Monomial Ideals and Duality

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These are lecture notes, in progress, on monomial ideals. The point of view is that monomial ideals are best understood by drawing them and looking at their corners, and that a combinatorial duality satisfied by these corners, *Alexander duality*, is key to understanding the more algebraic duality theories at play in algebraic geometry and commutative algebra. Sections written so far cover Alexander duality, corners, Möbius inversion and Poincaré series, minimal free resolutions, and the Cohen-Macaulay condition. The sections planned but not yet written will cover the *Stanley-Reisner* monomial ideals associated to simplicial complexes, Reisner's criteria for such an ideal to be Cohen-Macaulay, injective resolutions, local cohomology, and Serre duality.

1 Alexander duality

Recall the statement of Alexander duality for spheres (compare [Mun84, Theorem 71.1]):

Theorem 1.1 (Alexander duality) *Let X be a proper, nonempty subset of the sphere S^{n-2}. Suppose that the pair (S^{n-2}, X) is triangulable. There are isomorphisms*

$$H_i(X;G) \;\cong\; H^{n-i-3}(S^{n-2}\backslash X;G)$$

and

$$H^i(X;G) \;\cong\; H_{n-i-3}(S^{n-2}\backslash X;G)$$

where H denotes reduced singular (co)homology.

For example, take $n = 4$, and consider a circle S^1 in the sphere S^2, as shown in Figure 1. The circle has homology $H_1(S^1;\mathbb{Z}) \cong \mathbb{Z}$. Its complement is the union of two open hemispheres, which has cohomology $H^0(S^2\backslash S^1;\mathbb{Z}) \cong \mathbb{Z}$.

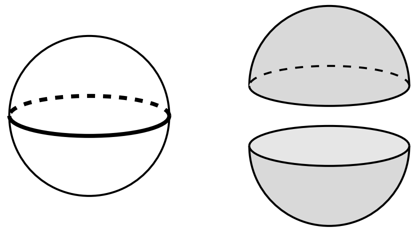

Figure 1: Alexander duality in a 2-sphere.

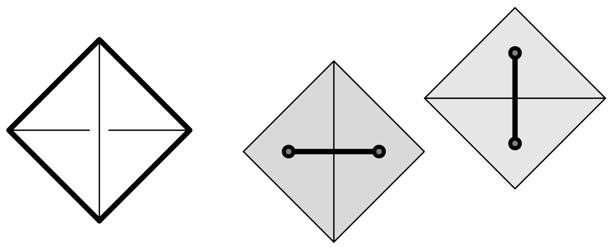

Figure 2: Alexander duality in a simplicial 2-sphere.

Let Δ denote the full $(n-1)$-simplex on $\{1,\ldots,n\}$. Alexander duality has a simplicial analogue, where we identify the sphere S^{n-2} with the subcomplex of Δ formed by deleting its unique $(n-1)$-face. Like poor computer code, the boundary conditions *proper, nonempty* and the restriction to S^{n-2} are symptoms of definitions which need repair. Therefore, take a simplicial complex to be any collection of subsets of $\{1,\ldots,n\}$ which is closed with respect to taking subsets. In particular, distinguish between the *void* simplicial complex $\{\}$, which is acyclic, and the *empty* simplicial complex $\{\emptyset\}$, which is not.

Our example is viewed simplicially in Figure 2 and Figure 3. The two hemispheres become two squares, each a union of two triangles. These squares can be retracted to two line segments, preserving (co)homology. This relationship can be

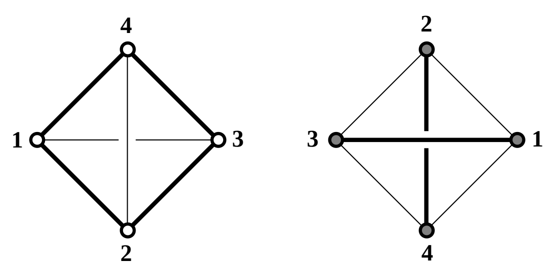

Figure 3: Two Alexander dual simplicial complexes.

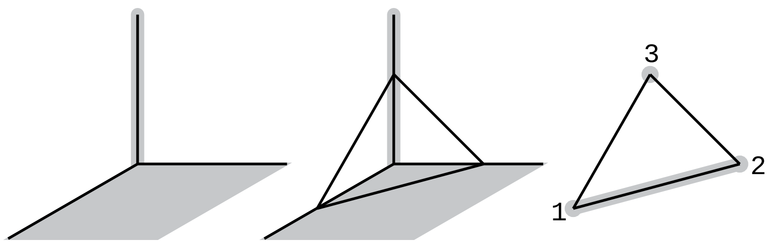

Figure 4: Two geometric realizations of $X = \{\emptyset, 1, 2, 3, 12\}$.

understood systematically as *polarity*: Let Δ^* denote a second copy of Δ, whose faces are in $1:1$ correspondence with the complementary faces of Δ. Under polarity, the complement of a simplicial complex $X \subset \Delta$ corresponds to a *dual* simplicial complex $X^\vee \subset \Delta^*$. In Figure 2, the vertices of the line segments are polar to the triangles, and the line segments themselves are polar to the edges gluing pairs of triangles. This process is similar to taking the dual of a planar graph, and yields Figure 3, showing two dual simplicial complexes.

With this motivation, define the *dual* subcomplex $X^\vee \subset \Delta^*$ of a subcomplex $X \subset \Delta$ to be the simplicial complex obtained by complementing the subsets of X and the collection X, in either order. In other words, define

$$X^\vee \;=\; \{\, F \mid F^c \notin X \,\} \;=\; \{\, F \mid F \notin X^c \,\}$$

where ^{c} denotes set complementation. We have

Theorem 1.2 (Alexander duality) *Let $X \subset \Delta$ be a simplicial complex, and let $X^\vee$ be its dual complex. There are isomorphisms*

$$H_i(X;G) \;\cong\; H^{n-i-3}(X^\vee;G) \quad \textit{and} \quad H^i(X;G) \;\cong\; H_{n-i-3}(X^\vee;G)$$

where H denotes reduced simplicial (co)homology.

This theorem is proved as Theorem 6.2 in an appendix on simplicial complexes. Essentially, one gets homology to look like cohomology by relabeling and tweaking signs; the duality operator $^\vee$ embodies the relabeling that takes place.

The *linear realization* of a simplicial complex X is a union of coordinate planes in $\mathbb{R}^n$, corresponding to the faces of X. Each face $F \in X$ is represented by the linear span of the coordinate vectors $\{\, e_i \mid i \in F \,\}$, where e_i is the ith unit basis vector of $\mathbb{R}^n$. The *Stanley-Reisner ring* $k[X]$ of X is the affine coordinate ring of this algebraic variety, for a field k. The *topological realization* of a simplicial complex X is the familiar union of geometric simplices in $\mathbb{R}^n$. Each face $F \in X$ is represented by the convex hull of the coordinate vectors $\{\, e_i \mid i \in F \,\}$, viewed as points. The

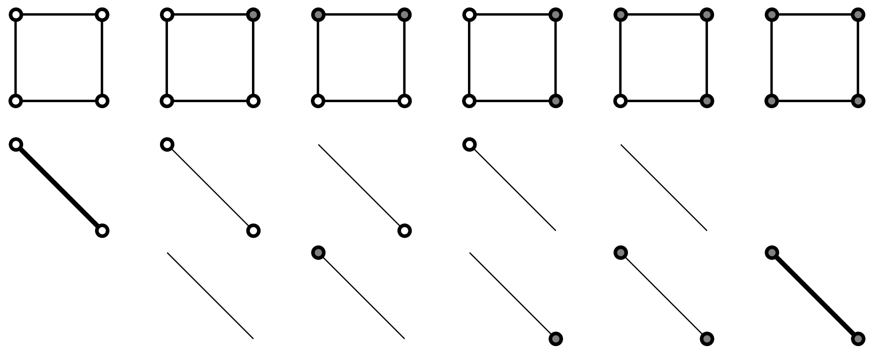

Figure 5: The simplicial complexes on 2 vertices.

relationship between these objects is shown in Figure 4, for the simplicial complex $X = \{\emptyset, 1, 2, 3, 12\}$. We obtain the topological realization by intersecting the linear realization with the standard $(n-1)$-simplex which is the convex hull of $\{e_1, \dots e_n\}$.

Working with these geometric realizations of a simplicial complex is analogous to working with the projective variety defined by a homogeneous ideal. The linear realization is like the affine cone cut out by a homogeneous ideal. The topological realization is like an affine piece of the projective variety, obtained by intersecting the affine cone with a hyperplane. The topological realization hides the distinction between the void complex $\{\}$ and the empty complex $\{\emptyset\}$, exactly as working projectively hides the distinction between the unit ideal (1) and the irrelevant ideal $(x_1, \dots, x_n)$.

For this reason, it is helpful in visualizing dual complexes to also picture the subsets of $\{1, \dots, n\}$ as the vertices of the n-cube $\{0,1\} \times \cdots \times \{0,1\}$. Draw a subcomplex $X \subset \Delta$ by marking the cube vertices which correspond to the characteristic vectors of the faces of X. Looking down from the $(1, \dots, 1)$-corner instead of up from the $(0, \dots, 0)$-corner, and considering the complementary vertices, one sees the dual complex $X^\vee$.

For example, there are 6 simplicial complexes on 2 vertices, as shown in Figure 5. They are drawn left to right from the full simplex Δ to the void complex $\{\}$; the vertices corresponding to faces of X are white, and the vertices corresponding to faces of $X^\vee$ are black. In the second row we have drawn topological realizations of X using white dots for vertices, and in the third row we have drawn topological realizations of $X^\vee$ using black vertices. The void complexes are drawn as nothing, and the empty complexes are drawn as empty frameworks for the faces of Δ.

There are 10 simplicial complexes up to symmetry on 3 vertices, as shown in Figure 6. They form 6 dual pairs; the last two are self-dual. The first, third and

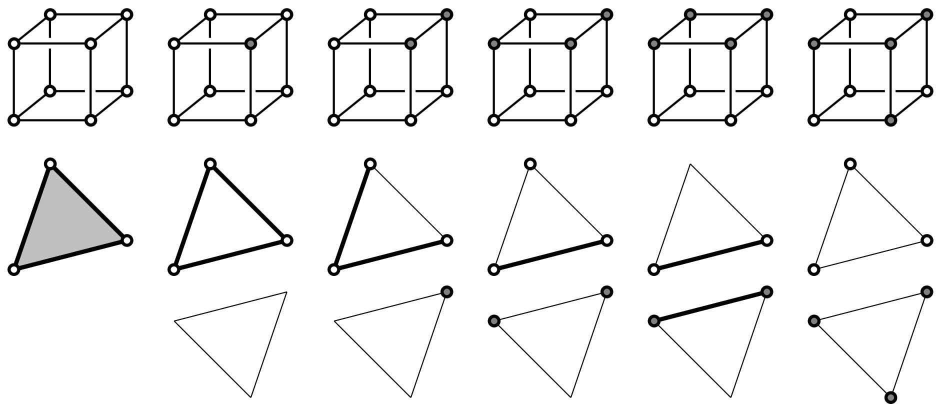

Figure 6: The simplicial complexes on 3 vertices.

fifth pairs are acyclic; the second pair satisfies

$$H_1(X;\mathbb{Z}) \;\cong\; H^{-1}(X^\vee;\mathbb{Z}) \;\cong\; \mathbb{Z},$$

and the fourth and sixth pairs satisfy

$$H_0(X;\mathbb{Z}) \;\cong\; H^0(X^\vee;\mathbb{Z}) \;\cong\; \mathbb{Z},\; \mathbb{Z}^2 \quad \text{respectively.}$$

There are 30 simplicial complexes on 4 vertices, up to symmetry, as shown in Figure 7. They form 17 dual pairs, four of which are self-dual.

It is often illuminating to dualize a complex in its span, rather than in the full simplex on all vertices. For example, the four simplicial complexes shown in Figure 8 have homology, yet involve fewer than 4 vertices. We compare their full duals with the duals in their spans. Note the homological "red shift" as we drop vertices before dualizing: The same homology groups appear, but in lower dimensions.

2 Corners

Our habitats abound with corners, yet the profusion of corner types barely has a chance to get started in the dimensions we inhabit. Corners are characterized by simplicial complexes; the essense of a corner is captured by the cube drawings of Section 1. If one steps through a corner and looks at it from the other side, one sees the dual simplicial complex. Indeed, isometric drawings of corners have a tendency to "pop out" the other way. As this happens, each corner appears to become its dual.

For example, up to symmetry there are five types of corners and 5 types of noncorners in 3 dimensions, as shown in Figure 9. By centering a *probe cube* around

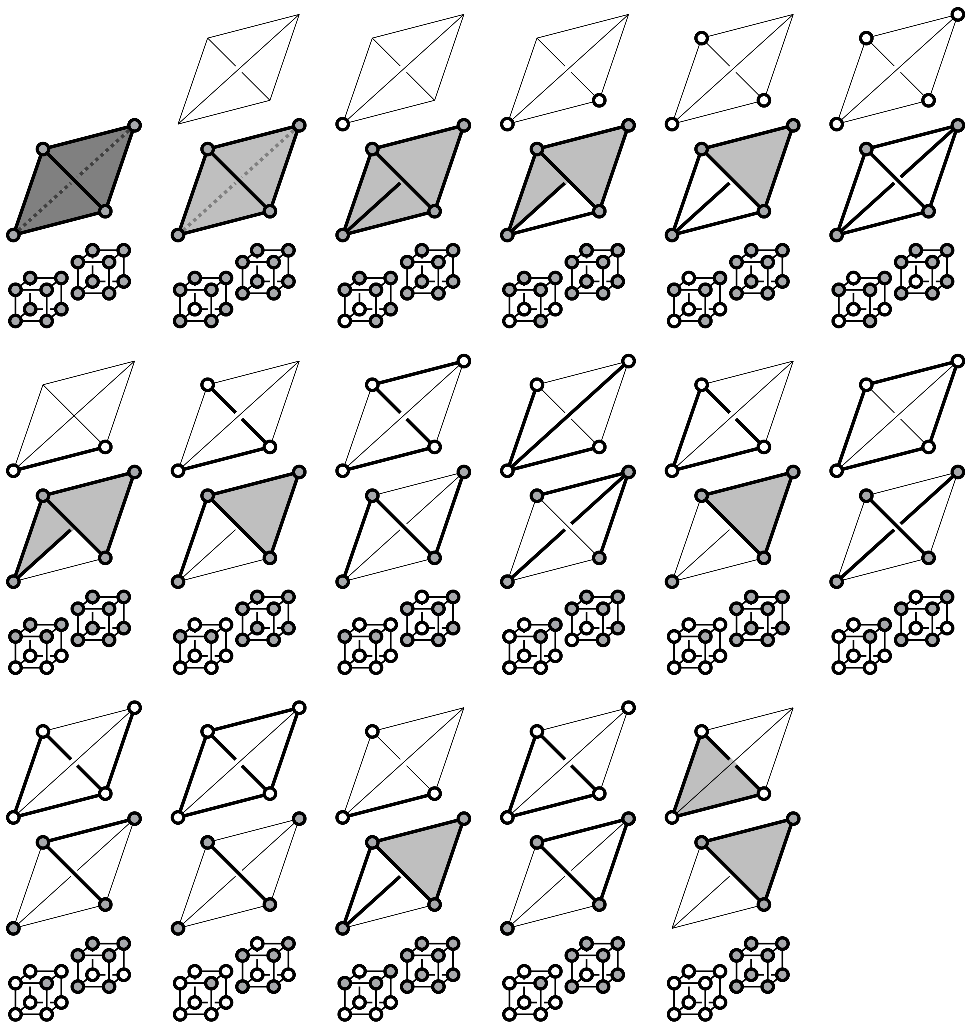

Figure 7: The simplicial complexes on 4 vertices.

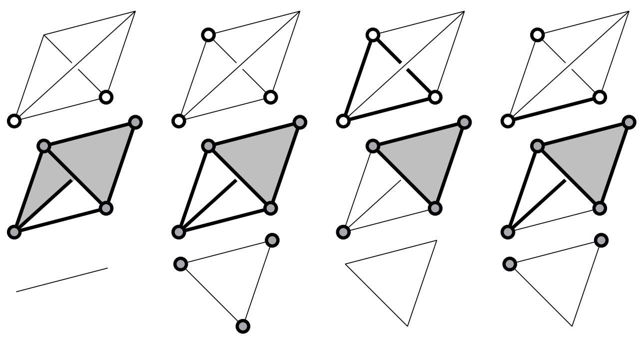

Figure 8: Dualizing in the span.

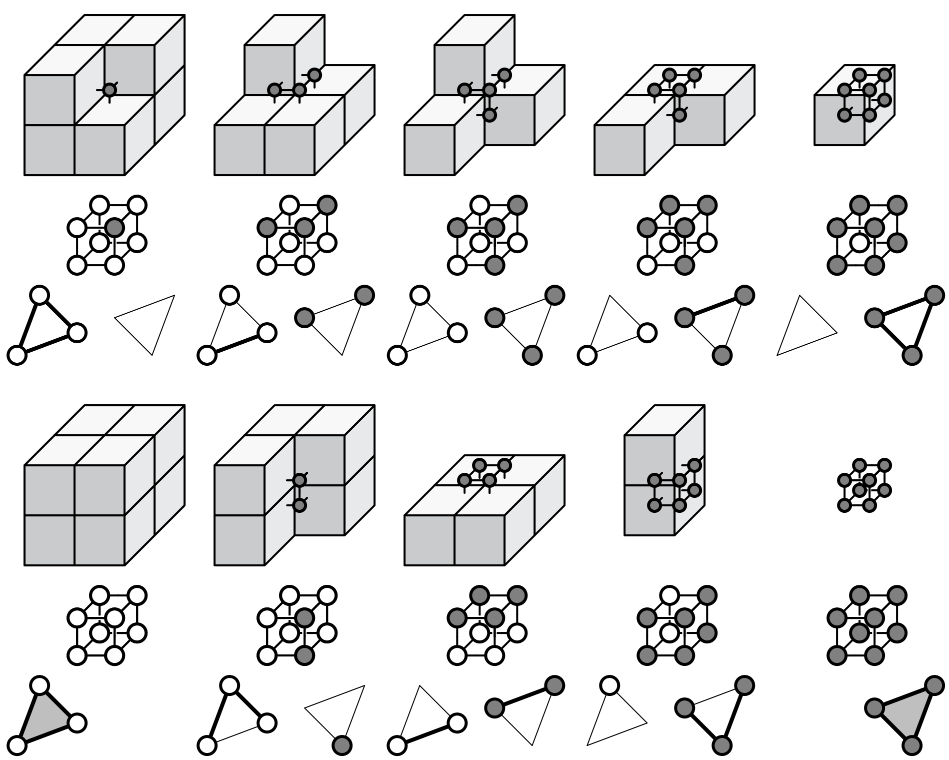

Figure 9: Corners and noncorners in 3 dimensions.

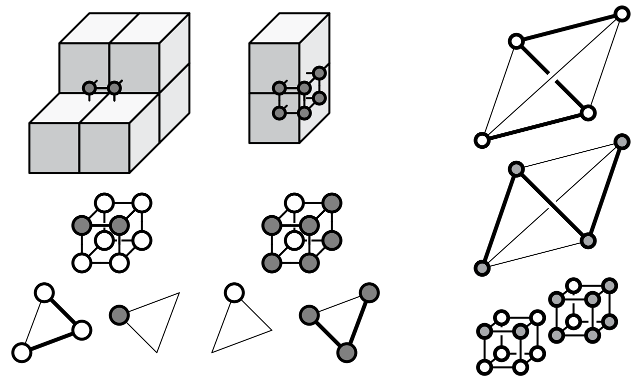

Figure 10: A corner with no homology, in 4 dimensions.

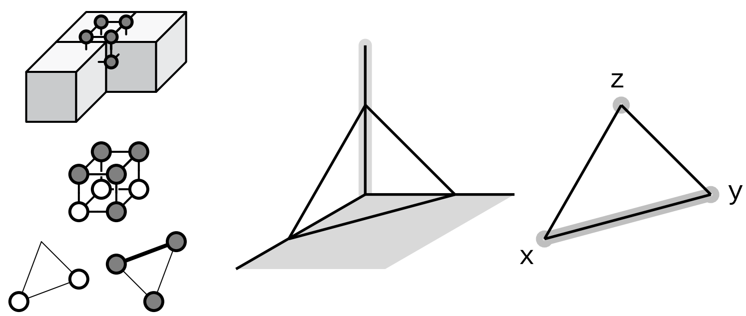

Figure 11: An observer's view of a corner.

each potential corner, we read off two dual simplicial complexes, according to which vertices are interior. Each of the 10 distinct simplicial complexes on 3 vertices makes two dual appearances; the cones are not corners. Notice that dualizing flips each row.

An interesting corner in 4 dimensions is shown in Figure 10, obtained by stacking two sets of blocks along a 4th dimension. This corner has no homology, which happens frequently in higher dimensions. After studying this model, one can return to the atlas of simplicial complexes on 4 vertices given by Figure 7, and intrepret each hypercube drawing as a probe cube for a corner or a noncorner in 4 dimensions.

How does an observer, perched locally at a corner, perceive the associated simplicial complexes? Consider the simplicial complex given by the upper, shaded vertices of a probe cube. As shown in Figure 11, the first orthant of its linear representation models the small nonnegative vectors which can be subtracted from the vantage point while remaining outside or on the boundary of the solid forming the corner. In this example, an observer can dangle an arm down parallel to the z-axis, or swing an arm down parallel to the xy-plane. The lower, dual simplicial complex is experienced by adding rather than subtracting, and staying inside or on the boundary of the solid. Here, an observer can reach up either parallel to the x-axis or the y-axis, yielding two vertices as the dual.

Let k be a field, and let $S = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over k. Visualizing a monomial ideal $I \subset S$ as a solid subset of $\mathbb{R}^n$, one also sees corners. These corners are critical to the study of the Poincaré series, or the minimal free resolution, or the minimal injective resolution of I, because the data for each of these algebraic objects is supported on the corners of I. More precisely, the homology of the simplicial complexes associated to the corners determines the numerical data for each of these objects.

In choosing how to view a corner of I, one is deciding which of two dual simplicial complexes to favor. Often, the relationship between a corner and properties of I is inscrutable viewed one way, but obvious viewed the other way. One wants to develop the reflex of always looking at corners both ways, rather than assuming that one's initial vantage point is preferable.

Let $C = [0,1] \times \cdots \times [0,1]$ be the standard unit cube in $\mathbb{R}^n$. We associate the monomial ideal I with the *solid* $N(I) \subset \mathbb{R}^n$ defined by "stacking blocks"

$$N(I) \;=\; \bigcup_{\mathbf{x}^{\mathbf{a}} \in I} \{\mathbf{a} + C\}.$$

The *corners* of I are the corners of $N(I)$. $N(I)$ is finitely generated by the minimal generating exponents A of I, for

$$\mathbf{b} \in N(I) \iff \mathbf{b} \geq \mathbf{a} \text{ for some } \mathbf{a} \in A.$$

Define the *dual solid* $N^\vee(I)$ to be the closure $\overline{\mathbb{R}^n \setminus N(I)}$ of the complement of $N(I)$. Sometimes one draws a monomial ideal by making the ideal solid, and sometimes by making its complement solid; replacing $N(I)$ by $N^\vee(I)$ is this change of perspective. $N(I)$ is stable under addition by $\mathbb{R}^n_+$, while $N^\vee(I)$ is stable under subtraction by $\mathbb{R}^n_+$. Thus, $N^\vee(I)$ can only be considered finitely generated if we allow exponents of ∞ to appear. Then, one can give a finite generating set $B \subset (\mathbb{R} \cup \{\infty\})^n$ so

$$\mathbf{a} \in N^\vee(I) \iff \mathbf{a} \leq \mathbf{b} \text{ for some } \mathbf{b} \in B.$$

Fix a lattice point $\mathbf{b} \in \mathbb{Z}^n$. Among the integer translations of C are 2^n lattice blocks which are incident on $\mathbf{b}$. Each of these blocks is either contained in $N(I)$ or in $N^\vee(I)$. Locally around $\mathbf{b}$, we see only the nearest corners of these blocks, emanating from $\mathbf{b}$ as the 2^n orthants of $\mathbb{R}^n$ emanate from the origin. A probe cube centered at $\mathbf{b}$ will have one vertex in each block, so the partition of incident blocks

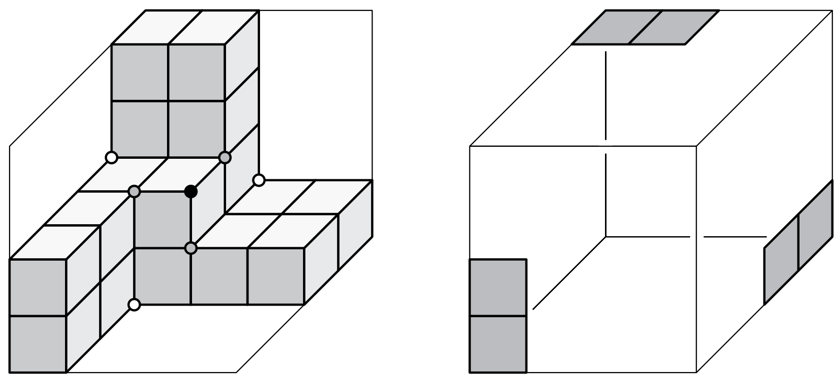

Figure 12: The monomial ideal $(x^2y,\, y^2z,\, xz^2)$.

into those contained in $N(I)$ and in $N^\vee(I)$ corresponds to a partition of the vertices of the probe cube into two dual simplicial complexes.

With this geometric visualization in mind, define

$$\begin{aligned} \Delta_{\mathbf{b}}(I) &= \left\{ F \in \Delta \;\middle|\; \mathbf{x}^{\mathbf{b}-F} \in I \right\}, \\ \Delta^\vee_{\mathbf{b}}(I) &= \left\{ F \in \Delta \;\middle|\; \mathbf{x}^{\mathbf{b}-F^c} \notin I \right\}, \end{aligned}$$

identifying each face F with its characteristic vector for the subtractions. One checks that these are indeed dual simplicial complexes. We have reversed the sense of which dual complex gets to wear the superscript $^\vee$, because $\Delta_{\mathbf{b}}(I)$ has a simpler definition and is encountered first when studying I.

How do we reconcile these definitions with the geometric interpretation given above? Let $\epsilon = (\frac{1}{2}, \ldots, \frac{1}{2})$. A unit probe cube centered at $\mathbf{b}$ will have vertices at $\mathbf{b} + \epsilon - F$ for $F \in \Delta$. Because $N(I)$ is closed,

$$\mathbf{x}^{\mathbf{b}-F} \in I \iff \mathbf{b} - F \in N(I) \iff \mathbf{b} + \epsilon - F \in N(I).$$

Thus, the faces of $\Delta_{\mathbf{b}}(I)$ correspond to the probe cube vertices contained in $N(I)$, and the faces of $\Delta^\vee_{\mathbf{b}}(I)$ correspond to the probe cube vertices contained in $N^\vee(I)$.

A *corner* of I is a point $\mathbf{b} \in \mathbb{R}^n$ so the local appearance of $N(I)$ near $\mathbf{b}$ changes in each coordinate direction. This local picture is invariant with respect to small translations in the coordinate direction i, iff $\Delta_{\mathbf{b}}(I)$ (equivalently, $\Delta^\vee_{\mathbf{b}}(I)$) is a cone over the vertex i. Thus, corners are characterized by dual pairs of simplicial complexes which aren't cones. Therefore, $\Delta_{\mathbf{b}}(I)$ and $\Delta^\vee_{\mathbf{b}}(I)$ can have nonzero reduced (co)homology, and nonzero reduced Euler characteristic, only if $\mathbf{b}$ is a corner of I.

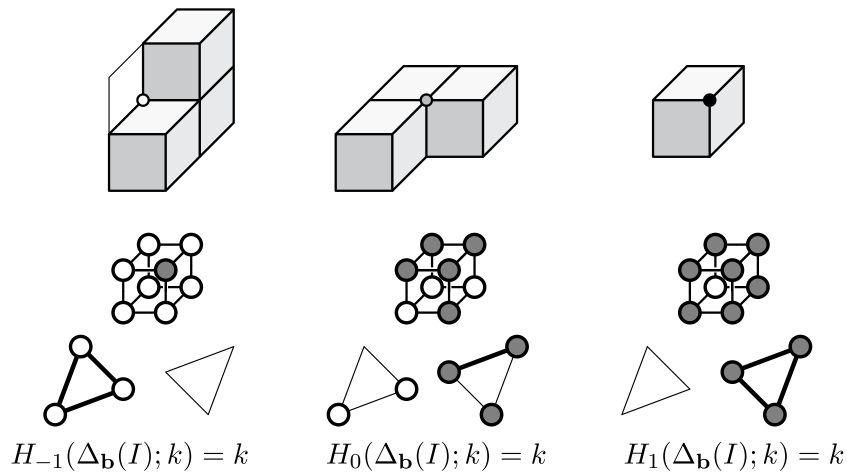

Figure 13: The corner types of $(x^2y,\ y^2z,\ xz^2)$.

Example 2.1 Let I be the monomial ideal $(x^2y,\ y^2z,\ xz^2) \subset S = k[x, y, z]$. I can be pictured as the solid $N(I)$ for the generating exponents $A = \{(2, 1, 0),\ (0, 2, 1),\ (1, 0, 2)\}$. We explore how the structure of I reveals itself in various drawings of I.

In Figure 12, we draw the intersection of $N^{\vee}(I)$ with a truncated first octant of $\mathbb{R}^3$. In this example, each corner has homology in a single dimension, so we are able to color them accordingly. In particular, the generators of I are the corners drawn with white dots. The drawing on the left can be interpretted as a picture of a minimal free resolution of I:

$$0 \leftarrow I \xleftarrow{\begin{bmatrix} x^2y & y^2z & xz^2 \end{bmatrix}} S^3 \xleftarrow{\begin{bmatrix} 0 & -z^2 & yz \\ xz & 0 & -x^2 \\ -y^2 & xy & 0 \end{bmatrix}} S^3 \xleftarrow{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} S \leftarrow 0.$$

The first syzygies of I have weights $(1, 2, 2)$, $(2, 1, 2)$, and $(2, 2, 1)$, which correspond to the gray corners. These are the multidegrees in which cancellation takes place when we multiply the first two matrices. The unique second syzygy of I has weight $(2, 2, 2)$, its cumulative multidegree, which corresponds to the black corner.

These three corner types are studied in Figure 13, with $\Delta_{\mathbf{b}}(I)$ given on the lower right for each corner. The generators, first syzygies, and second syzygies of I correspond to the homology of $\Delta_{\mathbf{b}}(I)$ in dimensions -1, 0, and 1. The picture for an arbitrary monomial ideal can be more complicated, because corners can have homology in several dimensions or none.

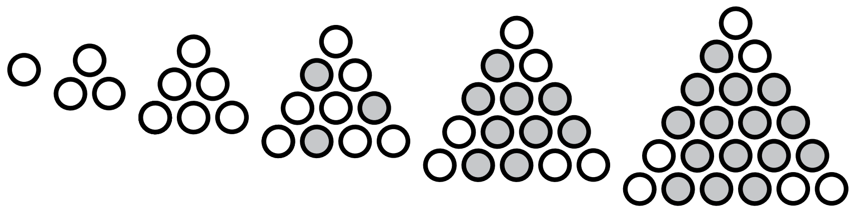

Figure 14: Degree slices of a monomial ideal.

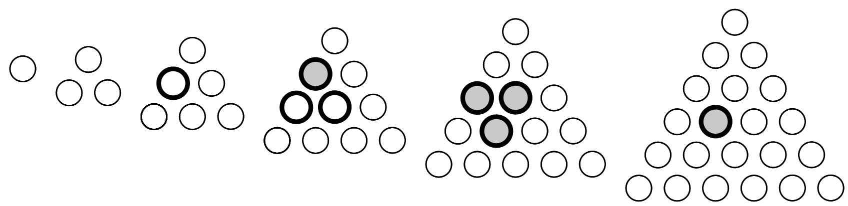

Figure 15: The probe cube descending from x^2yz^2.

From this free resolution one can compute the Poincaré series

$$\sum_d \dim I_d \; t^d \;=\; \frac{3t^3 - 3t^5 + t^6}{(1-t)^3}$$

of I. The terms in the numerator correspond to the 3 white, 3 gray, and 1 black corners shown. Möbius inversion is the most direct way to compute the Poincaré series of a monomial ideal; in this setting it specializes to computing the reduced Euler characteristic of $\Delta_{\mathbf{b}}(I)$ for each corner $\mathbf{b}$ of I.

We have truncated 6 rows of blocks that in fact continue out to infinity along each of the coordinate axis. The subscheme $X \subset P^2$ defined projectively by I consists of three double points; the faces of the truncation give us a good picture of this subscheme, as shown on the right in Figure 12.

If we picture monomials as balls rather than cubes, and organize them in antidiagonal slices according to their degree, we obtain Figure 14 as a different way to draw I. The monomials belonging to I have been shaded. Large degree slices give us another picture of the scheme $X \subset P^2$ consisting of three double points. The pattern in degree 3 does not look like a downward continuation of the pattern in higher degrees, for the center monomial xyz looks out of place. In fact, I is not *saturated* because it fails to contain xyz; the largest homogenous ideal cutting out X is $I^{\text{sat}} = (x^2y,\, y^2z,\, xz^2, xyz)$. A drawing of I^{sat} can be obtained from Figure 12 by removing the block for xyz, which doesn't spoil the picture in higher degrees.

Figure 14 also shows how the multiples of the three generators x^2y, y^2z, xz^2 first overlap 2 degrees later, giving the 3 first syzygies of I. For example, there

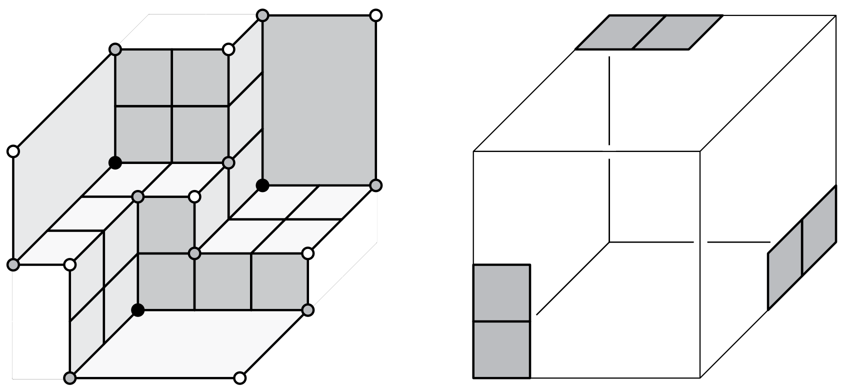

Figure 16: Dual generators for a monomial ideal.

is a first syzygy supported on the monomial x^2yz^2. Figure 15 restricts attention to the antidiagonal slices of a probe cube descending from x^2yz^2, whose homology corresponds to this syzygy. The vertices of the probe cube are the monomials having x^2yz^2 as a squarefree multiple. This is the same corner studied in the middle drawing of Figure 13.

If instead of descending from x^2yz^2 in Figure 15, we descend from a monomial not involving all of the variables, we face an interesting choice. The probe cube now extends beyond the monomials drawn, into negative exponents. The definitions of $\Delta_{\mathbf{b}}(I)$ and $\Delta^{\vee}_{\mathbf{b}}(I)$ indeed consider negative exponents, but these coordinate directions do not appear in the support of $\Delta_{\mathbf{b}}(I)$, for such monomials would never be shaded. Thus, we could just as well ignore these coordinates, changing the definition of $\Delta^{\vee}_{\mathbf{b}}(I)$. This corresponds to dualizing in the span, and is especially useful when studying Stanley-Reisner rings.

The dual solid $N^{\vee}(I)$ is generated by the exponent set

$$B = \{(2,2,2),\ (\infty,1,2),\ (2,\infty,1),\ (1,2,\infty),\ (\infty,\infty,0),\ (\infty,0,\infty),\ (0,\infty,\infty)\},$$

and can be drawn as shown in Figure 16. Here we take $\infty = 4$, an exponent that is larger than appears in any finite corner of I. Again, the dual generators of I are drawn with white dots, and the remaining corners are drawn according to their homology. This drawing can be interpretted as a picture of an injective resolution of S/I.

These dual generators give a primary decomposition of I, as shown in Figure 17. The point $(2,2,2)$ determines the embedded primary ideal (x^2,y^2,z^2), the points $(\infty,1,2)$, $(2,\infty,1)$, and $(1,2,\infty)$ determine the primary ideals (y,z^2), (x^2,z), and (x,y^2), and the points $(\infty,\infty,0)$, $(\infty,0,\infty)$, and $(0,\infty,\infty)$ keep the other octants of $\mathbb{R}^3$ out of our way.

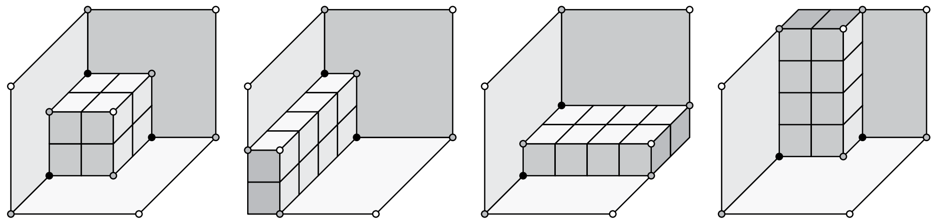

Figure 17: A primary decomposition.

In general, the dual generators give an irreducible decomposition of I, which can be finer than a minimal primary decomposition. Recall that an ideal is *irreducible* if it cannot be expressed as the proper intersection of two other ideals. By collecting primary components for each associated prime, we can consolidate an irreducible decomposition into a minimal primary decomposition.

In subsequent sections, we establish for arbitrary monomial ideals the relationships revealed in this example.

3 Möbius inversion

Let $I \subset S = k[x_1, \ldots, x_n]$ be a monomial ideal, and let $g : \mathbb{N}^n \to \mathbb{Z}$ be its characteristic function, so $g(\mathbf{b}) = 1$ if $\mathbf{x}^{\mathbf{b}} \in I$, and $g(\mathbf{b}) = 0$ otherwise. The multigraded Poincaré series for I is the rational power series

$$P(I, \mathbf{x}) \;=\; \sum_{\mathbf{b} \in \mathbb{N}^n} g(\mathbf{b})\, \mathbf{x}^{\mathbf{b}} \;=\; \frac{\sum_{\mathbf{b} \in \mathbb{N}^n} f(\mathbf{b})\, \mathbf{x}^{\mathbf{b}}}{\prod_{i=1}^{n} (1 - x_i)},$$

where $f : \mathbb{N}^n \to \mathbb{Z}$ is nonzero for only finitely many $\mathbf{b}$. We show below that

$$f(\mathbf{b}) \;=\; -\chi(\Delta_{\mathbf{b}}(I)),$$

where χ denotes the reduced Euler characteristic

$$\chi(X) \;=\; \sum_{F \in X} (-1)^{|F|-1} \;=\; \sum_{i=-1}^{n-2} (-1)^i \dim H_i(X; k).$$

Note that χ differs from the Euler characteristic used in topology, in that it counts -1 for the empty face of X. If X is acyclic, then $\chi(X) = 0$. It follows that $f(\mathbf{b})$ can only be nonzero when $\mathbf{b}$ is a corner. I has only finitely many corners, giving a proof that $P(I, \mathbf{x})$ is rational.

The Poincaré series $P(I, \mathbf{x})$ captures essential information about I. For example, the related Poincaré series

$$P(I,t) \;=\; \sum_{\mathbf{b}\in\mathbb{N}^n} g(\mathbf{b})\, t^{|\mathbf{b}|} \;=\; \frac{\sum_{\mathbf{b}\in\mathbb{N}^n} f(\mathbf{b})\, t^{|\mathbf{b}|}}{(1-t)^n},$$

where $|\mathbf{b}|$ is the total degree of $\mathbf{b}$, can be obtained from $P(I, \mathbf{x})$ by substituting t for each x_i. From $P(I,t)$, the Hilbert polynomial of I and its error terms can be easily computed: Write $P(I,t)$ as a Laurent polynomial with polar part in powers of $(1-t)$ and positive part in powers of t,

$$P(I,t) \;=\; \frac{a_{n-1}}{(1-t)^n} + \ldots + \frac{a_1}{(1-t)^2} + \frac{a_0}{(1-t)} + b_0 + b_1 t + \ldots + b_{m-1} t^{m-1}.$$

Then the Hilbert polynomial $p(I,d) = \dim(I_d)$, $d \gg 0$ of I is given by

$$p(I,d) \;=\; a_{n-1}\binom{d+n-1}{n-1} + \ldots + a_1\binom{d+1}{1} + a_0,$$

with error terms $b_d = \dim(I_d) - p(I,d)$ for $d = 0, \ldots, m-1$. In particular, $p(I,d)$ is correct for $d \geq m$. The Hilbert polynomial can be written more informatively as

$$p(I,d) \;=\; a_{n-1}\mathbb{P}^{n-1} + \ldots + a_1\mathbb{P}^1 + a_0\mathbb{P}^0$$

where $\mathbb{P}^i$ denotes projective i-space. This reads, "I has the same Hilbert polynomial as a_i copies of P^i for $i = n-1, \ldots, 0$."

Example 3.1 The Poincaré series for the ideal $I = (x^2y,\, y^2z,\, xz^2) \subset S = k[x,y,z]$ studied in Example 2.1 is

$$P(I,\mathbf{x}) \;=\; \frac{x^2y + y^2z + xz^2 - xy^2z^2 - x^2yz^2 - x^2y^2z + x^2y^2z^2}{(1-x)(1-y)(1-z)}$$

The monomials in the numerator are the corners of I. Substituting $x = t$, $y = t$, $z = t$ gives

$$P(I,t) \;=\; \frac{3t^3 - 3t^5 + t^6}{(1-t)^3}.$$

Substituting $t = 1-u$ gives

$$P(I,u) \;=\; \frac{3(1-u)^3 - 3(1-u)^5 + (1-u)^6}{u^3} \;=\; \frac{1}{u^3} - \frac{6}{u} + 7 - 3u^2 + u^3.$$

Substituting back $u = 1-t$ gives

$$P(I,t) \;=\; \frac{1}{(1-t)^3} - \frac{6}{1-t} + 5 + 3t - t^3.$$

This yields the Hilbert polynomial $p(I,d) = \mathbb{P}^2 - 6\mathbb{P}^0$, which says that projectively, I looks like S with 6 points notched out of it.

Tabulating this information for low degrees (compare with Figure 14),

d	0	1	2	3	4	5
$\dim I_d$	0	0	0	3	9	15
$p(I,d)$	-5	-3	0	4	9	15
error	5	3	0	-1	0	0

we see that $p(I,d)$ predicts impossible negative dimensions in degrees 0 and 1, when both of these dimensions are actually zero. This is a typical low degree squeezing effect; there simply isn't room to remove 6 points until degree 2. $p(I,d)$ also overestimates the dimension of I_3, for it extrapolates from high degree behavior, so it can't anticipate the missing monomial xyz. These differences are recorded by the error terms $5 + 3t - t^3$.

By hand or when programming a computer, the above computation can be carried out using repeated synthetic division, which takes a simple form:

$$
\begin{array}{rrrrrrr|r}
1 & -3 & 0 & 3 & 0 & 0 & 0 & \\
 & -1 & 2 & 2 & -1 & -1 & -1 & 1 \\
 & & 1 & -1 & -3 & -2 & -1 & 0 \\
 & & & -1 & 0 & 3 & 5 & -6
\end{array}
$$

Write the coefficients of the numerator of $P(I,t)$ in the first row, and draw a vertical bar. Working left to right, apply the recursion $c_{i,j} = c_{i,j-1} - c_{i-1,j-1}$ to generate each successive row. Crossing the bar, negate and stop. After computing n successive rows, the Hilbert polynomial is given by the column to the right of the bar, and the remainder terms are given by the last row to the left of the bar.

Each term $\mathbf{x}^{\mathbf{b}}$ in the numerator $\sum f(\mathbf{b})\,\mathbf{x}^{\mathbf{b}}$ of $P(I,\mathbf{x})$ sums the monomials in the principal ideal $(\mathbf{x}^{\mathbf{b}})$, because

$$
\begin{aligned}
\sum_{\mathbf{x}^{\mathbf{a}} \in (\mathbf{x}^{\mathbf{b}})} \mathbf{x}^{\mathbf{a}} &= \mathbf{x}^{\mathbf{b}}\,(1 + x_1 + x_1^2 + \ldots) \cdots (1 + x_n + x_n^2 + \ldots) \\
&= \mathbf{x}^{\mathbf{b}}\,\frac{1}{1-x_1} \cdots \frac{1}{1-x_n} = \frac{\mathbf{x}^{\mathbf{b}}}{\prod_{i=1}^{n}(1-x_i)}.
\end{aligned}
$$

We can think of these terms, with their coefficients, as taking inventory of the monomials of I by inclusion-exclusion counting: For $I = (x^2y,\, y^2z,\, xz^2)$, we separately add in every monomial which is a multiple of x^2y, y^2z, or xz^2. This counts the multiples of $x^2y^2z = \mathrm{lcm}(x^2y, y^2z)$ twice, so we subtract them back out, and likewise for x^2yz^2, xy^2z^2. This counts the multiples of $x^2y^2z^2 = \mathrm{lcm}(x^2y, y^2z, xz^2)$ one too few times, so we add them back in. The general formula for $I = (\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_\ell})$ is

$$\sum_{\mathbf{b}\in\mathbb{N}^n} f(\mathbf{b})\,\mathbf{x}^{\mathbf{b}} \;=\; \sum_{\emptyset\neq A\subset\{\mathbf{x}^{\mathbf{a}_1},\ldots,\mathbf{x}^{\mathbf{a}_\ell}\}} (-1)^{|A|-1}\,\mathrm{lcm}(A),$$

where $|A|$ is the cardinality of A.

In general, a monomial can occur as several different lcm's, and their inclusion-exclusion coefficients can cancel out to zero. Möbius inversion is a far-reaching generalization of inclusion-exclusion counting which computes these coefficients from local data near each monomial. In our setting, Möbius inversion can be easily carried out using generating functions:

$$\begin{aligned}
\sum_{\mathbf{b}\in\mathbb{N}^n} f(\mathbf{b})\,\mathbf{x}^{\mathbf{b}} &= \prod_{i=1}^{n}(1-x_i)\sum_{\mathbf{b}\in\mathbb{N}^n} g(\mathbf{b})\,\mathbf{x}^{\mathbf{b}} \\
&= \sum_{\mathbf{b}\in\mathbb{N}^n}\left(\sum_{F\in\Delta}(-1)^{|F|}\,g(\mathbf{b}-F)\right)\mathbf{x}^{\mathbf{b}} \\
&= \sum_{\mathbf{b}\in\mathbb{N}^n} -\chi(\Delta_{\mathbf{b}}(I))\,\mathbf{x}^{\mathbf{b}}.
\end{aligned}$$

If $\Delta_{\mathbf{b}}(I)$ is a cone, then $\chi(\Delta_{\mathbf{b}}(I)) = 0$. Thus, f is supported on the corners of I. In particular, $f(\mathbf{b}) = 0$ for any monomial $\mathbf{x}^{\mathbf{b}}$ interior to the solid $N(I)$. This proves

Proposition 3.2 *The Poincaré series of the monomial ideal $I \subset k[x_1, \ldots, x_n]$ is given by*

$$P(I,\mathbf{x}) \;=\; \frac{\sum_{\mathbf{b}\in\mathbb{N}^n} f(\mathbf{b})\,\mathbf{x}^{\mathbf{b}}}{\prod_{i=1}^{n}(1-x_i)}$$

with

$$f(\mathbf{b}) \;=\; -\chi(\Delta_{\mathbf{b}}(I)),$$

where χ is the reduced Euler characteristic. ∎

Lemma 3.3 *If I is a monomial ideal generated by m monomials, then I has at most $2^m - 1$ corners.*

Proof. If $\mathbf{b}$ is a corner of I, then each coordinate of $\mathbf{b}$ agrees with the same coordinate in the exponent of some generator of I. Otherwise, there would be a coordinate direction in which $\mathbf{b}$ could be increased or decreased without changing the set of generators dividing $\mathbf{x}^{\mathbf{b}}$, and $\Delta_{\mathbf{b}}(I)$ would be a cone over that vertex. Thus, $\mathbf{x}^{\mathbf{b}}$ is the least common multiple of a nonempty subset of the generators for I. There are $2^m - 1$ such subsets, giving the stated bound. ∎

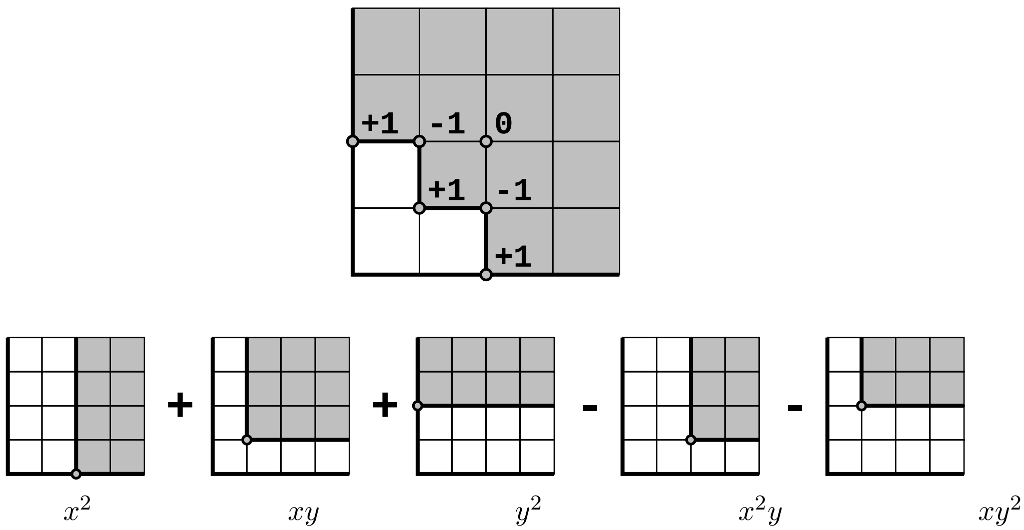

Figure 18: Möbius inversion coefficients for (x^2, xy, y^2).

Note that Example 2.1 achieves this bound. A comparison of our two derivations of $\sum_{\mathbf{b}\in\mathbb{N}^n} f(\mathbf{b})\,\mathbf{x}^{\mathbf{b}}$ also shows that every corner is the lcm of a nonempty subset of the generators.

Corollary 3.4 *$P(I, \mathbf{x})$ is a rational power series.*

Proof. We need to show that $f : \mathbb{N}^n \to \mathbb{Z}$ is nonzero for only finitely many $\mathbf{b}$. By Lemma 3.3, I has only finitely many corners. By Proposition 3.2, f can only be nonzero when $\mathbf{b}$ is a corner of I. Thus, $P(I, \mathbf{x})$ is rational. ∎

Example 3.5 Let $I = (x^2, xy, y^2) \subset k[x, y]$; its Poincaré series is

$$P(I, \mathbf{x}) \;=\; \frac{x^2 + xy + y^2 - x^2y - xy^2}{(1-x)(1-y)}.$$

The monomial x^2y^2 appears in inclusion-exclusion counting both as $\operatorname{lcm}(x^2, y^2)$ and as $\operatorname{lcm}(x^2, xy, y^2)$, canceling to zero. This monomial is interior to $N(I)$, as shown in Figure 18.

See [Sta86] for a beautiful account of Möbius inversion, where Proposition 3.8.6 gives another relationship with χ. The following is a synopsis of how Proposition 3.2 can be viewed more conceptually in terms of Möbius inversion:

The *incidence algebra* $\mathbf{I}(\mathbb{N}^n, \mathbb{Z})$ is the $\mathbb{Z}$-algebra of formal sums

$$h \;=\; \sum h(\mathbf{a}, \mathbf{b})\,[\mathbf{a}, \mathbf{b}]$$

of symbols $[\mathbf{a}, \mathbf{b}]$ for $\mathbf{a} \le \mathbf{b}$ in $\mathbb{N}^n$. Multiplication is defined by $[\mathbf{a}, \mathbf{b}][\mathbf{b}, \mathbf{c}] = [\mathbf{a}, \mathbf{c}]$, and by $[\mathbf{a}, \mathbf{b}][\mathbf{c}, \mathbf{d}] = 0$ if $\mathbf{b} \ne \mathbf{c}$. $\mathbf{I}(\mathbb{N}^n, \mathbb{Z})$ has $1 = \sum_{\mathbf{a}\in\mathbb{N}^n} [\mathbf{a}, \mathbf{a}]$ as its identity element, and acts on the set of functions $\{f : \mathbb{N}^n \to \mathbb{Z}\}$ by

$$(fh)(\mathbf{b}) \;=\; \sum_{\mathbf{a}\le\mathbf{b}} f(\mathbf{a})\, h(\mathbf{a}, \mathbf{b}).$$

The *zeta function* is defined as $\zeta = \sum_{\mathbf{a}\le\mathbf{b}} [\mathbf{a}, \mathbf{b}]$. Via ζ, we have $g = f\zeta$. The inverse of ζ is the *Möbius function*

$$\mu \;=\; \sum_{\mathbf{a}\in\mathbb{N}^n} \sum_{F\in\Delta} (-1)^{|F|}\,[\mathbf{a} - F, \mathbf{a}],$$

where $|F|$ denotes the cardinality of F. Thus $f = g\mu$, so

$$f(\mathbf{b}) \;=\; (g\mu)(\mathbf{b}) \;=\; \sum_{F\in\Delta} g(\mathbf{b} - F)\,(-1)^{|F|} \;=\; \sum_{F\in\Delta_{\mathbf{b}}(I)} (-1)^{|F|} \;=\; -\chi(\Delta_{\mathbf{b}}(I)).$$

4 Koszul homology

Let

$$0 \longleftarrow I \longleftarrow \oplus_j\, Se_{0j} \longleftarrow \oplus_j\, Se_{1j} \longleftarrow \ldots \longleftarrow \oplus_j\, Se_{mj} \longleftarrow 0$$

be a minimal free resolution of the monomial ideal $I \subset S = k[x_1, \ldots, x_n]$; we have $m \le n-1$. The multigraded *Betti number* $\beta_{i,\mathbf{b}}$ of I is the number of e_{ij} of degree $\mathbf{b}$, which is the number of degree $\mathbf{b}$ ith syzygies of I.

Any free resolution of I is an algebraic analogue to inclusion-exclusion counting: The module $\oplus_j\, Se_{0j}$ surjects onto I with some redundancy. The module $\oplus_j\, Se_{1j}$ accounts for this redundancy but creates some of its own, and so forth. Literally mimicking inclusion-exclusion counting leads to the Taylor resolution of I, a usually nonminimal resolution whose e_{ij} correspond to the lcm's of $i+1$ generators of I at a time. On the other hand, a minimal resolution of I gives finer information about I than Möbius inversion, exactly as the homology of a simplicial complex gives finer information than the Euler characteristic.

Any free resolution of I can be used to compute the Poincaré series of I, as

$$P(I, \mathbf{x}) \;=\; \frac{\sum_{i,j} (-1)^i\, \mathbf{x}^{\mathbf{b}_{ij}}}{\prod_{i=1}^n (1 - x_i)}$$

where $\mathbf{b}_{ij}$ is the multidegree of e_{ij}. This is a direct analogue of the formula for computing $P(I, \mathbf{x})$ by inclusion-exclusion. In particular, from the Betti numbers of I we can compute

$$P(I, \mathbf{x}) \;=\; \frac{\sum_{\mathbf{b}\in\mathbb{N}^n} \left(\sum_i (-1)^i \beta_{i,\mathbf{b}}\right) \mathbf{x}^{\mathbf{b}}}{\prod_{i=1}^n (1-x_i)}.$$

This invites comparison with Proposition 3.2, yielding the conclusion

$$-\chi(\Delta_{\mathbf{b}}(I)) \;=\; \sum_{i=0}^{n-1} (-1)^i \beta_{i,\mathbf{b}}.$$

In fact, an alternate way to compute the Euler characteristic is

$$\chi(\Delta_{\mathbf{b}}(I)) \;=\; \sum_{i=-1}^{n-2} (-1)^i \dim H_i(\Delta_{\mathbf{b}}(I), k)$$

where H is reduced simplicial homology. In the remainder of this section, we confirm that $\beta_{i,\mathbf{b}} = \dim H_{i-1}(\Delta_{\mathbf{b}}(I), k)$, showing that minimal free resolutions of monomial ideals are supported on their corners.

The Betti numbers $\beta_{i,\mathbf{b}}$ can be obtained by *Koszul homology* without computing a free resolution of I. Koszul homology was extensively studied in [Gre84], where "Green's conjecture" for canonical curves is made. See also [BH93].

We find the Betti numbers of I by computing $\mathrm{Tor}_*(I, k)$ two different ways, where $k = S/(x_1, \ldots, x_n)$. Tor is both the homology of a resolution of I after tensoring with k, and the homology of a resolution of k after tensoring with I.

First, use a minimal free resolution of I to expand in the first variable, computing Tor_* as the homology of the complex

$$(0 \longleftarrow \oplus_j\, Se_{0j} \longleftarrow \oplus_j\, Se_{1j} \longleftarrow \ldots \longleftarrow \oplus_j\, Se_{mj} \longleftarrow 0) \otimes_S\, k.$$

Since none of the maps in a minimal resolution involve constants, all of the maps become zero after tensoring with k, so

$$\mathrm{Tor}_i(I, k) \;=\; \oplus_j\, k\, e_{ij}.$$

In particular, the $\mathbf{b}$th graded piece of $\mathrm{Tor}_i(I, k)$ has dimension $\beta_{i,\mathbf{b}}$.

Now, use a minimal free resolution of k to expand instead in the second variable. Such a resolution is provided by the *Koszul complex*

$$0 \longleftarrow k \longleftarrow \wedge^0 V \otimes S \longleftarrow \wedge^1 V \otimes S \longleftarrow \ldots \longleftarrow \wedge^n V \otimes S \longleftarrow 0,$$

where V is the subspace of degree one forms of S, with basis $x_1, \ldots, x_n$. The maps are given by the rule

$$x_{i_0} \wedge \ldots \wedge x_{i_\ell} \;\mapsto\; \sum_{j=0}^{\ell} (-1)^j \, x_{i_j}(x_{i_0} \wedge \ldots \wedge \widehat{x_{i_j}} \wedge \ldots \wedge x_{i_\ell}),$$

where $\widehat{\ }$ omits the indicated term. Tor_* can also be computed as the homology of the complex

$$0 \longleftarrow \wedge^0 V \otimes I \longleftarrow \wedge^1 V \otimes I \longleftarrow \ldots \longleftarrow \wedge^n V \otimes I \longleftarrow 0.$$

The multigraded piece of degree $\mathbf{b}$ of this complex looks like $\mathcal{C}_*(\Delta_{\mathbf{b}}(I)) \otimes k$, where $\mathcal{C}_*(\Delta_{\mathbf{b}}(I))$ is the augmented oriented chain complex

$$0 \longleftarrow \mathcal{C}_{-1}(\Delta_{\mathbf{b}}(I)) \longleftarrow \mathcal{C}_0(\Delta_{\mathbf{b}}(I)) \longleftarrow \ldots \longleftarrow \mathcal{C}_{n-1}(\Delta_{\mathbf{b}}(I)) \longleftarrow 0.$$

To see this, note that $x_{i_0} \wedge \ldots \wedge x_{i_\ell}$ has degree $x_{i_0} \cdots x_{i_\ell}$, so it contributes to the degree $\mathbf{b}$ piece of the tensor product $\wedge^{\ell+1} V \otimes I$ iff

$$\mathbf{x}^{\mathbf{b}} / x_{i_0} \cdots x_{i_\ell} \in I \iff \{i_0, \ldots, i_\ell\} \in \Delta_{\mathbf{b}}(I).$$

Thus, we have

Proposition 4.1 *The Betti number $\beta_{i,\mathbf{b}}$ of the monomial ideal I is given by* $\dim H_{i-1}(\Delta_{\mathbf{b}}(I), k)$, *where H denotes reduced simplicial homology.*

Proof. By the above computations, $\beta_{i,\mathbf{b}} = \dim \mathrm{Tor}_i(I,k)_b$, and $\mathrm{Tor}_i(I,k)_b \cong H_{i-1}(\Delta_{\mathbf{b}}(I), k)$. ∎

5 Cohen-Macaulay rings

Let $I \subset S = k[x_1, \ldots, x_n]$ be a homogeneous ideal. $I = (f_1, \ldots, f_\ell)$ is a *complete intersection* if I defines a variety X of codimension ℓ in the affine space $\mathbb{A}^n_k$. Most ideals are not complete intersections. For example, the twisted cubic $I = (b^2 - ac,\ bc - ad,\ c^2 - bd) \subset k[a,b,c,d]$ defines a surface $X \subset \mathbb{A}^4_k$ of codimension 2.

If I is a complete intersection, then I has a minimal free resolution of length ℓ given by the Koszul complex

$$0 \longleftarrow I \longleftarrow \wedge^1 V \otimes S \longleftarrow \wedge^2 V \otimes S \longleftarrow \ldots \longleftarrow \wedge^\ell V \otimes S \longleftarrow 0,$$

where V is now the subspace of S with basis $\{f_1, \ldots, f_\ell\}$.

A polynomial $h \in S$ is *generic* with respect to an ideal I if h is not contained in any associated prime of I, or equivalently, if h is not a zero divisor in the quotient ring S/I. Geometrically, this means that the hypersurface $h = 0$ slices each component of X properly. If such an h is linear, then (I, h) is called a *generic hyperplane section* of I.

If I is a complete intersection of codimension ℓ, and h is generic with respect to I, then (I, h) is also a complete intersection, of codimension $\ell + 1$. In particular, (I, h) has no embedded components.

The Cohen-Macaulay condition is a generalization of a complete intersection. Let I be a homogeneous ideal, and for comparison, let J be a complete intersection of the same codimension. I is *Cohen-Macaulay* iff minimal free resolutions of I and J have the same length.

This is the most accessible definition out of a daunting array of equivalent technical conditions in modern use, and it is the definition we shall use in applications. To understand what it means for an ideal to be Cohen-Macaulay, it helps to go back to the original source [Mac16] to see what Macaulay was thinking.

Example 5.1 Consider the monomial ideal $I = (ab, bc, cd, ad) \subset S = k[a, b, c, d]$. I has as its primary decomposition $(a, c) \cap (b, d)$, so I cuts out a codimension 2 variety $X \subset \mathbb{A}^4_k$, consisting of the bd- and ac-coordinate planes meeting at the origin. I has as a minimal free resolution

$$0 \leftarrow I \xleftarrow{\begin{bmatrix} ab & bc & cd & ad \end{bmatrix}} S^4 \xleftarrow{\begin{bmatrix} 0 & 0 & -d & c \\ d & 0 & 0 & -a \\ -b & a & 0 & 0 \\ 0 & -c & b & 0 \end{bmatrix}} S^4 \xleftarrow{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}} S \leftarrow 0.$$

For comparison, $J = (a, b)$ is a complete intersection of the same codimension 2, cutting out the cd-coordinate plane in $\mathbb{A}^4_k$. J has as a minimal free resolution the Koszul complex

$$0 \leftarrow J \xleftarrow{\begin{bmatrix} a & b \end{bmatrix}} S^2 \xleftarrow{\begin{bmatrix} b \\ -a \end{bmatrix}} S \leftarrow 0$$

which is one step shorter, so I is not Cohen-Macaulay.

The inclusion $\mathbb{A}^3_k \hookrightarrow \mathbb{A}^4_k$ defined by $(a, b, c) \mapsto (a, b, c, c)$ has as its image a hyperplane H which is generic with respect to X, because H doesn't contain either of the associated primes of I. Restricting to H corresponds to taking the ring quotient $S/(c - d) \cong T = k[a, b, c]$; the image of I under this map is the ideal $I_H = (ab, bc, c^2, ac)$. Because H is generic, the minimal resolution of I restricts to a

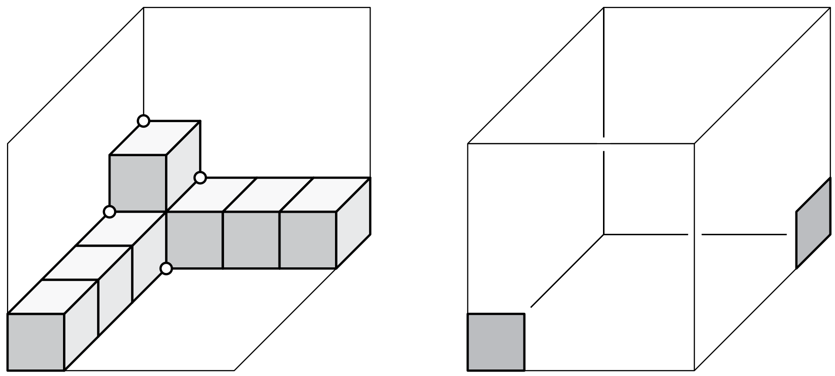

Figure 19: An embedded component that appears after slicing.

minimal resolution of I_H,

$$0 \leftarrow I_H \xleftarrow{\begin{bmatrix} ab & bc & c^2 & ac \end{bmatrix}} T^4 \xleftarrow{\begin{bmatrix} 0 & 0 & -c & c \\ c & 0 & 0 & -a \\ -b & a & 0 & 0 \\ 0 & -c & b & 0 \end{bmatrix}} T^4 \xleftarrow{\begin{bmatrix} a \\ b \\ c \\ c \end{bmatrix}} T \leftarrow 0.$$

On the other hand, taking primary decompositions doesn't commute with this map. (a, c) and (b, d) restrict to (a, c) and (b, c), with intersection $(ab, c) \neq I_H$. Instead, I_H has as a primary decomposition $(a, c) \cap (b, c) \cap (a, b, c^2)$, as can be seen in Figure 19. The hyperplane section of X has acquired an embedded component of codimension 3. The extra step in the minimal resolution of I reflects the presence of this "ghost" component. In Macaulay's terminology, I fails to be unmixed.

An ideal is *unmixed* if all of its primary components have the same codimension, and if this remains true after taking repeated generic hyperplane sections. The prototype of an unmixed ideal is a complete intersection. More generally, I is unmixed iff I is Cohen-Macaulay.

In general, a minimal free resolution of I is at least as long as resolutions of its primary components would be, if they were complete intersections of the same codimension. Thus, the highest codimension primary components of I impose the strictest lower bound on this length. Because this length is preserved by taking generic hyperplane sections, it must also satisfy the lower bounds imposed by primary components that only appear after taking repeated sections. The content of the definition we use for a Cohen-Macaulay ring is that minimal resolutions are as short as possible while satisfying these constraints.

The preceding calculations can be done by hand or machine. The following example requires a machine, and was one of the first practical applications of the computer algebra system *Macaulay* [BS].

Example 5.2 (HAL's song) Let $I \subset k[x_{11}, \ldots, x_{33}, y_{11}, \ldots, y_{33}]$ be the homogeneous ideal generated by the entries of $AB - BA$, where

$$A = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \text{ and } B = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix}.$$

I defines the variety $X \subset \mathbb{A}^{18}_k$ of commuting pairs (A, B) of 3×3 matrices, which has codimension 6 and degree 31. If we intersect I with a generic linear subspace of dimension 6 (by adjoining 12 random linear forms), we get a multiple point supported at the origin, also of degree 31.

Thus, I is Cohen-Macaulay, because any embedded components acquired by slicing would persist under further slicing, contributing to this latter multiplicity. I is not a complete intersection, for it has 8 minimal generators; the trace of $AB - BA$ is zero, giving a linear dependency among the 9 entries.

Appendix

6 Simplicial complexes

This section is a review of simplicial homology and cohomology, leading to a proof of Alexander duality in its simplicial form. Simplicial homology arises in commutative algebra in many guises. The level of detail given here is intended to aid in letting problems find their most natural forms.

We associate to each simplicial complex $X \subset \Delta$ a *dual simplicial complex* $X^\vee$, whose homology groups are *Alexander dual* to the cohomology groups of X. The proof involves reduced relative simplicial cohomology, which is a combinatorial analogue to local cohomology.

Let $\mathbf{n}$ denote the set $\{1, \ldots, n\}$, or the ordered list $[1, \ldots, n]$, as needed by the context. We write H (without a tilde) for *reduced* homology or cohomology. We also distinguish between the *void* simplicial complex $\{\}$, which is the empty collection of subsets of $\mathbf{n}$, and the *empty* simplicial complex $\{\emptyset\}$, which is the collection consisting only of the empty subset $\emptyset \subset \mathbf{n}$. These definitions reflect the patterns that arise naturally in commutative algebra, and are essential to defining Alexander duality simplicially.

6.1 Dual simplicial complexes

Let Δ denote the $(n-1)$-simplex $2^{\mathbf{n}}$ of all subsets of $\mathbf{n}$. Let F^c denote the complement $\mathbf{n} \backslash F$ of a subset $F \subset \mathbf{n}$, and let X^c denote the complement $\Delta \backslash X$ of a collection of subsets $X \subset \Delta$. Let $|F|$ denote the cardinality of $F \subset \mathbf{n}$; call F an i-face of Δ if $|F| = i + 1$.

The following terms give us a vocabulary for describing various collections of index sets that can arise in multigraded algebra.

Definition 6.1 *A* simplicial complex *$X \subset \Delta$ is a collection of subsets of $\mathbf{n}$ which is closed with respect to taking subsets, i.e. so $F' \in X$ whenever $F' \subset F$ for $F \in X$.*

A simplicial cocomplex *$U \subset \Delta$ is a collection of subsets of $\mathbf{n}$ which is closed with respect to taking supersets, i.e. so $F' \in U$ whenever $F' \supset F$ for $F \in U$.*

More generally, a simplicial pair *$(Y, X) \subset \Delta$ is the difference $Y \backslash X$ of two simplicial complexes $X \subset Y \subset \Delta$. Equivalently, a simplicial pair $Z \subset \Delta$ is a collection of subsets of $\mathbf{n}$ which is closed with respect to taking intervals, i.e. so $F'' \in Z$ whenever $F \subset F'' \subset F'$ for $F, F' \in Z$.*

Note that the definition of a simplicial complex does not require that the subsets F be nonempty, or that the collection X be nonempty. The empty subset $\emptyset$ of dimension -1 is treated like any other subset, and is required to be a member of any nonempty simplicial complex. The void complex $\{\}$ is acyclic, while the empty complex $\{\emptyset\}$ has nontrivial reduced simplicial homology.

One checks that if Z is a simplicial pair, then Z can be uniquely described as the difference $Y \backslash X$ of two simplicial complexes $X \subset Y$, allowing the notation $Z = (Y, X)$. Z is the collection of subsets of $\mathbf{n}$ not belonging to either the "floor" complex X or the "ceiling" cocomplex $U = Y^c$.

A complex X is a special case $(X, \emptyset)$ of a simplicial pair; it models a closed subspace of a topological space. A cocomplex U is a special case (Δ, U^c) of a simplicial pair; it models an open subspace of a topological space. The complement of a complex is a cocomplex, and vice-versa.

Let Δ^* denote another copy of Δ, called the *polar* of Δ. Define a correspondence called *polarity* between the faces of Δ and the complementary faces of Δ^*, by defining the polar $F^* \in \Delta^*$ of a face $F \in \Delta$ to be $F^* = F^c$, taken as an element of Δ^*. In particular, each vertex of Δ is polar to a facet of Δ^*, and vice-versa.

Make polarity symmetric by identifying Δ^{**} with Δ, so Δ is also the polar of Δ^*, and F is also the polar of F^*. We use polarity to define polar simplicial pairs, and dual complexes:

Definition 6.2 *Let* $(Y, X) \subset \Delta$ *be a simplicial pair. The* polar simplicial pair $(Y, X)^* \subset \Delta^*$ *is the collection of polar faces*

$$(Y, X)^* \;=\; \{\ F^* \mid F \in (Y, X)\ \}.$$

In particular, polarity takes complexes to cocomplexes, and cocomplexes to complexes. It is immediate that $(Y, X)^{**} = (Y, X)$.

Definition 6.3 *Let* $X \subset \Delta$ *be a simplicial complex. The* dual complex $X^\vee \subset \Delta^*$ *is the collection of subsets*

$$X^\vee \;=\; \{\ F^* \mid F \notin X\ \}.$$

In other words, $X^\vee$ is the polar in Δ^* of the complementary cocomplex X^c of X. It follows that $X^\vee$ is a simplicial complex, and that $X^{\vee\vee} = X$.

Polarity, complementation, and duality can be extended to arbitrary subsets of Δ and Δ^*. They all commute, so the three operators $*$, c, and $\vee$ act like the three nontrivial elements of the Klein four group:

$$\begin{array}{ccccccc} X^{**} & = & X^{cc} & = & X^{\vee\vee} & = & X \\ X^{*c} & = & X^{c*} & = & X^\vee & & \\ X^{*\vee} & = & X^{\vee*} & = & X^c & & \\ X^{\vee c} & = & X^{c\vee} & = & X^* & & \end{array}$$

The orbit of a complex X under the action of this group can be understood via the Cayley diagram for the generators $*$, c

$$\begin{array}{ccc} X & \stackrel{*}{\longleftrightarrow} & X^* \\ {\scriptstyle c}\updownarrow & & {\scriptstyle c}\updownarrow \\ X^c & \stackrel{*}{\longleftrightarrow} & X^\vee \end{array}$$

where the left column lives in Δ, and the right column lives in Δ^*. The identity and $\vee$ preserve complexes and cocomplexes, while $*$ and c interchange complexes and cocomplexes, corresponding to the quotient

$$\{1, *, c, \vee\} \;/\; \{1, \vee\} \;\cong\; \{1, -1\}.$$

Lemma 6.4 *$(Y, X)^*$ is the simplicial pair $(X^\vee, Y^\vee)$.*

Proof. $(Y, X)^* = (Y \backslash X)^* = Y^* \backslash X^* = X^\vee \backslash Y^\vee = (X^\vee, Y^\vee)$. ∎

It follows that polarity takes the complex $X = (X, \emptyset)$ to the cocomplex $X^* = (\Delta^*, X^\vee)$, and takes the cocomplex $U = (\Delta, U^c)$ to the complex $X^* = (U^*, \emptyset)$.

Example 6.5 Let $\mathbf{n} = 1, 2, 3$. Then

$$\begin{array}{rcl} \{\}^\vee & = & \{\emptyset,\ 1,\ 2,\ 3,\ 12,\ 13,\ 23,\ 123\}, \\ \{\emptyset\}^\vee & = & \{\emptyset,\ 1,\ 2,\ 3,\ 12,\ 13,\ 23\}, \\ \{\emptyset,\ 1\}^\vee & = & \{\emptyset,\ 1,\ 2,\ 3,\ 12,\ 13\}, \\ \{\emptyset,\ 1,\ 2\}^\vee & = & \{\emptyset,\ 1,\ 2,\ 3,\ 12\}, \\ \{\emptyset,\ 1,\ 2,\ 3\}^\vee & = & \{\emptyset,\ 1,\ 2,\ 3\}. \end{array}$$

We use a compressed notation, for example writing the face $\{1\}$ as 1 and the face $\{1, 2\}$ as 12. This list can be interpretted either as giving the duals in Δ^* of subcomplexes of Δ, or vice-versa. It is therefore complete, taking into account that $X^{\vee\vee} = X$. Note that in general, the void complex is dual to the full $(n-1)$-simplex.

It can be helpful to have a geometric picture of polarity. Equip Δ with a realization $|\Delta| \subset \mathbb{R}^{n-1}$, such that the interior of $|\Delta|$ contains the origin. The polar Δ^* can then be equipped with the realization

$$|\Delta^*| \;=\; \left\{\, \mathbf{y} \in (\mathbb{R}^{n-1})^* \;\mid\; \langle \mathbf{x}, \mathbf{y} \rangle \leq 1 \text{ for all } \mathbf{x} \in |\Delta| \,\right\}$$

(compare [Zie94, 2.3]). For each face $F \subset \Delta$, its polar face $F^* \subset \Delta^*$ is determined by the rule

$$|F^*| \;=\; \{\, \mathbf{y} \in |\Delta^*| \mid \langle \mathbf{x}, \mathbf{y} \rangle = 1 \text{ for all } \mathbf{x} \in |F| \,\}.$$

The combinatorial relationship between Δ and Δ^* is independent of the choice of realization $|\Delta|$: If we number each vertex of $|\Delta^*|$ by the complement of its polar facet in $|\Delta|$, then $F^* = F^c$ for each face $F \in \Delta$, agreeing with our previous definition for F^*. Polarity defined this way is a correspondence between closed faces of $|\Delta|$ and of $|\Delta^*|$.

Now view $|\Delta|$ as a cell complex by associating the faces of Δ with the relative interiors of the corresponding faces of $|\Delta|$; do the same for $|\Delta^*|$. $|\Delta|$ and $|\Delta^*|$ are now the disjoint unions of their cells; redefine polarity to be a correspondence between the cells of $|\Delta|$ and of $|\Delta^*|$. The realization $|(Y, X)|$ of a simplicial pair is the union of the corresponding cells, and the realization $|(Y, X)^*|$ of its polar is therefore the union of the polar cells. If $X \subset \Delta$ is a complex, then $|X|$ is a closed subset of $|\Delta|$, and $|X^*|$ is an open subset of $|\Delta^*|$.

In this language, the dual complex $|X^\vee|$ is the complement of the polar cocomplex $|X^*|$ in $|\Delta^*|$. Thus, $X^\vee$ lives in Δ^*.

6.2 Homology and cohomology

We recall the construction of the reduced simplicial homology and cohomology groups, adapted to our setting from [BH93, 5.3], [Mun84], and [Spa66]. We define a *polar pairing* between Δ and Δ^*, leading to Theorem 6.2 which identifies cohomology computations in Δ with homology computations in Δ^*.

Write $F = [j_0, \ldots, j_i]$ if $F = \{j_0, \ldots, j_i\}$ is an i-face of Δ with $j_0 < \ldots < j_i$, and write $F = [\,]$ if $F = \emptyset$. Given a simplicial complex $X \subset \Delta$, define $X_i = \{\, F \in X \mid |F| = i + 1 \,\}$ to be the i-faces of X, and for $-1 \le i \le n - 1$ define the chain groups

$$\mathcal{C}_i(X) = \bigoplus_{F \in X_i} \mathbb{Z}F.$$

For faces F, F' of Δ with $|F| = |F'| + 1$, define the *incidence function* ε by $\varepsilon(F, F') = 0$ if $F \not\supset F'$, and by $\varepsilon(F, F') = \sigma(F \backslash F', F')$ if $F \supset F'$, where σ gives the sign of the permutation that sorts the concatenated sequence $F \backslash F', F'$ into ascending order.

For $i \ge 0$, define the differential $\partial : \mathcal{C}_i(X) \longrightarrow \mathcal{C}_{i-1}(X)$ by

$$\partial[j_0, \ldots, j_i] \;=\; \sum_{\ell=0}^{i} (-1)^\ell \, [j_0, \ldots, \widehat{j_\ell}, \ldots, j_i].$$

The differential ∂ can equivalently be defined by

$$\partial F \;=\; \sum_{F' \in \Delta_{i-1}} \varepsilon(F, F') \, F'.$$

Note that these definitions makes no reference to X. In other words, if $F \in \mathcal{C}_i(\Delta)$ is supported on X_i, then $\partial F \in \mathcal{C}_{i+1}(\Delta)$ is supported on X_{i+1}, so ∂ is compatible with the inclusions $\mathcal{C}_i(X) \hookrightarrow \mathcal{C}_i(\Delta)$.

Define the *augmented oriented chain complex of X* to be

$$\mathcal{C}(X) \;:\; 0 \longrightarrow \mathcal{C}_{n-1}(X) \xrightarrow{\partial} \mathcal{C}_{n-2}(X) \longrightarrow \ldots \longrightarrow \mathcal{C}_0(X) \xrightarrow{\partial} \mathcal{C}_{-1}(X) \longrightarrow 0.$$

By the preceding discussion, $\mathcal{C}(X) \hookrightarrow \mathcal{C}(\Delta)$ is a natural inclusion of chain complexes.

Let G be an abelian group. The *i-th reduced simplicial homology of X with values in G* is defined by

$$H_i(X;G) \;=\; H_i(\mathcal{C}(X) \otimes G).$$

The *i-th reduced simplicial cohomology of X with values in G* is defined by

$$H^i(X;G) \;=\; H_i(\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}(X), G)).$$

Write $H_i(X) = H_i(X;\mathbb{Z})$ and $H^i(X) = H^i(X;\mathbb{Z})$. If G is a field k, then $H_i(X;k)$ and $H^i(X;k)$ are dual k-vector spaces for each i, and in particular have the same dimension.

We elaborate the definition of $H^i(X;G)$. For $-1 \le i \le n-1$, define

$$\mathcal{C}^i(X) = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}_i(X), \mathbb{Z}).$$

For $i < n-1$, the differential $\partial : \mathcal{C}^i(X) \longrightarrow \mathcal{C}^{i+1}(X)$ can be understood from the diagram

$$\begin{array}{ccc} \mathcal{C}_{i+1}(X) & \xrightarrow{\partial} & \mathcal{C}_i(X) \\ {\scriptstyle (\partial\alpha)}\downarrow & & {\scriptstyle \alpha}\downarrow \\ \mathbb{Z} & = & \mathbb{Z} \end{array}$$

In other words, ∂ is defined by its effect on elements $\alpha \in \mathcal{C}^i(X)$ and $F \in \mathcal{C}_{i+1}(X)$:

$$\begin{aligned} (\partial\alpha)(F) &= \alpha(\partial F) \\ &= \sum_{F' \in \Delta_i} \varepsilon(F, F')\, \alpha(F'). \end{aligned}$$

Written out, this looks like

$$(\partial\alpha)([j_0, \ldots, j_{i+1}]) = \sum_{\ell=0}^{i+1} (-1)^\ell\, \alpha([j_0, \ldots, \widehat{j_\ell}, \ldots, j_{i+1}]),$$

which is a pattern arising, for example, in the definition of Čech cohomology.

Define the *augmented oriented cochain complex of* X to be

$$\mathcal{C}^*(X) \;:\; 0 \longrightarrow \mathcal{C}^{-1}(X) \xrightarrow{\partial} \mathcal{C}^0(X) \longrightarrow \ldots \longrightarrow \mathcal{C}^{n-2}(X) \xrightarrow{\partial} \mathcal{C}^{n-1}(X) \longrightarrow 0.$$

The *i-th reduced simplicial cohomology of X with values in G* can equivalently be defined by

$$H^i(X;G) \;=\; H^i(\mathcal{C}^*(X) \otimes G).$$

Note that the equivalent definitions for $\partial\alpha$ also make no reference to X. Because X is a complex, the faces F' used in the definition of $(\partial\alpha)(F)$ all belong to X. However, if we view α as an element of $\mathcal{C}^i(\Delta)$ by defining $\alpha(F) = 0$ for $F \notin X$, then $\partial\alpha$ need not be supported on X. In other words, $\partial\alpha$ may fail to vanish on faces $F \notin X$.

Instead, if an element $\alpha \in \mathcal{C}^i(\Delta)$ is supported on a cocomplex $U \subset \Delta$, then $\partial\alpha$ is also supported on U. Thus, it makes sense to define the cochain complex $\mathcal{C}^*(U)$ with groups $\mathcal{C}^i(U)$. Then $\mathcal{C}^*(U) \hookrightarrow \mathcal{C}^*(\Delta)$ is a natural inclusion of chain complexes, and $\mathcal{C}^*(X)$ is naturally the quotient $\mathcal{C}^*(\Delta)/\mathcal{C}^*(X^c)$.

Make identical definitions for the polar Δ^* of Δ. We use polarity to give explicit bases for the groups of $\mathcal{C}^*(X)$:

Definition 6.1 *The* polar pairing $\langle\, , \rangle : \Delta \times \Delta^* \to \{0, 1, -1\}$ *is defined by*

$$\langle F, F^* \rangle \;=\; (-1)^{\left\lfloor \frac{|F|}{2} \right\rfloor} \sigma(F, F^c),$$

and by $\langle F, F' \rangle \;=\; 0$ *if* $F' \neq F^*$.

Again, σ gives the sign of the permutation F, F^c. Via this pairing, the polar faces $\{\; F^* \mid F \in X_i \;\}$ form a basis for $\mathcal{C}^i(X)$, so we can write

$$\mathcal{C}^i(X) = \bigoplus_{F \in X_i} \mathbb{Z}F^*.$$

This cochain group for Δ also looks like a chain group for Δ^*. Indeed, it looks like a chain group for the polar cocomplex $X^* \subset \Delta^*$.

Theorem 6.2 *The polar pairing induces an arrow-reversing isomorphism between the cochain complex $\mathcal{C}^*(\Delta)$ and the chain complex $\mathcal{C}_*(\Delta^*)$, identifying $\mathcal{C}^i(\Delta)$ with $\mathcal{C}_{n-i-2}(\Delta^*)$ for $-1 \leq i \leq n-1$.*

Proof. Because the polar F^* of an i-face F of Δ is an $(n-i-2)$-face of Δ^*, the polar pairing identifies $\mathcal{C}^i(\Delta)$ with $\mathcal{C}_{n-i-2}(\Delta^*)$. We need to show that the cochain differential of Δ and the chain differential of Δ^* agree under this construction. In other words, we need to show that the diagram

$$\begin{array}{ccc} \mathcal{C}^i(\Delta) & \xrightarrow{\partial} & \mathcal{C}^{i+1}(\Delta) \\ \downarrow & & \downarrow \\ \mathcal{C}_{n-i-2}(\Delta^*) & \xrightarrow{\partial} & \mathcal{C}_{n-i-3}(\Delta^*) \end{array}$$

commutes for each i. The sign pattern $(-1)^{\left\lfloor \frac{|F|}{2} \right\rfloor} = +,+,-,-,\ldots$ for $|F| = 0,1,2,3,\ldots$ keeps this diagram from anticommuting for odd i.

Working one coefficient at a time, it suffices to check that

$$\varepsilon(F,F') \;=\; \langle F,F^*\rangle\,\langle F',F'^*\rangle\,\varepsilon(F'^c,F^c)$$

for faces $F \supset F'$ with $|F| = |F'|+1$. What we instead establish is that

$$\varepsilon(F,F') \;=\; (-1)^{|F'|}\,\sigma(F,F^c)\,\sigma(F',F'^c)\,\varepsilon(F'^c,F^c),$$

leaving the problem of how to divvy up the sign $(-1)^{|F'|}$ between F and F'. This is accomplished by the term $(-1)^{\left\lfloor \frac{|F|}{2} \right\rfloor}$ in $\langle F,F^*\rangle$.

We compute these incidence functions by embedding them in permutations of $\mathbf{n}$, and canceling off the sign of the unwanted part of these permutations. Let $[j] = F\backslash F' = F'^c\backslash F'$. Then

$$\begin{aligned} \varepsilon(F,F') &= \sigma([j],F',F^c)\,\sigma(F,F^c), \\ \varepsilon(F'^c,F^c) &= \sigma([j],F^c,F')\,\sigma(F'^c,F') \\ &= (-1)^{|F'||F^c|+|F'||F'^c|}\,\sigma([j],F',F^c)\,\sigma(F',F'^c). \end{aligned}$$

The exponent $|F'||F^c|+|F'||F'^c|$ reduces mod 2 to $|F'|$, as desired. ∎

This theorem can be understood more conceptually as the duality of the exterior algebra; see [BH93, Prop. 1.6.10] and [BH95, Lemma 1.2]. In short, one identifies $\mathcal{C}_i(\Delta)$ with $\wedge^{i+1}\mathbb{Z}^n$ by identifying the basis element $F \in \mathcal{C}_i(\Delta)$ with the exterior product $e_F = \wedge_{j\in F}\, e_j$ for a basis $e_1,\ldots,e_n$ of $\mathbb{Z}^n$. Wedging e_F with e_{F^c} gives an element of $\mathcal{C}_{n-1}(\Delta) = \wedge^n\mathbb{Z}^n \cong \mathbb{Z}$, so the orientation map $\sigma : e_1\wedge\ldots\wedge e_n \mapsto 1$ induces an isomorphism $\mathcal{C}^i(\Delta) \cong \mathcal{C}_{n-i-2}(\Delta^*)$ via $F^*(F') = (-1)^{\left\lfloor \frac{|F|}{2} \right\rfloor}\sigma(e_{F'}\wedge e_{F^c})$. Phrased this way, Theorem 6.2 starts to resemble Serre duality, for example, where the orientation map takes the form of an isomorphism of sheaf cohomology $H^n(X,\omega_X) \cong k$ for the canonical sheaf $\omega_X = \wedge^n\,\Omega_{X/k}$; see [Har77, Thm. III.7.1].

Polarity renders cohomology as elementary as homology. It is now apparent that the cochain complex $\mathcal{C}^*(U)$ for a cocomplex $U \subset \Delta$ can be identified with the chain complex $\mathcal{C}(U^*)$ for the polar simplicial complex $U^* \subset \Delta^*$. Conversely, we can define the chain complex $\mathcal{C}(U)$ via polarity as the quotient $\mathcal{C}(\Delta)/\mathcal{C}(U^c)$ polar to the quotient

$$\mathcal{C}^*(U^*) = \mathcal{C}^*(\Delta^*)/\mathcal{C}^*(U^\vee).$$

This is a special case of relative homology, which is the subject of the next section.

6.3 Relative homology and cohomology

We recall the construction of the reduced homology and cohomology groups of a simplicial pair (Y, X). Define $\mathcal{C}(Y, X)$ by $\mathcal{C}_i(Y, X) = \mathcal{C}_i(Y)/\mathcal{C}_i(X)$. The *$i$-th reduced relative simplicial homology of* (Y, X) *with values in* G is defined by

$$H_i(Y, X; G) \;=\; H_i(\mathcal{C}(Y, X) \otimes G).$$

Note that $H_i(Y, X; G)$ specializes to $H_i(Y; G)$ when X is the void complex $\{\}$.

Lemma 6.1 *Let $(Y, X) \subset \Delta$ be a simplicial pair. If Y is acyclic, then there are natural isomorphisms $H_i(Y, X; G) \cong H_{i-1}(X; G)$ for all i.*

Proof. The short exact sequence of complexes

$$0 \longrightarrow \mathcal{C}(X) \longrightarrow \mathcal{C}(Y) \longrightarrow \mathcal{C}(Y, X) \longrightarrow 0$$

is split, because the images of the groups of $\mathcal{C}(X)$ are direct summands of the corresponding groups of $\mathcal{C}(Y)$. Thus, it yields the long exact *homology sequence of the pair* (Y, X)

$$\ldots \longrightarrow H_i(Y; G) \longrightarrow H_i(Y, X; G) \longrightarrow H_{i-1}(X; G) \longrightarrow H_{i-1}(Y; G) \longrightarrow \ldots$$

When Y is acyclic, this long exact sequence yields the sequences

$$0 \longrightarrow H_i(Y, X; G) \longrightarrow H_{i-1}(X; G) \longrightarrow 0$$

for each i, giving the desired isomorphisms. ∎

The definition of $\mathcal{C}^*(Y, X)$ is dual to that of $\mathcal{C}(Y, X)$. The groups $\mathcal{C}^i(Y, X)$ are kernels, defined by the split short exact sequence of cochain complexes

$$0 \longrightarrow \mathcal{C}^*(Y, X) \longrightarrow \mathcal{C}^*(Y) \longrightarrow \mathcal{C}^*(X) \longrightarrow 0$$

The *i-th reduced relative simplicial cohomology of* (Y, X) *with values in* G is defined by

$$H^i(Y, X; G) \;=\; H^i(\mathcal{C}^*(Y, X) \otimes G).$$

In particular, the relative cohomology of the simplicial pair (Δ, X) is the cohomology of the cochain complex $\mathcal{C}^*(X^c) \otimes G$.

Lemma 6.2 *Let $(Y, X) \subset \Delta$ be a simplicial pair. If Y is acyclic, then there are natural isomorphisms $H^i(Y, X; G) \cong H^{i-1}(X; G)$ for all i.*

Proof. Reversing arrows in the proof of Lemma 6.1, we obtain the long exact *cohomology sequence of the pair* (Y, X)

$$\ldots \longrightarrow H^{i-1}(Y; G) \longrightarrow H^{i-1}(X; G) \longrightarrow H^i(Y, X; G) \longrightarrow H^i(Y; G) \longrightarrow \ldots$$

giving the corresponding isomorphisms. ∎

6.4 The homology and cohomology of dual complexes

Returning to polarity, we are now able to relate the cohomology of a simplicial pair (Y, X) to the homology of its polar pair $(Y, X)^* = (X^\vee, Y^\vee)$, and the cohomology of a simplicial complex X to the homology of its dual complex $X^\vee$.

The following is a duality theorem for polar pairs:

Theorem 6.1 *Let $(Y, X) \subset \Delta$ be a simplicial pair. There are isomorphisms*

$$H_i(Y, X; G) \cong H^{n-i-2}(X^\vee, Y^\vee; G)$$

and

$$H^i(Y, X; G) \cong H_{n-i-2}(X^\vee, Y^\vee; G).$$

Proof. The two isomorphisms are polar; we prove the second. It follows immediately from Theorem 6.2, and the identifications

$$\mathcal{C}^i(Y, X) \otimes G \;=\; \bigoplus_{F \in (Y \setminus X)_i} GF^* \;=\; \bigoplus_{F^* \in (X^\vee \setminus Y^\vee)_{n-i-2}} GF^* \;=\; \mathcal{C}_{n-i-2}(X^\vee, Y^\vee) \otimes G.$$

∎

In other words, $\mathcal{C}^*(Y, X)$ is more easily understood in terms of the simplicial cocomplexes $U = X^c$ and $V = Y^c$. Rewriting

$$0 \longrightarrow \mathcal{C}^*(Y, X) \longrightarrow \mathcal{C}^*(Y) \longrightarrow \mathcal{C}^*(X) \longrightarrow 0$$

as

$$0 \;\longrightarrow\; \frac{\mathcal{C}^*(U)}{\mathcal{C}^*(V)} \;\longrightarrow\; \frac{\mathcal{C}^*(\Delta)}{\mathcal{C}^*(V)} \;\longrightarrow\; \frac{\mathcal{C}^*(\Delta)}{\mathcal{C}^*(U)} \;\longrightarrow\; 0$$

reveals $\mathcal{C}^*(Y, X)$ to be the quotient $\mathcal{C}^*(U)/\mathcal{C}^*(V)$. Polarity allows us to recognize this quotient, up to indexing and the direction of the arrows, as the quotient $\mathcal{C}^*(X^\vee)/\mathcal{C}^*(Y^\vee)$ which computes the relative homology of the polar pair $(X^\vee, Y^\vee)$.

The following is a duality theorem for dual complexes:

Theorem 6.2 (Alexander duality) *Let $X \subset \Delta$ be a simplicial complex, and let $X^\vee$ be its dual complex. There are isomorphisms*

$$H_i(X;G) \cong H^{n-i-3}(X^\vee;G)$$

and

$$H^i(X;G) \cong H_{n-i-3}(X^\vee;G).$$

Proof. The two isomorphisms are polar; we prove the second. Because Δ is acyclic, $H^i(X;G) \cong H^{i+1}(\Delta,X;G)$ by Lemma 6.2. The result follows because $H^{i+1}(\Delta,X;G) \cong H_{n-i-3}(X^\vee,\emptyset;G)$ by Theorem 6.1, and $(X^\vee,\emptyset) = X^\vee$. ∎

Theorem 6.1 and Theorem 6.2 can be understood together in terms of the diagram

$$\begin{array}{ccccccc}
\longrightarrow & H_i(Y) & \longrightarrow & H_i(Y,X) & \longrightarrow & H_{i-1}(X) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H^{n-i-3}(Y^\vee) & \longrightarrow & H^{n-i-3}(X^\vee,Y^\vee) & \longrightarrow & H^{n-i-3}(X^\vee) & \longrightarrow
\end{array}$$

Polarity turns the homology sequence of the pair (Y,X) into the cohomology sequence of the pair $(X^\vee,Y^\vee)$, and vice-versa.

Example 6.3 Returning to Example 6.5, we consider those subcomplexes of Δ for $n=3$ that have homology.

$$\begin{array}{rcl}
H_{-1}(\{\emptyset\}) & \cong & H^1(\{\emptyset,\,1,\,2,\,3,\,12,\,13,\,23\}) \;\cong\; \mathbb{Z}, \\
H_0(\{\emptyset,\,1,\,2\}) & \cong & H^0(\{\emptyset,\,1,\,2,\,3,\,12\} \;\cong\; \mathbb{Z}, \\
H_0(\{\emptyset,\,1,\,2,\,3\}) & \cong & H^0(\{\emptyset,\,1,\,2,\,3\}) \;\cong\; \mathbb{Z}^2.
\end{array}$$

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