Gröbner Bases and Extension of Scalars

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Dedicated to Professor Heisuke Hironaka on his sixtieth birthday.

1 Introduction

Let A be a Noetherian commutative ring with identity, let $A[\mathbf{x}] = A[x_1, \ldots, x_n]$ be a polynomial ring over A, and let $I \subset A[\mathbf{x}]$ be an ideal. Geometrically, I defines a family of schemes over the base scheme $\operatorname{Spec} A$; the fiber over each point $p \in \operatorname{Spec} A$ is a subscheme of the affine space $\mathbf{A}^n_{k(p)} = \operatorname{Spec} k(p)[\mathbf{x}]$, where $k(p) = A_p/p_p$ is the residue field of p.

Let $>$ be a total order on the monomials of $A[\mathbf{x}]$ satisfying $\mathbf{x}^E > \mathbf{x}^F \Rightarrow \mathbf{x}^G\mathbf{x}^E > \mathbf{x}^G\mathbf{x}^F$, and satisfying $x_i > 1$ for each i. For $f \in A[\mathbf{x}]$, define $\operatorname{in}(f)$ to be the initial (greatest) term $c\mathbf{x}^E$ of f with respect to the order $>$, where $c \in A$ is nonzero. For $I \subset A[\mathbf{x}]$, define the initial ideal $\operatorname{in}(I)$ to be the ideal $(\operatorname{in}(f) \mid f \in I)$ generated by all initial terms of elements of I. $\operatorname{in}(I) \subset A[\mathbf{x}]$ is generated by single terms of the form $c\mathbf{x}^E$; we call such an ideal a monomial ideal. $\{f_1, \ldots, f_r\} \subset I$ is a Gröbner basis for I if and only if $\{\operatorname{in}(f_1), \ldots, \operatorname{in}(f_r)\}$ generates $\operatorname{in}(I)$.

For an ideal I in a ring of formal power series $A[[\mathbf{x}]]$, Hironaka defined the corresponding notion (the standard basis of I) and proved a generalized Weierstrass division theorem.The relation between flatness and division was considered in [HLT 73] and later in [Gal 79] in order to obtain a presentation

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for the flattener of a germ of an analytical morphism. The algebraic situation is slightly different.

In this paper, we study the behavior of Gröbner bases with respect to an extension of scalars $A \to B$. When does a Gröbner basis for I map to a Gröbner basis for $IB[\mathbf{x}]$? It suffices to have $\text{in}(I)$ generate $\text{in}(IB[\mathbf{x}])$; we focus on this condition. Taking $B = k(p)$ for $p \in \operatorname{Spec} A$, we consider the relationship between a Gröbner basis for I, and the Gröbner bases of the fibers of the family defined by I. How much information about the fibers of this family can be inferred from knowledge of $\text{in}(I)$ alone?

Let $X \subset \operatorname{Spec} A$ be the support of the family defined by I. A Gröbner basis for I encodes considerable information about this family, even when X is nonreduced or reducible. To interpret $\text{in}(I)$ in such situations, we work with its *coefficient ideals*: The coefficient ideal for the monomial $\mathbf{x}^E$ vanishes on the support of those fibers where $\mathbf{x}^E$ fails to belong to $\text{in}(I)k(p)[\mathbf{x}]$. From this point of view, a point $p \in \operatorname{Spec} A$ is "good" if each coefficient ideal of $\text{in}(I)$ defines a scheme which either avoids p, or contains an open neighborhood of p in X.

In §2, we study coefficient ideals of monomial ideals. In §3, we prove that an extension of scalars commutes with taking the initial ideal of any ideal I, if and only if the extension is flat. We then prove that a Gröbner basis for I determines Gröbner bases for the localizations to dense open subsets of each isolated component of X. We also prove that for this family, $\text{in}(I)$ determines the fiber initial ideals over "good" points, as defined above. These results reveal that $\text{in}(I)$ carries generic information for each isolated component of X. In §4, we prove that if every point of X is "good", then the family defined by I is faithfully flat over X. Faithful flatness imposes strong conditions on the component structure of the total space of our family, so this result has geometric applications, such as the removal of unwanted components. In §5, we give two other applications of coefficient ideals, describing the locus where a morphism of schemes is an isomorphism, or a finite map.

If A is a finitely generated k-algebra for a field k, then we can write $A = k[a_1, \ldots, a_m]/J$ for some ideal J. We can reformulate our problem as $A = k[\mathbf{a}]$, with $I \subset A[\mathbf{x}]$ and $I \cap A \supset J$. A Gröbner basis for I can then be computed by the usual algorithm over a field, by combining orders $>_1$, $>_2$ into a product order

$$\mathbf{a}^D \mathbf{x}^E > \mathbf{a}^F \mathbf{x}^G \;\Leftrightarrow \mathbf{x}^E >_1 \mathbf{x}^G, \text{ or } \; \mathbf{x}^E = \mathbf{x}^G \text{ and } \; \mathbf{a}^D >_2 \mathbf{a}^F.$$

In this setting, I defines a subscheme $Y \subset \mathbf{A}_k^{m+n}$ which projects to $X \subset \mathbf{A}_k^m$.

More generally, the computational relevance of this work depends on our ability to compute in the base ring A. Specifically, A needs to be a ring where linear equations are solvable; see Trager, Gianni, and Zacharias ([GTZ 88]) for background material and references on Gröbner bases in this setting. Our paper continues their study of families of Gröbner bases; we would like to thank each of them, David Eisenbud, and an anonymous referee, for many helpful conversations and suggestions.

2 Monomial Ideals

Let $J \subset A[\mathbf{x}]$ be a monomial ideal, i.e., an ideal generated by single terms of the form $c\mathbf{x}^E$, with $c \in A$.

When A is a field, J is easily understood: its structure is realized by the subset $L = \{E \mid \mathbf{x}^E \in J\}$ of $\mathbf{N}^n$, where $\mathbf{N}$ denotes the nonnegative integers. $\mathbf{N}^n$ admits a natural partial order $\leq$ defined by $E \leq F$ iff $\mathbf{x}^E$ divides $\mathbf{x}^F$. The characteristic function of L can be viewed as a poset homomorphism from $\mathbf{N}^n$, ordered by $\leq$, to the set of ideals $\{(0), (1)\}$ of A, ordered by inclusion.

To understand J when A is not a field, it is helpful to consider the *coefficient ideals* of J: Define $J_E = J_{\mathbf{x}^E} = (c \in A \mid c\mathbf{x}^E \in J)$. Alternatively, J_E is the ideal quotient $(J : \mathbf{x}^E) \cap A$. This construction defines a poset homomorphism $E \mapsto J_E$ from $\mathbf{N}^n$, ordered by $\leq$, to the set of ideals of A, ordered by inclusion. Conversely, any such poset homomorphism determines a monomial ideal J, so we can think of this construction as describing an equivalence of categories.

Viewing a monomial ideal as its collection of coefficient ideals is thus a purely tautological change of perspective; any operation on A can be viewed as acting on J via the inclusion $J \subset A[\mathbf{x}]$, or equivalently as acting on the set of coefficient ideals of J. In particular, if $v : A \to B$ is a ring homomorphism, then v extends naturally to a homomorphism $v : A[\mathbf{x}] \to B[\mathbf{x}]$. The image under v of any ideal $I \subset A[\mathbf{x}]$ generates the extension ideal $I^e = IB[\mathbf{x}]$. For monomial ideals, we have the immediate proposition

Proposition 2.1 *Let $v : A \to B$ be a ring homomorphism. Let $JB[\mathbf{x}]$ denote the monomial ideal obtained by extension of scalars from A to B, and let $J_E B$ also be obtained by extension of scalars, for an exponent E. Then*

$$J_E B = (JB[\mathbf{x}])_E. \quad \blacksquare$$

In particular, if B is the residue field $k(p)$ of a prime ideal $p \subset A$, then Proposition 2.1 asserts that $\mathbf{x}^E \in Jk(p)[\mathbf{x}]$ if and only if the point p does not belong to the subscheme of $\operatorname{Spec} A$ defined by J_E. This subscheme is the support of the A-module $A\mathbf{x}^E \subset A[\mathbf{x}]/J$. The monomial $\mathbf{x}^E$ appears only with a coefficient of zero in $J(A/J_E)[\mathbf{x}]$; J_E is the intersection of all ideals $K \subset A$ with the property that $(J(A/K)[\mathbf{x}])_E = (0)$.

In other words, we can view J as a family of monomial ideals over $\operatorname{Spec} A$. The monomial ideals corresponding to each fiber of the family defined by J are defined over fields, and can be visualized combinatorially: Each monomial $\mathbf{x}^E$ either belongs or does not belong to a given fiber monomial ideal, which in turn is determined by this data. The coefficient ideal J_E defines the subscheme of $\operatorname{Spec} A$ over which $\mathbf{x}^E$ does not belong to the fiber monomial ideals. J is determined by these subschemes.

Example 2.2 Let $A = \mathbf{Z}$, let $A[\mathbf{x}] = A[x, y]$, and let $J = (9x,\ 2y,\ x^2,\ y^2)$. The coefficient ideals for 1, x, y, x^2, xy, and y^2 respectively are (0), (9), (2), (1), (1), and (1); this is summarized in the diagram

$$
\begin{array}{r|lll}
y & (1) & & \\
 & (2) & (1) & \\
 & (0) & (9) & (1) \\
\hline
 & & & x
\end{array}
$$

J specializes to $(x,\ y)$ in each fiber over the open subset of $\operatorname{Spec} \mathbf{Z}$ which is the complement of the points (2) and (3). In the fiber over (2), J specializes to $(x,\ y^2)$. Over the double point at (3), J specializes to $(x^2,\ y)$.

A is an integral domain. The union of the monomials with a nonzero coefficient ideal spans the monomial ideal $(x,\ y)$, which occurs generically. On the other hand, the union of the monomials with coefficient ideal (1) spans the monomial ideal $(x^2,\ xy,\ y^2)$. There is no specialization which produces this monomial ideal, but it is contained in every monomial ideal obtained by specialization.

Example 2.3 Modify the preceding example by taking $A = \mathbf{Z}/18\mathbf{Z}$. $\operatorname{Spec} A$ is no longer reduced or irreducible. The union of the monomials with nonzero coefficient ideals again spans the monomial ideal $(x,\ y)$. There is no specialization to a field which produces this monomial ideal; this can happen whenever A is not an integral domain.

3 Initial Ideals

Let $I \subset A[\mathbf{x}]$ be an ideal. For monomial ideals, Proposition 2.1 asserts that the formation of coefficient ideals commutes with extension of scalars. In contrast, the formation of the initial ideal of an arbitrary ideal I need not commute with extension of scalars: If $v : A \to B$ is a ring homomorphism, then it can happen that $\mathrm{in}(I)B[\mathbf{x}] \neq \mathrm{in}(IB[\mathbf{x}])$. This is because if v maps the leading coefficient of a polynomial $f \in I$ to zero, then the first surviving term of the image of f will contribute to $\mathrm{in}(IB[\mathbf{x}])$. When this happens, $\mathrm{in}(IB[\mathbf{x}])$ cannot be predicted from knowledge of $\mathrm{in}(I)$ alone.

Let $\{f_1, \dots f_r\}$ be a Gröbner basis for I. If $\mathrm{in}(I)B[\mathbf{x}] = \mathrm{in}(IB[\mathbf{x}])$, then the images $\{v(f_1), \dots v(f_r)\}$ form a Gröbner basis for I. We shall study the behavior of initial ideals with respect to various hypotheses on I and B; as a consequence we obtain sufficient conditions for the construction of a Gröbner basis for I to commute with the extension of scalars $v : A \to B$.

Example 3.1 Let $A = k[a]$ for a field k, let $A[\mathbf{x}] = A[x, y]$, and let $I = (ax - y) \subset A[\mathbf{x}]$. In this example, as in all subsequent examples involving the variables x and y, we use the lexicographic order extending $x > y$. Choose a prime $p \subset A$, and let $k(p)$ be the residue field A_p/p_p of p. When $p \neq (a)$, $\mathrm{in}(I)k(p)[\mathbf{x}] = \mathrm{in}(Ik(p)[\mathbf{x}]) = (x)$. However, when $p = (a)$, $\mathrm{in}(I)k(p)[\mathbf{x}] = (0)$, but $\mathrm{in}(Ik(p)[\mathbf{x}]) = (y)$. Nevertheless, the image of $\{ax - y\}$ is a Gröbner basis for $Ik(p)[\mathbf{x}]$.

I defines a faithfully flat family over $\operatorname{Spec} A$, because $A[\mathbf{x}]/I \simeq A[x]$. The total space is given by the surface $ax - y = 0$, and each fiber consists of a line through the origin in $\mathbf{A}^2_k$, with slope a. $\mathrm{in}(Ik(p)[\mathbf{x}])$ momentarily flips from (x) to (y) as this slope passes through zero.

Example 3.2 Let $A = k[a, b]$, let $A[\mathbf{x}] = A[x, y]$, let $I = (ax^2 + y,\, by^2 + y + 1)$, and let $B = A/(a, b)$. Then $\mathrm{in}(I)B[\mathbf{x}] = (0)$, but $\mathrm{in}(IB[\mathbf{x}]) = (1)$. $\{ax^2 + y,\, by^2 + y + 1\}$ is a Gröbner basis for I, but its image $\{y,\, y + 1\}$ in $B[\mathbf{x}]$ is not a Gröbner basis for $IB[\mathbf{x}]$.

Example 3.3 Let $A = k[a]$, let $A[\mathbf{x}] = A[x]$, let $I = (ax - 1)$, and let $p \subset A$ be a prime. When $p \neq (a)$, $\mathrm{in}(I)k(p)[\mathbf{x}] = \mathrm{in}(Ik(p)[\mathbf{x}]) = (x)$. However, when $p = (a)$, $\mathrm{in}(I)k(p)[\mathbf{x}] = (0)$, but $\mathrm{in}(Ik(p)[\mathbf{x}]) = (1)$.

I defines a flat family which is not faithfully flat: The fiber over $a = 0$ is empty, so $(a) \subset A$ extends to the unit ideal in $A[\mathbf{x}]/I$.

As suggested by these examples, we do have an inclusion in one direction:

Proposition 3.4 *For any ring homomorphism $v : A \to B$, and for any ideal $I \subset A[\mathbf{x}]$, we have*

$$\mathrm{in}(I)B[\mathbf{x}] \subset \mathrm{in}(IB[\mathbf{x}]).$$

Proof. It is enough to show that each generator of $\mathrm{in}(I)B[\mathbf{x}]$ also belongs to $\mathrm{in}(IB[\mathbf{x}])$. $\mathrm{in}(I)B[\mathbf{x}]$ is generated by $v(\mathrm{in}(f))$ for $f \in I$. For each $f \in I$, either $\mathrm{in}(f)$ maps to zero in $B[\mathbf{x}]$, or else $v(\mathrm{in}(f)) = \mathrm{in}(v(f)) \in \mathrm{in}(IB[\mathbf{x}])$. ∎

The following theorem asserts that taking initial ideals universally commutes with an extension of scalars if and only if the extension is flat. We apply the criterion for flatness given in [Mat 86], Thm. 7.6, which asserts that $v : A \to B$ is flat iff the syzygies in B of a set of elements from A can always be generated by syzygies from A:

Lemma 3.5 *Let $v : A \to B$ be a ring homomorphism. v is flat if and only if for each sequence $a_i \in A$ and $b_i \in B$ for $1 \leq i \leq r$ so*

$$\sum_i b_i v(a_i) = 0,$$

then for some s we can choose $c_{ij} \in A$ and $d_j \in B$ for $1 \leq j \leq s$ so

$$\sum_i c_{ij} a_i = 0 \text{ for each } j, \text{ and } b_i = \sum_j d_j v(c_{ij}) \text{ for each } i.$$

∎

Theorem 3.6 *Let $v : A \to B$ be a ring homomorphism. The following two conditions are equivalent:*

(a) for any number of variables $x_1, \ldots, x_n$, and for any ideal $I \subset A[\mathbf{x}]$, $\mathrm{in}(I)B[\mathbf{x}] = \mathrm{in}(IB[\mathbf{x}])$;

(b) B is a flat A-algebra.

Proof. First, suppose that (b) holds, and let $I \subset A[\mathbf{x}]$ be an ideal. We need to show that $\mathrm{in}(IB[\mathbf{x}]) \subset \mathrm{in}(I)B[\mathbf{x}]$. Given $c\mathbf{x}^E \in \mathrm{in}(IB[\mathbf{x}])$, consider expressions of the form $c\mathbf{x}^E = \mathrm{in}(\sum_i b_i v(f_i))$, where $b_i \in B[\mathbf{x}]$ and $f_i \in I$. By expanding out each b_i and absorbing variables into each f_i, we need only consider expressions for which each $b_i \in B$. Among all such expressions, choose one for which the greatest monomial appearing in any of the f_i is minimal. We claim that this greatest monomial is $\mathbf{x}^E$. Letting c_i be the coefficient of $\mathbf{x}^E$ in each f_i, we have $c = \sum_i b_i v(c_i)$, so $c\mathbf{x}^E \in \mathrm{in}(I)B[\mathbf{x}]$.

Suppose otherwise, that the greatest monomial of a minimal expression is $\mathbf{x}^D > \mathbf{x}^E$. Let a_i be the coefficient of $\mathbf{x}^D$ in each f_i. Then $\sum_i b_i v(a_i) = 0$. Choosing c_{ij}, d_j as in Lemma 3.5, define $g_j = \sum_i c_{ij} f_i \in I$ for $1 \leq j \leq s$. Then

$$\sum_j d_j v(g_j) = \sum_i b_i v(f_i),$$

and $\mathrm{in}(g_j) < \mathbf{x}^D$ for each j, contradicting the minimality of our expression. This proves (a).

Now, suppose instead that (b) does not hold. Choose a sequence $a_i \in A$ and $b_i \in B$ for $1 \leq i \leq r$ with r minimal, so $\sum_i b_i v(a_i) = 0$ but the b_i cannot be expressed as in Lemma 3.5. We construct an example in two variables $x > y$ for which v does not commute with taking initial ideals: Let

$$f_i = a_i x^r + x^{r-i} y^i \quad \text{for} \quad 1 \leq i \leq r,$$

and let $I \subset A[x, y]$ be defined by $I = (f_1, \ldots, f_r)$. Then

$$\sum_i b_i v(f_i) = b_1 x^{r-1} y + \ldots + b_r y^r.$$

Moreover, for $c_{ij} \in A$,

$$\sum_i c_{ij} a_i = 0 \;\Leftrightarrow\; c_{1j} x^{r-1} y + \ldots + c_{rj} y^r = \sum_i c_{ij} f_i \in I.$$

Because the b_i cannot be expressed as in Lemma 3.5, and because r was chosen to be minimal, already b_1 is not in the ideal generated by the images of all c_{1j} for c_{ij} as above. Because $x^r > x^{r-1}y > \ldots > y^r$, this ideal is the coefficient ideal of $\mathrm{in}(I)B[\mathbf{x}]$ with respect to the monomial $x^{r-1}y$, so

$$(\mathrm{in}(I)B[\mathbf{x}])_{x^{r-1}y} \neq \mathrm{in}(IB[\mathbf{x}])_{x^{r-1}y}.$$

This proves that (a) does not hold. ∎

The following corollary asserts that taking initial ideals commutes with taking rings of fractions, and is due to Gianni, Trager, and Zacharias ([GTZ 88], Prop. 3.4).

Corollary 3.7 *Let $I \subset A[\mathbf{x}]$ be an ideal, and let $B = S^{-1}A$ for a multiplicatively closed set $S \subset A$. Then*

$$\mathrm{in}(I)B[\mathbf{x}] = \mathrm{in}(IB[\mathbf{x}]).$$

Proof. A ring of fractions is a flat extension ([Mat 86], Thm. 4.5). ∎

Suppose that we want to work with the extension A/J, for an ideal $J \subset A$. If J arises as the kernel of a map $A \to S^{-1}A$ for some multiplicatively closed set $S \subset A$, then we can apply Corollary 3.7 if we instead work with the extension $S^{-1}A$. Viewing $S^{-1}A$ as a ring of fractions of A/J, this extension retains generic information along the scheme defined by J, but loses primes annihilated by elements of S. Such primes can prevent the taking of initial ideals from commuting with the extension A/J, as illustrated by the following example.

Example 3.8 Let $A = k[a,b]/(ab)$, let $A[\mathbf{x}] = A[x]$, let $I = (ax+1)$, and let $J = (a)$. Then $\mathrm{in}(I) = (ax, b)$. Taking $B = A/J$, we have $\mathrm{in}(I)B[\mathbf{x}] = (b)$, and $\mathrm{in}(IB[\mathbf{x}]) = (1)$. Instead taking $B = A_b$, we have $\mathrm{in}(I)B[\mathbf{x}] = \mathrm{in}(IB[\mathbf{x}]) = (1)$. Thus taking initial ideals commutes with the extension A_b, but does not commute with the extension A/J.

$\mathrm{Spec}\, A_b$ differs from $\mathrm{Spec}\, A/J$ only by the removal of the prime $p = (b)$. This prime obstructs good behavior for the extension A/J: $\mathrm{in}(I)k(p)[\mathbf{x}] = (0)$, and $\mathrm{in}(Ik(p)[\mathbf{x}]) = (1)$.

Which kernels arise from taking rings of fractions? From the proof of [AtMa 69], Thm. 4.10, one sees that these kernels are precisely the ideals $q \subset A$ which arise as the intersection of primary components corresponding to an isolated set of associated primes of (0). For each such q, it is enough to choose a multiplicatively closed subset $S \subset A$ which intersects $\mathrm{Ann}(q)$, for $B = S^{-1}A/S^{-1}q$ to equal $S^{-1}A$.

More generally, we may wish to consider components of the subscheme defined by $I \cap A$, when this ideal is nonzero. The following lemma reduces us to the above setting.

Lemma 3.9 *Let $I \subset A[\mathbf{x}]$ be an ideal, and let $B = A/(I \cap A)$. Then*

$$\mathrm{in}(I)B[\mathbf{x}] = \mathrm{in}(IB[\mathbf{x}]).$$

Proof. Let $v : A \to B$ be the quotient map. We need to show that $\text{in}(IB[\mathbf{x}]) \subset \text{in}(I)B[\mathbf{x}]$. Given $c\mathbf{x}^E \in \text{in}(IB[\mathbf{x}])$, choose $f \in IB[\mathbf{x}]$ so $\text{in}(f) = c\mathbf{x}^E$. Among all $g \in I$ so $v(g) = f$, choose one with miminal leading term $\text{in}(g)$. We claim that $v(\text{in}(g)) = \text{in}(f)$, so $c\mathbf{x}^E \in \text{in}(I)B[\mathbf{x}]$.

Suppose otherwise, that $\text{in}(g) = b\mathbf{x}^D$ with $\mathbf{x}^D > \mathbf{x}^E$, and $v(b) = 0$. Then $b \in I \cap A$, so $b\mathbf{x}^D \in I$, and $g - b\mathbf{x}^D \in I$ has image f. This element has a lower leading term than g, contradicting the minimality of our choice for g. ∎

Proposition 3.10 *Let $I \subset A[\mathbf{x}]$ be an ideal, let the ideal $q \subset A$ be an isolated primary component of $I \cap A$, and let the ideal $p \subset A$ be its associated minimal prime. Define $B = A_p/q_p$. Then*

$$\text{in}(I)B[\mathbf{x}] = \text{in}(IB[\mathbf{x}]).$$

Proof. Let $q \cap q_2 \cap \ldots \cap q_s$ be a minimal primary decomposition of $I \cap A$, with associated primes $p, p_2, \ldots, p_s$. Then $q_2 \cap \ldots \cap q_s \not\subset p$, for otherwise we would have $q_i \subset p$ for some i, and thus $p_i \subset p$, contradicting the minimality of p. Choose an element $r \notin p$ such that $r \in q_2 \cap \ldots \cap q_s$. Then $rq \subset q \cap q_2 \cap \ldots \cap q_s = I \cap A$, so $r \in (I \cap A : q)$. Thus $A_p/q_p = A_p/(I \cap A)_p$.

By Lemma 3.9, taking initial ideals commutes with taking the quotient by $I \cap A$. By Corollary 3.7, taking initial ideals commutes with forming a ring of fractions. The proposition follows by combining these results. ∎

Proposition 3.10 affirms the utility of Gröbner bases when $\operatorname{Spec} A$ is reducible: Enough information is encoded in such a basis to determine the corresponding Gröbner bases over dense open subsets of each isolated component of $\operatorname{Spec} A/(I \cap A)$.

It is of interest computationally to be able to replace multiplicatively closed sets by powers of a single element. In the proof of Proposition 3.10, we have chosen an element r which vanishes on every primary component of $\operatorname{Spec} A/(I \cap A)$ except the one defined by q. By construction, the product of r with any element of q vanishes everywhere. We observe that our choice of a single element r differs from the construction of single elements to replace multiplicatively closed sets in [GTZ 88], Prop. 3.7:

Example 3.11 Let $A = k[a, b]/(ab)$, let $A[\mathbf{x}] = A[x]$, and let $I = (a(a-1)x,\, x^2)$. If p is chosen to be the minimal prime $(b) \subset A$, then $a \notin p$ and $a \in \text{Ann}((b))$. If we let $r = a$, and let $B = A_r/p_r$, we have

$$\text{in}(I)B[\mathbf{x}] = \text{in}(IB[\mathbf{x}]) = ((a-1)x,\, x^2).$$

On the other hand,

$$\text{in}(I)k(p)[\mathbf{x}] = \text{in}(Ik(p)[\mathbf{x}]) = (x).$$

Following [GTZ 88], if we take $s = a(a-1)$, then

$$IA_p[\mathbf{x}] \cap A[\mathbf{x}] = IA_s[\mathbf{x}] \cap A[\mathbf{x}] = (x).$$

r cannot replace s in this role.

In other words, the extension and contraction of an ideal with respect to a local ring strips away all but generic behavior along the corresponding prime, while it is possible to specialize a Gröbner basis in the sense of Proposition 3.10 and still retain some information about nongeneric behavior along the primary component.

Proposition 3.10 makes no claims about the relationship between initial ideals and their specializations to specific primes. It can happen that no specialization to a prime is well-behaved, as is illustrated by the following example.

Example 3.12 We modify example Example 3.1. Let $A = k[a,b]/(a^2)$, let $A[\mathbf{x}] = A[x,y]$, and let $I = (ax - y)$. For any prime $p \subset A$, $\text{in}(I)k(p)[\mathbf{x}] = (y^2)$, but $\text{in}(Ik(p)[\mathbf{x}]) = (y)$.

We would like to associate a set of monomials with each fiber of the family defined by I, corresponding over each prime p to the monomials of $\text{in}(Ik(p)[\mathbf{x}])$, and then assert that $\text{in}(I)$ encodes enough information to determine these sets generically, i.e. along a dense open subscheme of the base. When the family has a nonreduced base, as in Example 3.12, what should we do over the fuzz? One feels in this example that the fiber monomial ideals are generically (x), i.e. along the open set away from the subscheme cut out by (a). Alas, this open set is empty. This same phenomenon can be observed in studying the "open nature of flatness", where for a nonreduced base scheme, the open set along which a family is flat can be empty. One could think of such open sets as being supported on the fuzz away from a proper subscheme.

The following proposition characterizes those primes which are certain to be well behaved with respect to specialization of a given Gröbner basis.

Proposition 3.13 *Let $I \subset A[\mathbf{x}]$ be an ideal, let $p \subset A$ be a prime, and let $B = A_p/(I \cap A)_p$. If for each monomial $\mathbf{x}^E$, $\operatorname{in}(I)_E B$ is either (0) or (1), then*

$$\operatorname{in}(I)k(p)[\mathbf{x}] = \operatorname{in}(Ik(p)[\mathbf{x}]).$$

Proof. If $I \cap A \not\subset p$, then B is the zero ring, and $\operatorname{in}(I)k(p)[\mathbf{x}] = \operatorname{in}(Ik(p)[\mathbf{x}]) = (1)$. Otherwise, using Lemma 3.9, we can reduce to the case where $I \cap A = (0)$, so $B = A_p$. By Corollary 3.7, we know in any case that

$$\operatorname{in}(I)A_p[\mathbf{x}] = \operatorname{in}(IA_p[\mathbf{x}]).$$

Let $J = IA_p[\mathbf{x}]$; we need to show that

$$\operatorname{in}(Jk(p)[\mathbf{x}]) \subset \operatorname{in}(J)k(p)[\mathbf{x}].$$

Given $\mathbf{x}^E \in \operatorname{in}(Jk(p)[\mathbf{x}])$, choose $f \in Jk(p)[\mathbf{x}]$ so $\operatorname{in}(f) = \mathbf{x}^E$. Among all $g \in J$ with image f in $k(p)[\mathbf{x}]$, choose one with minimal leading term $\operatorname{in}(g)$. We claim that $\operatorname{in}(g) = (1+c)\mathbf{x}^E$ with $c \in p$, so $\mathbf{x}^E \in \operatorname{in}(J)\, k(p)[\mathbf{x}]$.

Suppose otherwise, that $\operatorname{in}(g) = c\mathbf{x}^D$ with $\mathbf{x}^D > \mathbf{x}^E$. Then $c \in p$, and $\operatorname{in}(I)_D A_p = (1)$. Choose $h \in J$ so $\operatorname{in}(h) = \mathbf{x}^D$. Then $g - ch$ also has image f in $k(p)[\mathbf{x}]$, and has a lower leading term than g, contradicting the minimality of our choice for g. ∎

Let $X \subset \operatorname{Spec} A$ be the support of the family defined by I; X is cut out by $I \cap A$. Geometrically, the criterion of Proposition 3.13 is satisfied if the zero locus of each coefficient ideal of $\operatorname{in}(I)$ either avoids the point p, or contains an open neighborhood of p in X.

This criterion is sufficient, but not necessary, for taking initial ideals to commute with specialization. For example, if I is a monomial ideal, it can have arbitrary coefficient ideals, yet $\operatorname{in}(Ik(p)[\mathbf{x}]) = \operatorname{in}(I)k(p)[\mathbf{x}]$ for all primes p.

4 Faithful Flatness

The criterion of Proposition 3.13 gives a sufficient condition for $A[\mathbf{x}]/I$ to be faithfully flat over $A/(I \cap A)$. We will apply the following criterion for faithful flatness; see Matsumura [Mat 86], Atiyah and MacDonald [AtMa 69], or Bourbaki [Bou 89], for full expositions.

Lemma 4.1 *Let M be an A-module. If for each prime $p \subset A$, M_p is a nontrivial, free A_p-module, then M is faithfully flat over A.* ∎

The following lemma will be used in two different proofs.

Lemma 4.2 *Let $I \subset A[\mathbf{x}]$ be an ideal, and define $M = A[\mathbf{x}]/I$. Let $V = \{\mathbf{x}^E \mid \mathrm{in}(I)_E \neq (1)\}$, and let $N \subset M$ be the A-submodule generated by V. Then $N = M$.*

Proof. Suppose that $N \neq M$, and choose a nonzero element $f \in M/N$. Among all $g \in A[\mathbf{x}]$ with image f in M/N, choose one with minimal leading term $\mathrm{in}(g)$. Let $\mathrm{in}(g) = c\mathbf{x}^F$. If $\mathbf{x}^F \in V$, then $c\mathbf{x}^F \in N$, so $g - c\mathbf{x}^F$ also represents f, contradicting the minimality of our choice for g. On the other hand, if $\mathbf{x}^F \notin V$, then $\mathrm{in}(I)_F = (1)$, so $\mathrm{in}(h) = \mathbf{x}^F$ for some $h \in I$. Then $g - ch$ also represents f, again contradicting the minimality of our choice for g. ∎

Proposition 4.3 *Let $I \subset A[\mathbf{x}]$ be a proper ideal, and define $M = A[\mathbf{x}]/I$. If for each prime $p \subset A$ and for each monomial $\mathbf{x}^E$, $\mathrm{in}(I)_E B$ is either (0) or (1) where $B = A_p/(I \cap A)_p$, then M is a faithfully flat $A/(I \cap A)$-module.*

Proof. Disregard primes $p \not\supset I \cap A$. Using Lemma 3.9, we can reduce to the case where $I \cap A = (0)$. We want to show that M is a faithfully flat A-module.

Given a prime $p \subset A$, let $V = \{\mathbf{x}^E \mid \mathrm{in}(I)_E A_p = (0)\}$, and let $N \subset M_p$ be the A_p-submodule generated by V. $N = M_p$ by Lemma 4.2; we claim that N is nontrivial and free. The result then follows from Lemma 4.1.

N is nontrivial because $I \cap A = (0)$, so $\mathrm{in}(I)_1 = (0)$, and $1 \in V$. N is free, because any relation among its generators would be an element of $IA_p[\mathbf{x}]$, whose lead term belongs to V. Since $\mathrm{in}(IA_p[\mathbf{x}])_E = \mathrm{in}(I)_E A_p$ by Corollary 3.7, this would contradict the definition of V. ∎

Corollary 4.4 *Given a Gröbner basis $\{f_1, \ldots, f_r\}$ for $I \subset A[\mathbf{x}]$, let T denote the finite set of exponents E for which $\mathrm{in}(I)_E \neq I \cap A$, and which occur as the exponent of some $\mathrm{in}(f_i)$. If*

$$s \in \bigcap_{E \in T} \{\sqrt{(J^2 : J)} \mid J = \mathrm{in}(I)_E\}$$

and $s \notin \sqrt{I \cap A}$, then M_s is faithfully flat over $A_s/(I \cap A)_s$.

Proof. We have the equality of sets

$$\{\mathrm{in}(I)_E \mid \mathrm{in}(I)_E \neq I \cap A\} \ = \{\sum_i \mathrm{in}(I)_{E_i} \mid E_i \in T \text{ for each } i\}.$$

Thus, we get the same intersection of ideals if we replace the index set T by the infinite set of exponents

$$\{E \mid \mathrm{in}(I)_E \neq I \cap A\}.$$

Reduce to the case where $I \cap A = (0)$. The second condition, that s is not nilpotent, insures that A_s is not the zero ring.

For each $J = \mathrm{in}(I)_E$, $(J^2 : J)$ is supported on precisely those primes p so JA_p is neither (0) nor (1):

$$(J^2 : J)A_p = (1) \Leftrightarrow JA_p = J^2 A_p \Leftrightarrow JA_p = (0) \,\mathrm{or}\, (1).$$

∎

If the scheme defined by $I \cap A$ has any reduced components, then such s exist. Corollary 4.4 remains true for

$$s \in \bigcap_{E \in T} \{\sqrt{J} \mid J = \mathrm{in}(I)_E\},$$

but nontrivial such s need not exist when $I \cap A$ has more than one reduced component, as is illustrated by the following example.

Example 4.5 Let $A = k[a,b,c,d]/((a,b) \cap (c,d))$, let $A[\mathbf{x}] = A[x,y]$, let $I = (ax+b, cy+d)$, and let $M = A[\mathbf{x}]/I$. $\{ax+b, cy+d\}$ is a Gröbner basis for I, and we have $\mathrm{in}(I)_x = (a)$, $\mathrm{in}(I)_y = (c)$, $(a^2 : a) = (a,c,d)$, and $(c^2 : c) = (a,b,c)$. For any $s \in (a,c,d) \cap (a,b,c) = (a,c)$, and any prime $p \subset A$ such that $s \notin p$, $\mathrm{in}(I)_x A_p$ and $\mathrm{in}(I)_y A_p$ are each either (0) or (1). Thus M_s is faithfully flat over A_s for any such s.

$\operatorname{Spec} A$ consists of the union of the two planes $a = b = 0$ and $c = d = 0$, meeting at a common origin. $\mathrm{in}(I)_x = (a)$ vanishes identically on the plane $a = b = 0$, and is nonzero away from the line $a = 0$ on the plane $c = d = 0$. Thus $(a^2 : a)$ is supported on this line, which is the locus where $\mathrm{in}(I)_x$ is locally neither (0) nor (1). Analogous statements hold for $\mathrm{in}(I)_y$.

In this example,

$$\sqrt{\mathrm{in}(I)_x} \cap \sqrt{\mathrm{in}(I)_y} = (a) \cap (c) = (0),$$

so we cannot simplify the criterion of Corollary 4.4.

Compare Proposition 4.3 with Proposition 3.13, and Example 3.1. Summarizing, if the set of monomials of $J = \mathrm{in}(I)k(p)[\mathbf{x}]$ is locally constant as a function of p, then Gröbner bases are well behaved with respect to specialization, and moreover, M is a faithfully flat A-module. However, away from the locus where J is locally constant, M can remain faithfully flat over A.

The next proposition gives some geometric consequences of faithful flatness. Recall that a morphism of schemes $f : X \to Y$ is said to be surjective if for every point $p \in Y$, there exists a point $P \in X$ such that $f(P) = p$. A morphism of schemes is said to be dominant if for every point $P \in X$, the induced map $f_P^{\#} : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is injective. While the first condition is purely topological, the second condition considers the effect of f on the sheaves of rings $\mathcal{O}_X$, $\mathcal{O}_Y$, and does not imply the first.

Proposition 4.6 *Let $I \subset A[\mathbf{x}]$ be an ideal, and let $X = \mathrm{Spec}\, A[\mathbf{x}]/I$, $Y = \mathrm{Spec}\, A$. If $M = A[\mathbf{x}]/I$ is a faithfully flat A-module, then the corresponding morphism of schemes $X \to Y$ is surjective and dominant, and maps each primary component of X to a primary component of Y.*

Proof. Once a map is known to be flat, the surjectivity of $X \to Y$ is an equivalent definition of faithful flatness ([Mat 86], Thm. 7.3; [AtMa 69], Ch. 3., Ex. 16). Given any prime $P \subset M$, let $p = P \cap A$. The local homomorphism $A_p \to M_P$ is faithfully flat, and thus injective, because M is flat over A ([AtMa 69], Ch. 3., Ex. 18). Thus $X \to Y$ is dominant. It remains to prove that if P is an associated prime of (0) in M, then $p = P \cap A$ is an associated prime of (0) in A. This follows from [Bou 89], Ch. 4, §2.6, Cor. 1 to Thm. 2. ∎

Combining Proposition 4.3 with Proposition 4.6, Gröbner bases can be used to manipulate the component structure of the total space of a family of schemes. In [GTZ 88], this problem is approached differently, via rings of fractions. For example, in Cor. 3.8 of [GTZ 88], for A an integral domain with quotient field K, the coefficients of $\mathrm{in}(I)$ are used to find an $s \in A$ so $I\, A_s[\mathbf{x}] \cap A[\mathbf{x}]$ computes $I\, K[\mathbf{x}] \cap A[\mathbf{x}]$; the resulting ideal is then the intersection of the components of I which surject onto the base $\mathrm{Spec}\, A$. In this setting, our choice of s in Corollary 4.4 represents a modest improvement over their choice, and Proposition 4.6 illuminates the connection between these approaches.

When M is finitely generated as an A-module, one could determine the point set in $\mathrm{Spec}\, A$ over which faithful flatness fails, by studying a presentation matrix of M as an A-module; the maximal minors of this matrix generate one of the *Fitting ideals* of M. The use of Gröbner bases is more efficient than a brute force study of these minors, as is evidenced by the following example:

Example 4.7 This example is a modification of Example 3.1. Let $A = k[a]$, let $A[\mathbf{x}] = A[x,y]$, and let $I = (ax+y, x^3, x^2y, xy^2, y^3) \subset A[\mathbf{x}]$. The coefficient ideals of $\mathrm{in}(I)$ are given by the following diagram:

$$\begin{array}{r|llll}
y & (1) & & & \\
 & (0) & (1) & & \\
 & (0) & (a) & (1) & \\
 & (0) & (a) & (a) & (1) \\
\hline
 & & & & x
\end{array}$$

As an A-module, $M = A[\mathbf{x}]/I$ is finitely generated by the set of monomials having nonunit coefficient ideals, $\{x^2, xy, y^2, x, y, 1\}$. These generators have as relations the multiples $ax+y$, $x(ax+y)$, and $y(ax+y)$ of the ideal generator $ax+y$. We organize this data into the following presentation matrix for M:

$$\begin{array}{r|cccccc|}
 & x^2 & xy & y^2 & x & y & 1 \\
\hline
ax+y & 0 & 0 & 0 & a & 1 & 0 \\
x(ax+y) & a & 1 & 0 & 0 & 0 & 0 \\
y(ax+y) & 0 & a & 1 & 0 & 0 & 0 \\
\hline
\end{array}$$

The $(xy,\, y^2,\, y)$-minor of this matrix is nonsingular, so M is flat over A. However, the leading nonzero minor, on columns $(x^2,\, xy,\, x)$, has determinant a^3. This minor demonstrates that M_a is flat over A_a, but leaves open the question of what happens over $a = 0$.

The coefficient ideals of $\mathrm{in}(I)$ determine a locus away from which this leading minor is nonsingular. Since one only needs to consider coefficient ideals corresponding to minimal generators of $\mathrm{in}(I)$, Gröbner bases can be used to find this locus without explicitly considering every row of the presentation matrix: a single element of the Gröbner basis stands in for many rows of the presentation matrix.

Note also that the coefficient ideals and the determinant give different scheme structures for the set where this leading minor loses rank. The coefficient ideals describe the support of the module defined by the leading minor, while the determinant describes a thicker scheme enjoying a universal property with respect to base change; see [Eis 89], Ch. 10.

5 Fibers

The following pair of propositions concern the behavior of coefficient ideals with respect to the geometry of the fibers of a family.

Proposition 5.1 *Let $I \subset A[\mathbf{x}]$ be an ideal, let $M = A[\mathbf{x}]/I$, and let $p \subset A$ be a prime ideal. The following two statements are equivalent:*

(a) $f : A_p \to M_p$ is surjective;

(b) $\mathrm{in}(I)_{x_i} A_p = (1)$ for each i.

Proof. If $\mathrm{in}(I)_{x_i} A_p = (1)$ for each i, then M_p admits a relation of the form $x_i - f_i(x_{i+1}, \ldots, x_n)$, for each i. This proves that f is surjective. Conversely, if f is surjective, then M_p admits a relation of the form $x_i - c_i$ with $c_i \in A_p$, for each i. Thus, the corresponding coefficient ideals are unit ideals. ∎

Corollary 5.2 *Let $I \subset A[\mathbf{x}]$ be an ideal, let $X = \operatorname{Spec} A/(I \cap A)$, and let $Y = \operatorname{Spec} A[\mathbf{x}]/I$. If $I \cap A$ is a prime ideal, and each coefficient ideal $\mathrm{in}(I)_{x_i} \neq I \cap A$, then the induced morphism of schemes $Y \to X$ is an isomorphism over a nonempty open subset of X.* ∎

Let $\mathrm{in}(I)_{x_i^\infty}$ denote the stationary limit of the ascending chain of coefficient ideals $\mathrm{in}(I)_{x_i} \subset \mathrm{in}(I)_{x_i^2} \subset \ldots$. Each $\mathrm{in}(I)_{x_i^\infty}$ can be computed as the ideal generated by the leading coefficients of all Gröbner basis elements having a leading monomial of the form x_i^e.

Proposition 5.3 *Let $I \subset A[\mathbf{x}]$ be an ideal, let $M = A[\mathbf{x}]/I$, and let $p \in A$ be a prime ideal. The following two statements are equivalent:*

(a) $f : A_p \to M_p$ is a finite map;

(b) $\mathrm{in}(I)_{x_i^\infty} A_p = (1)$ for each i.

Proof. Suppose that (a) holds, so M_p is a finite A_p-module. Fix i, and let $N \subset M_p$ be the subalgebra generated by x_i. Then N can be generated as an A_p-module by the finite set $\{1, x_i, \ldots, x_i^{e-1}\}$ for some e. x_i^e can be expressed in terms of these generators, yielding an expression in $IA_p[\mathbf{x}]$ with leading term x_i^e. Thus $\mathrm{in}(IA_p[\mathbf{x}])_{x_i^e} = (1)$.

Conversely, assume (b). For each i, choose e_i so $\text{in}(IA_p[\mathbf{x}])_{x_i^e} = (1)$, and let V denote the finite set of monomials which do not belong to the ideal $(x_1^{e_1}, \ldots, x_n^{e_n})$. If $N \subset M_p$ is the A_p-submodule generated by V, then N is finitely generated, and $N = M_p$ by Lemma 4.2. ∎

Geometrically, let $X = \operatorname{Spec} A[\mathbf{x}]/I$, let $Y = \operatorname{Spec} A/(I \cap A)$, and let $g : X \to Y$ be the morphism of schemes induced by the inclusion $A/(I \cap A) \subset A[\mathbf{x}]/I$. If we let $U \subset Y$ be the complement of the union of the subschemes of Y cut out by the coefficient ideals $\text{in}(I)_{x_i}$, then Proposition 5.1 asserts that U is the largest open set with the property that $g^{-1}(U) \to U$ is an isomorphism. If instead we let $U \subset Y$ be the complement of the union of the subschemes of Y cut out by the coefficient ideals $\text{in}(I)_{x_i^\infty}$, then Proposition 5.3 asserts that U is the largest open set with the property that $g^{-1}(U) \to U$ is a finite morphism. Thus, while Gröbner bases can only generically detect faithful flatness, they are capable of detecting precisely the locus where a morphism restricts to an isomorphism, or to a finite map.

Detection of quasi-finite morphisms is more subtle; a family which restricts to a finite family over an open subset of the base need not be quasi-finite:

Example 5.4 Let $A = k[a, b]$, let $A[\mathbf{x}] = A[x]$, let $I = (ax - b)$, and let $M = A[\mathbf{x}]/I$. The localization M_a is finite over A_a. However, the total space $ax - b = 0$ is irreducible, and consists of a line over $a = b = 0$.

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