{41} - x_{21}x_{42}x_{14}\,,\; x_{13}x_{34}x_{41} - x_{31}x_{43}x_{14}, \\ & x_{23}x_{34}x_{42} - x_{32}x_{43}x_{24}\,, \quad x_{12}x_{23}x_{34}x_{41} - x_{21}x_{32}x_{43}x_{14}\,, \\ & x_{13}x_{32}x_{24}x_{41} - x_{31}x_{23}x_{42}x_{14}\,, \quad x_{12}x_{24}x_{43}x_{31} - x_{21}x_{42}x_{34}x_{13} \,\big\rangle \end{aligned}$$

Since the matroid of K_4 is self-dual, that is, K_4 is self-dual under planar duality of graphs, it follows that the cographic ideal $J_{K_4}^*$ is isomorphic to the graphic ideal J_{K_4} whose minimal resolution is depicted in Figure 4. For $d \geq 5$, however, the cographic ideal $J_{K_d}^*$ is different from and more complicated than the graphic ideal J_{K_d}.

A special case of interest is the complete bipartite graph $K_{d,e}$. Its cographic ideal $J^*_{K_{d,e}}$ is the Lawrence liftings of the ideal of 2×2-minors of a generic matrix. While the minimal free resolution of the ideals of 2×2-minors depends on the characteristic of the base field, a result of Hashimoto [RW], their Lawrence liftings have minimal free resolutions which are cellular and characteristic free, by Corollary 3.6.

6 The diagonal embedding of a unimodular toric variety

We next present a different geometric interpretation of the Lawrence ideal J_L. It concerns the diagonal embeddings of a toric variety, written in the homogeneous coordinates of Audin-Cox [Au],[Cox]. This connection first appeared in [Oda]. As an application, Beilinson's spectral sequence is presented for unimodular toric varieties.

Let Σ be a complete fan in the lattice $\mathbb{Z}^m$ and X the associated complete normal toric variety; see e.g. [Fu]. Let $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{Z}^m$ be the primitive generators of the one-dimensional cones of Σ, and let B be the $n \times m$-matrix with row vectors $\mathbf{b}_i$. Each $\mathbf{b}_i$ determines a torus-invariant Weil divisor D_i on X, and the group $\mathrm{Cl}(X)$ of torus-invariant Weil divisors modulo linear equivalence has the presentation

$$0 \longrightarrow \mathbb{Z}^m \xrightarrow{B} \mathbb{Z}^n \xrightarrow{\pi} \mathrm{Cl}(X) \longrightarrow 0,$$

where π takes the i-th standard basis vector of $\mathbb{Z}^n$ to the linear equivalence class $[D_i]$ of the corresponding divisor D_i. If the divisor class group is torsion free, so that $\mathrm{Cl}(X) = \mathbb{Z}^{n-m}$, then we express π by a matrix A, and we are back in (2.1).

Audin [Au] and Cox [Cox] define the *homogeneous coordinate ring* of X to be the polynomial ring $R = k[x_1, \ldots, x_n]$, equipped with a grading by the abelian group $\mathrm{Cl}(X)$ via the morphism π above. Hence a monomial $\mathbf{x}^{\mathbf{a}} \in R$ has degree $[\sum_{i=1}^n a_i D_i] \in \mathrm{Cl}(X)$. The fan Σ is encoded in the *irrelevant ideal* $B_\Sigma \subset R$ whose monomial generators correspond to complements of facets of Σ. Coherent sheaves on the toric variety X are represented by B_Σ-torsion-free $\mathrm{Cl}(X)$-graded modules over R; see [Cox] for the case where Σ is simplicial, and [Mu] for the general case. Closed subschemes of X are defined by B_Σ-saturated $\mathrm{Cl}(X)$-graded ideals of R. Furthermore, for any $\mathrm{Cl}(X)$-graded R-module T, and any $\mathbf{a} \in \mathrm{Cl}(X)$, we define its *twist* $T(\mathbf{a})$ to be the graded R-module with components $T(\mathbf{a})_{\mathbf{b}} := T_{\mathbf{a}+\mathbf{b}}$. Let $\mathcal{O}_X(\mathbf{a})$ denote the coherent sheaf on X corresponding to the twisted R-module $R(\mathbf{a})$.

The toric variety $X \times X$ has the homogeneous coordinate ring

$$S \quad = \quad R \otimes_k R \quad = \quad k[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

The diagonal embedding $X \subset X \times X$ defines a closed subscheme, and is represented by a $\mathrm{Cl}(X) \times \mathrm{Cl}(X)$-graded ideal I_X in S. That ideal I_X is the kernel of the natural map $S = R \otimes R \to k[\mathrm{Cl}(X)] \otimes R$, $\mathbf{x}^{\mathbf{u}}\mathbf{y}^{\mathbf{v}} = \mathbf{x}^{\mathbf{u}} \otimes \mathbf{x}^{\mathbf{v}} \mapsto [\mathbf{u}] \otimes \mathbf{x}^{\mathbf{u}+\mathbf{v}}$. Explicitly,

$$I_X \quad = \quad \langle\, \mathbf{x}^{\mathbf{u}}\mathbf{y}^{\mathbf{v}} - \mathbf{x}^{\mathbf{v}}\mathbf{y}^{\mathbf{u}} \mid \pi(\mathbf{u}) = \pi(\mathbf{v}) \text{ in } \mathrm{Cl}(X)\, \rangle \quad \subset \quad S.$$

This formula proves the following observation.

Proposition 6.1 *The ideal $I_X \subset S$ defining the diagonal embedding $X \subset X \times X$ equals the Lawrence ideal J_L for the lattice $L = \mathrm{im}(B) = \ker(\pi)$ of principal divisors.*

The basic example is projective space $X = \mathbb{P}^m$. Here $n = m+1$, $L = \ker(\,1\ 1\ 1\ \cdots\ 1\,)$, $\mathrm{Cl}(X) = \mathbb{Z}^1$, and $I_X = J_L$ is the ideal (1.1) of 2×2-minors of a $2 \times n$-matrix of indeterminates. The minimal free resolution of $I_X = J_L$ is an Eagon-Northcott complex. Example 10 in [Oda] demonstrates how Beilinson's spectral sequence for $\mathbb{P}^m$ can be derived from this observation.

The results in this paper provide a cellular model for this Eagon-Northcott complex as a hypersimplicial complex (Figure 1). We will show that this derivation of Beilinson's spectral sequence extends to the general setting of Theorem 3.5. Thus our cellular resolutions provide a partial solution to [Oda, Problem 6].

We say that a toric variety X is *unimodular* if its lattice L of principal divisors is a unimodular sublattice of $\mathbb{Z}^n$. This condition is equivalent, by [Stu, Remark 8.10], to saying that the toric variety X is smooth and any other variety obtained by toric flips and flops is smooth as well. Toric varieties with this property appear frequently in representation theory and in integer programming. For example, the toric varieties associated with *transportation polytopes* and *products of minors* are unimodular. These toric varieties were studied recently by Babson and Billera [BB].

Suppose now that X is a unimodular toric variety. Then Theorem 3.5 constructs an m-dimensional cellular model for the $\mathrm{Cl}(X) \times \mathrm{Cl}(X)$-graded minimal resolution for the ideal $I_X \subset S$ of the diagonal embedding of X in $X \times X$:

$$L_\bullet : \quad 0 \longrightarrow L_m \longrightarrow \ldots \longrightarrow L_2 \longrightarrow L_1 \longrightarrow L_0 = \mathcal{O}_{X\times X} \longrightarrow \mathcal{O}_{X\times X}/\mathcal{I}_X \longrightarrow 0.$$

Here we identify I_X with the associated coherent sheaf $\mathcal{I}_X \subset \mathcal{O}_{X\times X}$, which is the ideal sheaf of the diagonal embedding of X. Note that S/I_X is Cohen-Macaulay.

Since $\mathcal{I}_X$ is a $\mathrm{Cl}(X) \times \mathrm{Cl}(X)$-homogeneous ideal, we can write $L_i = \oplus_{j=1}^{m_i} \mathcal{O}(-\mathbf{a}_{ij}, -\mathbf{b}_{ij})$ with $\mathbf{a}_{ij}, \mathbf{b}_{ij} \in \mathrm{Cl}(X)$. Here $\mathcal{O}(-\mathbf{a}_{ij}, -\mathbf{b}_{ij}) = \mathcal{O}(-\mathbf{a}_{ij}) \boxtimes \mathcal{O}(-\mathbf{b}_{ij}) = p_1^*(\mathcal{O}(-\mathbf{a}_{ij})) \otimes p_2^*(\mathcal{O}(-\mathbf{b}_{ij}))$ denotes the exterior tensor product on $X \times X$ of $\mathcal{O}(-\mathbf{a}_{ij})$ and $\mathcal{O}(-\mathbf{b}_{ij})$, and p_i are the projections on the factors of $X \times X$.

Let $\mathcal{F}$ be any coherent sheaf on X. By tensoring the above free resolution $L_\bullet$ with $p_1^*(\mathcal{F})$ we obtain a resolution $C_\bullet$ for the restriction of $p_1^*(\mathcal{F})$ to the diagonal $X \subset X \times X$. As in the case of projective space [Bei], pushing via p_{2*} a Cartan-Eilenberg injective resolution of $C_\bullet$ yields a double complex of $\mathcal{O}_X$-modules, whose total chain complex has as cohomology the hyperdirect image $\mathbf{R}p_{2*}(C_\bullet)$. One of the spectral sequences belonging to the double complex $\mathbf{R}p_{2*}(C_\bullet)$ yields the following version of the *Beilinson Spectral Sequence*. It says that any coherent sheaf $\mathcal{F}$ can be reconstructed from the knowledge of the cohomology of a few of its $\mathrm{Cl}(X)$-twists.

Theorem 6.2 *Let $\mathcal{F}$ be a coherent sheaf on a complete unimodular toric variety X. Then there is a (third quadrant) spectral sequence with E_1 term*

$$E_1^{pq} \quad = \quad \bigoplus_j H^q(\mathcal{F} \otimes \mathcal{O}_X(-\mathbf{a}_{pj})) \otimes \mathcal{O}_X(-\mathbf{b}_{pj})$$

which converges to the associated graded sheaf of a filtration of $\mathcal{F}$. ∎

In the case of $\mathbb{P}^m$, Beilinson's spectral sequence induces an equivalence between the derived category $\mathbf{D}^b(\mathrm{Coh}(\mathbb{P}^m))$ of bounded complexes of sheaves with coherent cohomology, and the derived category of modules over the exterior algebra in $m+1$ generators. An analogous simple description of $\mathbf{D}^b(\mathrm{Coh}(X))$ does not hold over an arbitrary unimodular toric variety X, since the bundle $\mathcal{E} = \oplus_{i,j}\mathcal{O}(-\mathbf{a}_{ij})$ defined as the direct sum (without repetitions) of all "left" summands in the above resolution of the ideal sheaf of the diagonal is in general not exceptional, i.e. $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}) \neq 0$ for some $i > 0$. See [AH] for a related approach to describing $\mathbf{D}^b(\mathrm{Coh}(X))$.

References

[AH] K. Altmann, L. Hille: Strong exceptional sequences provided by quivers. *Algebr. Represent. Theory* **2** (1999) 1-17.

[Au] M. Audin: *The Topology of Torus Actions on Symplectic Manifolds*, Progress in Math. **93**, Birkhäuser, 1991.

[BB] E. Babson, L.J. Billera: The geometry of products of minors, *Discrete Comput. Geometry* **20** (1998) 231–249

[BPS] D. Bayer, I. Peeva, B. Sturmfels: Monomial resolutions, *Math. Res. Lett.* **5** (1998), no. 1-2, 31–46.

[BS] D. Bayer, B. Sturmfels: Cellular resolutions of monomial modules, *J. Reine Angew. Math.* **502** (1998), 123–140.

[Bei] A. Beilinson: Coherent sheaves on $\mathbb{P}^N$ and problems of linear algebra, *Funct. Anal. Appl.*, **12**, (1978), 214–216.

[BLSWZ] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler: *Oriented Matroids*, Cambridge University Press, 1993.

[BW] A. Björner, M. Wachs: Geometrically constructed bases for intersection lattices of real arrangements, preprint 1999.

[Cox] D.A. Cox: The homogeneous coordinate ring of a toric variety, *J. Algebraic Geom.* **4** (1995), no. 1, 17–50.

[Fu] W. Fulton: *Introduction to Toric Varieties*, Princeton Univ. Press, 1993.

[Mu] M. Mustaţă: Vanishing theorems on toric varieties, preprint 1999.

[Oda] T. Oda: Problems on Minkowski sums of convex lattice polytopes, Manuscript, 1997.

[PS] I. Peeva, B. Sturmfels: Generic lattice ideals, *J. Amer. Math. Soc.*, **11**, (1998), no. 2, 363–373.

[RW] J. Roberts, J. Weyman: A short proof of a theorem of M. Hashimoto. *J. Algebra* **134** (1990), no. 1, 144–156.

[Stu] B. Sturmfels: *Gröbner Bases and Convex Polytopes*, AMS University Lecture Series, Vol. **8**, Providence RI, 1995.

[Whi1] N. White: *Theory of Matroids*, Encyclopedia of Mathematics and its Applications, **26**, Cambridge University Press, 1986.

[Whi2] N. White: *Combinatorial Geometries*, Encyclopedia of Mathematics and its Applications, **29**, Cambridge University Press, 1987.

[Za] T. Zaslavsky: Facing up to arrangements: face-count formulas for partitions of space by hyperplanes, *Mem. Amer. Math. Soc.* **1** (1975), no. 154.

[Zi] G. Ziegler: *Lectures on Polytopes*, Graduate Text in Mathematics **152**, Springer Verlag, 1997.

Dave Bayer, Department of Mathematics, Barnard College, Columbia University, New York, NY 10027, USA, `bayer@math.columbia.edu`.

Sorin Popescu, Department of Mathematics, Columbia University, New York, NY 10027, USA, and Department of Mathematics, SUNY at Stony Brook, Stony Brook, NY 11794, `psorin@math.columbia.edu`.

Bernd Sturmfels, Department of Mathematics, University of California, Berkeley, CA 94720, USA, `bernd@math.berkeley.edu`.