Extremal Betti Numbers and Applications to Monomial Ideals

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Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field k, let M be a graded S-module, and let

$$F_\bullet : \quad 0 \longrightarrow F_r \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a minimal free resolution of M over S. As usual, we define the associated (graded) Betti numbers $\beta_{i,j} = \beta_{i,j}(M)$ by the formula

$$F_i = \oplus_j S(-j)^{\beta_{i,j}}.$$

Recall that the (Mumford-Castelnuovo) regularity of M is the least integer ρ such that for each i all free generators of F_i lie in degree $\leq i + \rho$, that is $\beta_{i,j} = 0$, for $j > i + \rho$. In terms of *Macaulay* [Mac] regularity is the number of rows in the diagram produced by the "betti" command.

A Betti number $\beta_{i,j} \neq 0$ will be called *extremal* if $\beta_{l,r} = 0$ for all $l \geq i$, $r \geq j+1$ and $r - l \geq j - i$, that is if $\beta_{i,j}$ is the nonzero top left "corner" in a block of zeroes in the *Macaulay* "betti" diagram. In other words, extremal Betti numbers account for "notches" in the shape of the minimal free resolution and one of them computes the regularity. In this sense, extremal Betti numbers can be seen as a refinement of the notion of Mumford-Castelnuovo regularity.

In the first part of this note we connect the extremal Betti numbers of an arbitrary submodule of a free S-module with those of its generic initial module. In the second part, which can be read independently of the first, we relate extremal multigraded Betti numbers in the minimal resolution of a square free monomial ideal with those of the monomial ideal corresponding to the Alexander dual simplicial complex.

Our techniques give also a simple geometric proof of a more precise version of a recent result of Terai [Te97] (see also [FT97] for a homological reformulation and related results), generalizing Eagon and Reiner's theorem [ER96] that a Stanley-Reisner ring is Cohen-Macaulay if and only if the homogeneous ideal corresponding to the Alexander dual simplicial complex has a linear resolution.

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1 GINs and extremal Betti numbers

A theorem of Bayer and Stillman ([BaSt87], [Ei95]) asserts that if M is a graded submodule of a free $S = k[x_1, \ldots, x_n]$-module F, and one considers the degree reverse lexicographic monomial order, then after a generic change of coordinates, the modules F/M and $F/\operatorname{In}(M)$ have the same regularity and the same depth (in this situation the module $\operatorname{In}(M)$ is known as $\operatorname{Gin}(M)$).

We generalize this result to show that corners in the minimal resolution of F/M correspond to corners in the minimal resolution of $F/\operatorname{Gin}(M)$ and that moreover the extremal Betti numbers of F/M and of $F/\operatorname{Gin}(M)$ match. The proof, inspired by the approach in [Ei95], shows that each extremal Betti number of F/M or respectively $F/\operatorname{Gin}(M)$ is computed by the unique extremal Betti number of a finite length submodule.

We use the same notation as in the introduction. Let M be a graded S-module, and let

$$F_\bullet : \quad 0 \longrightarrow F_r \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a minimal free resolution of M. As usual, we define $\operatorname{Syz}_l(M) := \operatorname{Ker}(F_l \longrightarrow F_{l-1})$ to be the l^{th} syzygy module of M.

We say that M is *(m,l)-regular* iff $\operatorname{Syz}_l(M)$ is $(m+l)$-regular (in the classical sense); that is to say that all generators of F_j for $l \le j \le r$ have degrees $\le j + m$.

We also define the *l-regularity* of M, denoted in the sequel as l-reg(M), to be the regularity of the module $\operatorname{Syz}_l(M)$; it is the least integer m such that M is (m,l)-regular.

It is easy to see that $\operatorname{reg}(M) = 0$-reg$(M)$ and l-reg$(M) \le (l-1)$-reg(M). Strict inequality occurs only at extremal Betti numbers, which thus pinpoint “jumps” in the regularity of the successive syzygy modules. In this case, if $m = l$-reg(M), we say that (l,m) is a *corner* of M and that $\beta_{l,m+l}(M)$ is an *extremal Betti number* of M.

Proposition 1.1 *M is (m,l)-regular iff*

$$\operatorname{Ext}^j(M,S)_k = 0 \quad \textit{for all } j \ge l \textit{ and all } k \le -m-j-1.$$

If moreover (l,m) is a corner of M, then $\beta_{l,m+l}(M)$ is equal to the number of minimal generators of $\operatorname{Ext}^l(M,S)$ in degree $(-m-l)$.

Proof. The first part follows from [Ei95, Proposition 20.16] since M is (m,l)-regular iff $\operatorname{Syz}_l(M)$ is m-regular. For the second part notice that by degree considerations the nonzero generators of F_l in degree $m+l$ correspond to nontrivial cycles of $\operatorname{Ext}^l(M,S)_{-m-l}$. ∎

Finite length modules have exactly one extremal Betti number:

Theorem 1.2 *If M is a finite length module, and $\beta_{l,m+l}$ is an extremal Betti number of M, then $l = n$ and $\beta_{n,m+n}$ is the last nonzero value in the Hilbert function of M.*

Proof. Since M has finite length it follows that $\mathrm{Ext}^j(M,S)=0$ for all $j<n$. On the other hand $\mathrm{Ext}^n(M,S)_{-n-t}=\mathrm{Ext}^n(M,S(-n))_{-t}\cong\mathrm{Hom}_k(M,k)_{-t}=\mathrm{Hom}_k(M_t,k)$, from which the claim follows easily. ∎

Corollary 1.3 *Let F be a graded free S-module with basis, and let M a graded submodule of F such that F/M has finite length. Then the extremal Betti number of F/M is equal to the extremal Betti number of $F/\operatorname{Gin}(M)$ (with respect to the graded reverse lexicographic order).*

Proof. F/M and $F/\operatorname{Gin}(M)$ have the same Hilbert function. ∎

Proposition 1.4 *If $0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0$ is a short exact sequence of graded finitely generated S-modules, then*
a) $l\text{-reg}(A)\le\max(l\text{-reg}(B),(l+1)\text{-reg}(C)+1)$.
b) $l\text{-reg}(B)\le\max(l\text{-reg}(A),l\text{-reg}(C))$.
c) $l\text{-reg}(C)\le\max((l-1)\text{-reg}(A)-1,l\text{-reg}(B))$.
d) If A has finite length, then $l\text{-reg}(B)=\max(l\text{-reg}(A),l\text{-reg}(C))$.

Proof. The proof follows by examining the appropriate l^{th} graded pieces of the long exact sequence in $\mathrm{Ext}(\cdot,S)$. See the analogue statement for regularity in [Ei95, Corollary 20.19]. ∎

Lemma 1.5 *If F is a finitely generated graded free S-module, M a graded submodule of F, and x a linear form of S such that the module $(M:x)/M$ has finite length, then*

$$l\text{-reg}(F/M)=\max(l\text{-reg}((M:x)/M),l\text{-reg}(F/(M:x))).$$

Proof. The claim follows from the short exact sequence

$$0\longrightarrow(M:x)/M\longrightarrow F/M\longrightarrow F/(M:x)\longrightarrow 0$$

and Proposition 1.4. ∎

We may now state and prove the analogue of Bayer and Stillman's ([BaSt87], [Ei95]) result on regularity:

Theorem 1.6 *Let F be a finitely generated graded free S-module with basis, let M be a graded submodule of F, and let $\beta_{i,j}$ denote the ith graded Betti number of F/M, and $\beta^{gin}_{i,j}$ the ith graded Betti number of $F/\operatorname{Gin}(M)$. Then*

$$l\text{-reg}(F/M)=l\text{-reg}(F/\operatorname{Gin}(M)).$$

If moreover (l,m) is a corner of F/M, then

$$\beta_{l,m+l}=\beta^{gin}_{l,m+l}.$$

Proof. We can assume that $\operatorname{In}(M) = \operatorname{Gin}(M)$. If x_n is a nonzero divisor of F/M the claims follow by induction on the number of variables: the Betti numbers of F/M over S are equal to the Betti numbers of $F/(x_nF, M)$ over S/x_n and the initial module of M over S is the same as the initial module of M/x_nM over S/x_n. Therefore we will assume in the sequel that x_n is a zero divisor of F/M.

We prove the first part of the theorem. Since $(M : x_n)/M$ is a finite length module, n-$\operatorname{reg}((M : x_n)/M) = \operatorname{reg}((M : x_n)/M) = \operatorname{reg}((\operatorname{In}(M) : x_n)/\operatorname{In}(M)) = n$-$\operatorname{reg}((\operatorname{In}(M) : x_n)/\operatorname{In}(M))$ and the first part of the theorem follows by Lemma 1.5 and induction on the sum of degrees of the elements in a reduced Gröbner basis of M. (Recall that with F and M as above, using the reverse lexicographic order, if $\{g_1, \ldots, g_t\}$ is a (reduced) Gröbner basis for M and $g_i' := g_i/\operatorname{GCD}(x_n, g_i)$, then $\{g_1', \ldots, g_t'\}$ is a (reduced) Gröbner basis for the module $(M : x_n)$.)

Assume first that (m, n) is a corner of F/M, so in particular $\operatorname{Ext}^n(F/M, S) \neq 0$. Let $N = H^0_{\mathbf{m}}(F/M)$ be the set of all elements in F/M that are annihilated by some power of the ideal $\mathbf{m} \subset S$ generated by the variables, and let $L := (F/M)/N$. From the short exact sequence

$$0 \longrightarrow H^0_{\mathbf{m}}(F/M) \longrightarrow F/M \longrightarrow L \longrightarrow 0,$$

we conclude that $\operatorname{Ext}^n(F/M, S) \cong \operatorname{Ext}^n(H^0_{\mathbf{m}}(F/M), S)$, since L has no torsion and thus $\operatorname{Ext}^n(L, S) = 0$. By the first part and Theorem 1.2 the last nonzero value, say w, of the Hilbert function of $(M : x_n)/M$ (or $(\operatorname{In}(M) : x_n)/\operatorname{In}(M)$) occurs in degree m. But this is also the last nonzero value of the Hilbert function of $H^0_{\mathbf{m}}(F/M)$. By Corollary 1.3 it follows that $\beta_{n,(m+n)} = w = \beta^{gin}_{n,(m+n)}$.

Finally consider a corner say (m, l), with $l < n$, in the resolution of F/M. From the short exact sequence in the proof of Lemma 1.5, it follows that $\operatorname{Ext}^l(F/M, S) \cong \operatorname{Ext}^l(F/(M : x_n), S)$ so we are done again by induction on the sum of degrees of the elements in a reduced Gröbner basis of M. ∎

Given an S-module P define $\operatorname{red}(P)$ to be $P/(H^0_{\mathbf{m}}(P) + xP)$, where x is a generic linear form. Let $P_0 = P$ and define $P_{i+1} = \operatorname{red}(P_i)$, for all $i \geq 1$.

Corollary 1.7 *Let L be a module over a polynomial ring S with free presentation $L = F/M$, and let $N = F/\operatorname{Gin}(M)$. Then for all $i \geq 0$:*

a) The Hilbert functions of $H^0_{\mathbf{m}}(L_i)$ and $H^0_{\mathbf{m}}(N_i)$ coincide,

b) The depths of L_i and N_i coincide.

c) The extremal Betti numbers of L correspond to jumps in the highest socle degrees of the L_is.

Proof. If $\operatorname{depth}(L) \geq 1$, then taking generic initial modules commutes with factoring out a generic linear form, up to semicontinuous numerical data such as depth and Hilbert function. The same thing is true if we factor out an element of highest degree of the socle of L since it corresponds to a corner by Theorem 1.6. Induction now proves *a)* and *b)*. ∎

2 Alexander duality and square-free monomial ideals

The minimal free resolution of a multigraded ideal in $S = k[x_1, \ldots, x_n]$, the polynomial ring in n variables over a field k, is obviously multigraded, and so it is natural to introduce and study in this context a multigraded analogue for "extremal Betti numbers".

We use the same notation as above. Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring let $[n] = \{1, \ldots, n\}$, and let Δ denote the set of all subsets of $[n]$. Given a simplicial complex $X \subseteq \Delta$, define the *Stanley-Reisner* ideal $I_X \subseteq S$ to be the ideal generated by the monomials corresponding to the nonfaces of X:

$$I_X \quad = \quad \langle\, \mathbf{x}^F \mid F \notin X \,\rangle.$$

I_X is a square-free monomial ideal, and every square-free monomial ideal arises in this way.

Define the *Alexander dual* simplicial complex $X^\vee \subseteq \Delta$ to be the complex obtained by successively complementing the faces of X and X itself, in either order. In other words, define

$$X^\vee \quad = \quad \{\, F \mid F^c \notin X \,\} \quad = \quad \Delta \setminus \{\, F \mid F^c \in X \,\}$$

where F^c denotes the complement $[n] \setminus F$. Defining also the *Alexander dual* ideal $I_{X^\vee}$, note the following pattern:

$$\begin{array}{ccc} X & \longleftrightarrow & I_X \\ \updownarrow & & \updownarrow \\ I_{X^\vee} & \longleftrightarrow & X^\vee \end{array}$$

The sets of faces which define X, $X^\vee$, I_X and $I_{X^\vee}$ are related horizontally by complementing with respect to Δ, and vertically by complementing with respect to $[n]$.

The following is a simplicial version of Alexander duality:

Theorem 2.1 *Let $X \subset \Delta$ be a simplicial complex. For any abelian group G, there are isomorphisms*

$$\widetilde{H}_i(X;G) \;\cong\; \widetilde{H}^{n-i-3}(X^\vee;G) \quad \text{and} \quad \widetilde{H}^i(X;G) \;\cong\; \widetilde{H}_{n-i-3}(X^\vee;G)$$

where $\widetilde{H}$ denotes reduced simplicial (co)homology.

Proof. First, suppose that X is a nonempty, proper subcomplex of the sphere $S^{n-2} = \Delta \setminus [n]$. Working with geometric realizations, Alexander duality asserts (compare [Mun84, Theorem 71.1]) that

$$\widetilde{H}_i(X;G) \;\cong\; \widetilde{H}^{n-i-3}(S^{n-2} \setminus X;G) \quad \text{and} \quad \widetilde{H}^i(X;G) \;\cong\; \widetilde{H}_{n-i-3}(S^{n-2} \setminus X;G).$$

The claim follows because $X^\vee$ is homotopy-equivalent to $S^{n-2} \setminus X$: Let X' denote the first barycentric subdivision of X. Complementing the faces of $X^\vee$ embeds $(X^\vee)'$ as a simplicial subcomplex of $(S^{n-2} \setminus X)'$. The straight-line homotopy defined by collapsing each face of $(S^{n-2} \setminus X)'$ onto its vertices not belonging to X' is a strong deformation retract of $(S^{n-2} \setminus X)'$ onto $(X^\vee)'$.

The remaining cases $X = \emptyset$, $\{\emptyset\}$, $\Delta \setminus [n]$, and Δ are easily checked by hand. ∎

Theorem 2.1 is also easily proved directly, modulo a subtle sign change. Define a pairing on faces F, $G \in \Delta$ by

$$\langle F, G \rangle \quad = \quad \begin{cases} (-1)^{\lfloor \frac{|F|}{2} \rfloor} \, \sigma(F,G), & \text{if } G = F^c \\ 0, & \text{otherwise} \end{cases}$$

where $\sigma(F,G)$ is the sign of the permutation that sorts the concatenated sequence F, G into order. This pairing allows us to reinterpret any i-chain as an $(n-i-2)$-cochain, identifying relative homology with relative cohomology. We compute

$$\widetilde{H}_i(X;G) \;\cong\; \widetilde{H}_{i+1}(\Delta, X; G) \;\cong\; \widetilde{H}^{n-i-3}(X^\vee, \emptyset; G) \;\cong\; \widetilde{H}^{n-i-3}(X^\vee; G);$$

the second isomorphism is similar. See [Bay96] for details. This formulation can also be understood as the self-duality of the Koszul complex; see [BH93, 1.6.10].

Given an arbitrary monomial ideal $I \subseteq S$, let

$$L_\bullet : \quad 0 \longrightarrow L_m \longrightarrow \ldots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow I \longrightarrow 0$$

be a minimal free resolution of I; we have $m \leq n-1$. The multigraded *Betti numbers* of I are the ranks $\beta_{i,\mathbf{b}} = \dim(L_i)_\mathbf{b}$ of the $\mathbf{b}^{\text{th}}$ graded summands $(L_i)_\mathbf{b}$ of L_i.

For each degree $\mathbf{b} \in \mathbb{N}^n$, define the following subcomplex of Δ:

$$K_\mathbf{b}(I) \quad = \quad \{ \, F \in \Delta \;|\; \mathbf{x}^{\mathbf{b}-F} \in I \, \}.$$

Here, we identify each face $F \in \Delta$ with its characteristic vector $F \in \{0,1\}^n$. The following is a characterization of the Betti numbers of I in terms of $K_\mathbf{b}(I)$:

Theorem 2.2 *The Betti numbers of a monomial ideal $I \subseteq S$ are given by*

$$\beta_{i,\mathbf{b}} \quad = \quad \dim \widetilde{H}_{i-1}(K_\mathbf{b}(I); k).$$

Proof. The groups $\mathrm{Tor}_i(I,k)$ can be computed either by tensoring a resolution of I by k, or by tensoring a resolution of k by I. Using the minimal resolution $L_\bullet$ of I, one sees that $\beta_{i,\mathbf{b}} = \dim \mathrm{Tor}_i(I,k)_\mathbf{b}$. Using the Koszul complex $K_\bullet$ of k, $\mathrm{Tor}_i(I,k)$ is also the ith homology of the complex

$$I \otimes K_\bullet : \quad 0 \longrightarrow I \otimes \wedge^n V \longrightarrow \ldots \longrightarrow I \otimes \wedge^1 V \longrightarrow I \otimes \wedge^0 V \longrightarrow 0,$$

where V is the subspace of degree one forms of S. Now, $(I \otimes \wedge^i V)_\mathbf{b}$ has a basis consisting of all expressions of the form

$$\mathbf{x}^\mathbf{b}/x_{j_1} \cdots x_{j_i} \;\otimes\; x_{j_1} \wedge \ldots \wedge x_{j_i}$$

where $\mathbf{x}^\mathbf{b}/x_{j_1} \cdots x_{j_i} \in I$. These expressions correspond $1:1$ to the $(i-1)$-faces $F = \{j_1, \ldots, j_i\}$ of $K_\mathbf{b}(I)$. Thus, one recognizes $(I \otimes K_\bullet)_\mathbf{b}$ as the augmented oriented chain complex used to compute $\widetilde{H}_{i-1}(K_\mathbf{b}(I); k)$. ∎

A striking reformulation of Theorem 2.2 for square-free monomial ideals is due to Hochster [Ho77], based on ideas of Reisner [Rei76]. For each $\mathbf{b} \in \{0,1\}^n$, let $X_\mathbf{b}$ denote the full subcomplex of X on the vertices in the support of $\mathbf{b}$.

Theorem 2.3 *Let $I_X \subseteq S$ be the square-free monomial ideal determined by the simplicial complex $X \subseteq \Delta$. We have $\beta_{i,\mathbf{b}} = 0$ unless $\mathbf{b} \in \{0,1\}^n$, in which case*

$$\beta_{i,\mathbf{b}} \quad = \quad \dim \widetilde{H}_{|\mathbf{b}|-i-2}(X_\mathbf{b}; k).$$

Proof. If $b_j > 1$ for some j then $K_\mathbf{b}(I)$ is a cone over the vertex j, so $\beta_{i,\mathbf{b}} = 0$ by Theorem 2.2. Otherwise, $X_\mathbf{b}$ is the dual of $K_\mathbf{b}(I)$ with respect to the support of $\mathbf{b}$: $F \in K_\mathbf{b}(I) \Leftrightarrow \mathbf{x}^{\mathbf{b}-F} \in I \Leftrightarrow \mathbf{b} - F \notin X$. By Theorem 2.1,

$$\widetilde{H}_{i-1}(K_\mathbf{b}(I); k) \quad \cong \quad \widetilde{H}^{|\mathbf{b}|-i-2}(X_\mathbf{b}; k).$$

Homology and cohomology groups with coefficients in k are (non-canonically) isomorphic, so the result follows by Theorem 2.2. ∎

This is essentially Hochster's original argument; he implicitly proves Alexander duality in order to interpret $\mathrm{Tor}_i(I,k)_\mathbf{b}$ as computing the homology of X_b.

Recall that the *link* of a face $F \in X$ is the set

$$\mathrm{lk}(F,X) \quad = \quad \{\, G \mid F \cup G \in X \text{ and } F \cap G = \emptyset \,\}.$$

Together with the restrictions $X_\mathbf{b}$, the links $\mathrm{lk}(F,X)$ are the other key ingredient in the study of square-free monomial ideals, dating to [Rei76]. They too have a duality interpretation, first made explicit in [ER96]: The Betti numbers $\beta^\vee_{i,\mathbf{b}}$ of $I_{X^\vee}$ can be computed using links in X.

For each $\mathbf{b} \in \{0,1\}^n$, let $\mathbf{b}^c$ denote the complement $(1,\ldots,1) - \mathbf{b}$.

Theorem 2.4 *Let $I_{X^\vee} \subseteq S$ be the square-free monomial ideal determined by the dual $X^\vee$ of the simplicial complex $X \subseteq \Delta$. We have $\beta^\vee_{i,\mathbf{b}} = 0$ unless $\mathbf{b} \in \{0,1\}^n$ and $\mathbf{b}^c \in X$, in which case*

$$\beta^\vee_{i,\mathbf{b}} \quad = \quad \dim \widetilde{H}_{i-1}(\mathrm{lk}(\mathbf{b}^c, X); k).$$

Proof. We have

$$F \in K_\mathbf{b}(I_{X^\vee}) \;\Leftrightarrow\; \mathbf{x}^{\mathbf{b}-F} \in I_{X^\vee} \;\Leftrightarrow\; (\mathbf{b}-F)^c \in X \;\Leftrightarrow\; F \in \mathrm{lk}(\mathbf{b}^c, X). \qquad ∎$$

In other words, looking at Betti diagrams (as *Macaulay* outputs) we have the following picture:
$\beta_{i,\mathbf{b}}(I_X)$:

degree					
0	1	$h_{-1}(X_{\bullet})$	$h_{-1}(X_{\bullet\bullet})$	$h_{-1}(X_{\bullet\bullet\bullet})$	$\ldots$
1	–	$h_0(X_{\bullet\bullet})$	$h_0(X_{\bullet\bullet\bullet})$	$h_0(X_{\bullet\bullet\bullet\bullet})$	$\ldots$
2	–	$h_1(X_{\bullet\bullet\bullet})$	$h_1(X_{\bullet\bullet\bullet\bullet})$	$h_1(X_{\bullet\bullet\bullet\bullet\bullet})$	$\ldots$
$\vdots$					

where for example $X_{..}$ stands for all full subcomplexes of X supported on two vertices, and
$\beta_{i,\mathbf{b}^c}(I_{X^\vee})$:

degree					
0	1	$h_{-1}(\mathrm{lk}(\bullet^c))$	$h_0(\mathrm{lk}(\bullet\bullet^c))$	$h_1(\mathrm{lk}(\therefore^c))$	$\ldots$
1	$-$	$h_{-1}(\mathrm{lk}(\bullet\bullet^c))$	$h_0(\mathrm{lk}(\therefore^c))$	$h_1(\mathrm{lk}(::^c))$	$\ldots$
2	$-$	$h_{-1}(\mathrm{lk}(\therefore^c))$	$h_0(\mathrm{lk}(::^c))$	$h_1(\mathrm{lk}(::\bullet^c))$	$\ldots$
$\vdots$					

where ^{c} stands for complementation, and again the number of dots stands for the number of vertices in the corresponding faces.

The main observation of this paper is that a simple homological relationship between restrictions and links has as a consequence the known duality results involving square-free monomial ideals. We apply it to give a sharper description of the relationship between the Betti numbers of the dual ideals I_X and $I_{X^\vee}$.

Theorem 2.5 *For each* $\mathbf{b} \in \{0,1\}^n$ *and any vertex* v *not in the support of* $\mathbf{b}$*, there is a long exact sequence*

$$\ldots \to \widetilde{H}_i(X_{\mathbf{b}}; k) \to \widetilde{H}_i(X_{\mathbf{b}+v}; k) \to \widetilde{H}_{i-1}(\mathrm{lk}(v, X_{\mathbf{b}+v}); k) \to \widetilde{H}_{i-1}(X_{\mathbf{b}}; k) \to \ldots.$$

Proof. This is the long exact homology sequence of the pair $(X_{\mathbf{b}+v}, X_{\mathbf{b}})$, in disguise; it is immediate that

$$\cdots \to \widetilde{H}_i(X_{\mathbf{b}}; k) \to \widetilde{H}_i(X_{\mathbf{b}+v}; k) \to H_i(X_{\mathbf{b}+v}, X_{\mathbf{b}}; k) \to \widetilde{H}_{i-1}(X_{\mathbf{b}}; k) \to \cdots.$$

Now, recall that $\mathrm{star}(F, X) = \{\, G \mid F \cup G \in X \,\}$; which is the acyclic subcomplex of X generated by all faces of X which contain F. It is also immediate that for all i,

$$H_i(X_{\mathbf{b}+v}, X_{\mathbf{b}}; k) \quad \cong \quad H_i(\mathrm{star}(v, X_{\mathbf{b}+v}), \mathrm{lk}(v, X_{\mathbf{b}+v}); k).$$

Because $\mathrm{star}(v, X_{\mathbf{b}+v})$ is acyclic, the long exact sequence of the second pair breaks up into isomorphisms

$$H_i(\mathrm{star}(v, X_{\mathbf{b}+v}), \mathrm{lk}(v, X_{\mathbf{b}+v}); k) \quad \cong \quad \widetilde{H}_{i-1}(\mathrm{lk}(v, X_{\mathbf{b}+v}); k)$$

for all i. Composing these isomorphisms yields the desired sequence. ∎

Theorem 2.5 can also be interpreted as the Mayer-Vietoris sequence of the two subcomplexes $X_{\mathbf{b}}$ and $\mathrm{star}(v, X_{\mathbf{b}+v})$ of $X_{\mathbf{b}+v}$, whose intersection is $\mathrm{lk}(v, X_{\mathbf{b}+v})$.

We shall exploit the exactness of this sequence at $\widetilde{H}_i(X_{\mathbf{b}})$. It is easy to observe this exactness at the level of cycles: Let α be an i-cycle supported on $X_{\mathbf{b}}$, representing a homology class in $\widetilde{H}_i(X_{\mathbf{b}}; k)$. If α maps to zero in $\widetilde{H}_i(X_{\mathbf{b}+v}; k)$, then there exist an $(i+1)$-cycle β supported on $X_{\mathbf{b}+v}$, whose boundary $\partial\beta = \alpha$. Express β as a sum $\beta_1 + \beta_2$, where β_1 is supported on $X_{\mathbf{b}}$ and every face of β_2 contains the vertex v. Define $\alpha' = \partial\beta_2 = \alpha - \partial\beta_1$. The cycle α' is supported on $\mathrm{lk}(v, X_{\mathbf{b}+v})$, and represents the same homology class as α in $\widetilde{H}_i(X_{\mathbf{b}}; k)$.

Corollary 2.6 *The Betti numbers of I_X and of $I_{X^\vee}$ satisfy the inequality*

$$\beta_{i,\mathbf{b}} \quad \leq \quad \sum_{\mathbf{b} \preceq \mathbf{c} \preceq [n]} \beta^\vee_{|\mathbf{b}|-i-1,\mathbf{c}}$$

for each $0 \leq i \leq n-1$ and each $\mathbf{b} \in \{0,1\}^n$.

Proof. The exactness at $\widetilde{H}_i(X_\mathbf{b};k)$ of the sequence of Theorem 2.5 yields the inequality

$$\dim \widetilde{H}_i(X_\mathbf{b};k) \quad \leq \quad \dim \widetilde{H}_i(\mathrm{lk}(v, X_{\mathbf{b}+v});k) \;+\; \dim \widetilde{H}_i(X_{\mathbf{b}+v};k).$$

Note that for any face F disjoint from $\mathbf{b}$, and any vertex v not in the support of $\mathbf{b}+F$,

$$\mathrm{lk}(v, \mathrm{lk}(F,X)_{\mathbf{b}+F+v}) \quad = \quad \mathrm{lk}(F+v, X_{\mathbf{b}+F+v}).$$

Applying Theorem 2.5 to $\mathrm{lk}(F,X)_{\mathbf{b}+F}$ in place of $X_\mathbf{b}$ yields the exact sequence

$$\widetilde{H}_i(\mathrm{lk}(F+v, X_{\mathbf{b}+F+v});k) \longrightarrow \widetilde{H}_i(\mathrm{lk}(F, X_{\mathbf{b}+F});k) \longrightarrow \widetilde{H}_i(\mathrm{lk}(F, X_{\mathbf{b}+F+v});k).$$

Combining the resulting inequalities while iteratively adding vertices yields

$$\dim \widetilde{H}_i(X_\mathbf{b};k) \quad \leq \quad \sum_{F \cap \mathbf{b} = \emptyset} \dim \widetilde{H}_i(\mathrm{lk}(F,X);k).$$

By Theorems 2.3 and 2.4 these dimensions can be interpreted as Betti numbers of I_X and $I_{X^\vee}$, respectively. ∎

In particular, summing up and collecting all terms of the same total degree we obtain:

Corollary 2.7 *The single graded Betti numbers of I_X and of $I_{X^\vee}$ satisfy the inequality*

$$\beta_{i,m} \quad \leq \quad \sum_{k=0}^{n-m} \binom{m+k}{k} \beta^\vee_{m-i-1,m+k},$$

for each $0 \leq i \leq n-1$ and each $m \geq i+1$.

The following consequence of Theorem 2.5 and Corollary 2.6 extends Terai's characterization of dual Stanley-Reisner ideals. Define a Betti number $\beta_{i,\mathbf{b}}$ to be *i-extremal* if $\beta_{i,\mathbf{c}} = 0$ for all $\mathbf{c} \succ \mathbf{b}$, that is all multigraded entries below $\mathbf{b}$ on the i-th column vanish in the Betti diagram as a *Macaulay* output [Mac]. Define $\beta_{i,\mathbf{b}}$ to be *extremal* if $\beta_{j,\mathbf{c}} = 0$ for all $j \geq i$, and $\mathbf{c} \succ \mathbf{b}$ so $|\mathbf{c}| - |\mathbf{b}| \geq j-i$. In other words, $\beta_{i,\mathbf{b}}$ corresponds to the "top left corner" of a box of zeroes in the multigraded Betti diagram, thus our definition agrees with the single graded one we've introduced in Section 1. Note that we have not assumed this time that $\beta_{i,\mathbf{b}} \neq 0$.

Theorem 2.8 *If $\beta^{\vee}_{i,\mathbf{b}}$ is i-extremal, then the inclusion* $\mathrm{lk}(\mathbf{b}^c, X) \subseteq X_b$ *induces an exact sequence*

$$\widetilde{H}_i(X_{\mathbf{b}}, \mathrm{lk}(\mathbf{b}^c, X); k) \longrightarrow \widetilde{H}_{i-1}(\mathrm{lk}(\mathbf{b}^c, X); k) \longrightarrow \widetilde{H}_{i-1}(X_{\mathbf{b}}; k) \longrightarrow 0$$

showing that

$$\beta^{\vee}_{i,\mathbf{b}} \geq \beta_{|\mathbf{b}|-i-1,\mathbf{b}}.$$

If $\beta^{\vee}_{i,\mathbf{b}}$ is extremal, then the above surjection is in fact an isomorphism, showing that $\beta^{\vee}_{i,\mathbf{b}} = \beta_{|\mathbf{b}|-i-1,\mathbf{b}}$.

Proof. The condition that $\beta^{\vee}_{i,\mathbf{b}}$ is i-extremal, means that the right hand sum in Corollary 2.6, applied for $|\mathbf{b}| - i - 1$ instead of i, has exactly one summand, which gives the first part of the theorem. If moreover $\beta^{\vee}_{i,\mathbf{b}}$ is extremal, then $\beta_{|\mathbf{b}|-i-1,\mathbf{b}}$ is in fact $(|\mathbf{b}| - i - 1)$-extremal so the second claim follows from the first part applied for $X^{\vee}$ instead of X. ∎

Looking at a Betti diagram as output by *Macaulay*, this result asserts in particular that any lower right corner flips via duality. Thus we can speak of "d-regularity in homological dimensions $\geq i$" and interpret it as a statement generalizing Terai's theorem [Te97] (compare [FT97, Corollary 3.2]):

Corollary 2.9 *The regularity of I_X equals the projective dimension of $S/I_{X^{\vee}}$, the Stanley-Reisner ring of its Alexander dual.*

Proof. The regularity of I_X is computed by the largest $|\mathbf{b}| - i$ such that $\beta_{i,\mathbf{b}} \neq 0$, while the projective dimension of $S/I_{X^{\vee}}$, is $1 + \max(j)$ such that $\beta^{\vee}_{j,\mathbf{c}} \neq 0$ for some $\mathbf{c} \in \{0,1\}^n$. Thus the claim follows from Theorem 2.8, because of the equality of the corresponding pairs of extremal Betti numbers. ∎

Moreover, Theorem 2.8 provides also easy proofs of classical criteria due to Reisner [Re76], and Stanley [Sta77] respectively:

Theorem 2.10 *The following conditions are equivalent:*

a) S/I_X *is a Cohen-Macaulay ring;*

b) $\widetilde{H}_i(\mathrm{lk}(F, X); k) = 0$, *for all* $F \in X$ *and* $i < \dim(\mathrm{lk}(F, X))$.

Proof. By Theorem 2.8 or Corollary 2.9, if S/I_X is Cohen-Macaulay, then $I_{X^{\vee}}$ is generated in degree $n - \dim(X) - 1$ and has a linear resolution. In other words $\beta^{\vee}_{i,\mathbf{b}} = 0$, for all $\mathbf{b}$ with $|\mathbf{b}| > n - \dim(X) - 1 + i$. By Theorem 2.4, this means that for all $F = \mathbf{b}^c \in X$, $\dim \widetilde{H}_i(\mathrm{lk}(F, X); k) = 0$ for $i < \dim(X) + |\mathbf{b}| - n = \dim(X) - |F| = \dim(\mathrm{lk}(F, X))$. To prove the implication $b) \Rightarrow a)$ it is enough to show that X is pure, and then the above argument reverses. Since $\mathrm{lk}(G, \mathrm{lk}(F, X)) = \mathrm{lk}(F \cup G, X)$, whenever $F \cup G \in X$ and $F \cap G = \emptyset$, we observe that the same cohomological vanishing holds for all proper links of X, hence by induction we may assume that they are pure. Now if $\dim(X) \geq 1$, then $b)$ also gives $\widetilde{H}_0(X; k) = \widetilde{H}_0(\mathrm{lk}(\emptyset, X); k) = 0$, so X is connected and this together with the purity of the links shows the purity of X. ∎

Since $S/I_X = S/I_{\mathrm{core}(X)}[X_i \mid v_i \in X \setminus \mathrm{core}(X)]$ (see for instance [BH93, p.232]), we have that S/I_X is Gorenstein iff $S/I_{\mathrm{core}(X)}$ is Gorenstein, thus it is enough to show the following

Theorem 2.11 *If $X = \mathrm{core}(X)$ (that is X is not a cone), then the following are equivalent:*

a) S/I_X is a Gorenstein ring (over k);

b) For all $F \in X$, $\widetilde{H}_i(\mathrm{lk}(F, X); k) = \begin{cases} k & \text{if } i = \dim(\mathrm{lk}(F, X)), \\ 0 & \text{otherwise} \end{cases}$;

Proof. To prove the implication $a) \Rightarrow b)$, we argue by induction on $|F|$: for any vertex $v \in F$, one has $S/I_{\mathrm{lk}(v,X)} = S/(I_X : (x_v))$, on the other hand S/I_X Gorenstein implies that $S/(I_X : (x_v)) = x_v S/I_X$ is also Gorenstein, whereas $\mathrm{lk}(G, \mathrm{lk}(v, X)) = \mathrm{lk}(v + G, X)$, for all $\{v\} \cup G \in X$ with $v \notin G$.

If condition $b)$ holds, then S/I_X is a Cohen-Macaulay ring by Theorem 2.10, and so X is pure. Moreover X is a pseudomanifold, that is every $(\dim(X) - 1)$-face of X lies in exactly two facets, and X is orientable, that is $\widetilde{H}_{\dim(X)}(X; k) = k$ since $X = \mathrm{lk}(\emptyset, X)$. In fact the same holds for every proper link of X. We observe next that X_{v^c} is also Cohen-Macaulay of the same dimension, for any vertex v of X. By Theorem 2.10, all we have to check is that $\widetilde{H}_i(\mathrm{lk}(F, X_{v^c}); k) = 0$, for all $F \in X_{v^c}$ and $i < \dim(\mathrm{lk}(F, X_{v^c}))$. If v is not a vertex of $\mathrm{lk}(F, X)$, or $i < \dim(\mathrm{lk}(F, X_{v^c})) - 1$ this is immediate from condition $b)$. If v is a vertex of $\mathrm{lk}(F, X)$ and $i = \dim(\mathrm{lk}(F, X_{v^c})) - 1$, this follows from the long exact sequence in Theorem 2.5

$$\to \widetilde{H}_{i+1}(\mathrm{lk}(F, X); k) \to \widetilde{H}_i(\mathrm{lk}(F \cup \{v\}, X); k) \to \widetilde{H}_i(\mathrm{lk}(F, X_{v^c}); k) \to \widetilde{H}_i(\mathrm{lk}(F, X); k)$$

where the leftmost arrow is nonzero being induced by the restriction of an orientation class. To prove that S/I_X is Gorenstein, it is enough to show that the canonical module of S/I_X is invertible, or equivalently that all generators of the canonical module lie in a single degree and S/I_X has a unique extremal Betti number whose value is one. The first condition follows now from the fact that X_{v^c} is also Cohen-Macaulay of the same dimension, for any vertex v of X, and thus $\beta_{n-\dim(X)-2,\mathbf{b}} = 0$, for all $\mathbf{b} \neq (1, 1, \ldots, 1)$. For the second condition observe that, by Corollary 2.9, $I_{X^\vee}$ has a linear resolution and thus by Theorem 2.8 the unique extremal Betti number of $I_{X^\vee}$, which is one by our hypothesis, coincides with the corresponding extremal Betti number of I_X. ∎

Remark 2.12 Recall that a simplicial complex X is called *doubly Cohen-Macaulay* if X is a Cohen-Macaulay complex and X_{v^c} is also Cohen-Macaulay of the same dimension, for any vertex v of X. The formula in Theorem 2.4 and the proof of Theorem 2.11 show that if X is doubly Cohen-Macaulay, then $I_{X^\vee}$ has a linear resolution and $\beta^\vee_{\dim(X)+1,(1,1,\ldots,1)}$ is the unique extremal Betti number.

3 Examples

We end with three examples illustrating the above described behavior of the extremal multigraded Betti numbers:

Example 3.1 Let X be a length five cycle, that is $I_X = (x_i x_{i+2})_{i\in\mathbb{Z}_5} \subset k[x_0,\ldots,x_4]$. Then $X^\vee$ is the triangulation of a Möbius band shown in Figure 1, and $I_{X^\vee} = (x_i x_{i+1} x_{i+2})_{i\in\mathbb{Z}_5}$.

degree	1	5	5	1
0	1	–	–	–
1	–	5	5	–
2	–	–	–	1

$\beta_{i,j}(I_X)$

degree	1	5	5	1
0	1	–	–	–
1	–	–	–	–
2	–	5	5	1

$\beta_{i,j}(I_{X^\vee})$

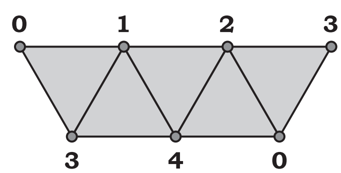

Figure 1

Example 3.2 It is easily seen that a triangulation of the torus T_1 has at least 7 vertices, and in case the triangulation has exactly 7 vertices, that the graph of its 1-skeleton is necessarily K_7, the complete graph on seven vertices. Such a triangulation X (first constructed in 1949 by Császár) is shown in Figure 2; it is unique up to isomorphism and has an automorphism group of order 42. The dual graph of its 1-skeleton divides the torus in the well known 7-colourable map (see [Wh] for more details). Thus up to a permutation, $I_X = (x_i x_{i+1} x_{i+2},\, x_i x_{i+1} x_{i+4},\, x_i x_{i+2} x_{i+4})_{i\in\mathbb{Z}_7}$. Then $I_{X^\vee} = (x_i x_{i+1} x_{i+2} x_{i+4},\, x_i x_{i+1} x_{i+2} x_{i+5})_{i\in\mathbb{Z}_7}$.

degree	1	21	49	42	15	2
0	1	–	–	–	–	–
1	–	–	–	–	–	–
2	–	21	49	42	14	2
3	–	–	–	–	1	–

$\beta_{i,j}(I_X)$

degree	1	14	21	9	1
0	1	–	–	–	–
1	–	–	–	–	–
2	–	–	–	–	–
3	–	14	21	7	1
4	–	–	–	2	–

$\beta_{i,j}(I_{X^\vee})$

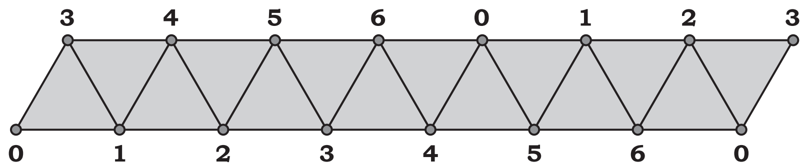

Figure 2

Example 3.3 In fact, one can construct examples of homogeneous modules with prescribed extremal graded Betti numbers, for example, by considering appropriate direct sums where each direct summand features exactly one extremal Betti number. Moreover, a classical result of Bruns [Br76] (see also [EG85, Corollary 3.13, p.56]) implies that all such possible extremal "shapes" and values of extremal Betti numbers in resolutions of modules may be realized also in minimal free resolutions of homogeneous ideals (generated by 3 elements). By passing to the generic initial ideal and then polarizing we may also construct examples of squarefree monomial ideals with the desired extremal Betti numbers.

Example 3.4 Extremal multigraded numbers need not to be also extremal in the total degree sense. For example, if X is the simplicial complex shown in Figure 3, then $I_X = (x_0x_2, x_0x_3, x_0x_4, x_1x_4)$, and $I_{X^\vee} = (x_0x_4, x_0x_1, x_2x_3x_4)$,

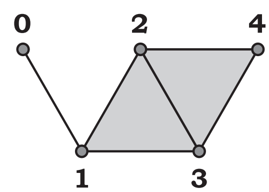

Figure 3

while the corresponding Betti diagrams are

degree	1	4	4	1
0	1	–	–	–
1	–	4	4	1

$\beta_{i,j}(I_X)$

degree	1	3	2
0	1	–	–
1	–	2	1
2	–	1	1

$\beta_{i,j}(I_{X^\vee})$

Both second order syzygies of $I_{X^\vee}$ are extremal (in the multigraded sense), but $\beta_{1,\{0,1,4\}} = \beta^\vee_{1,\{0,1,4\}}$, which is also extremal, is not extremal in the single graded sense.

References

[Bay96] Bayer, D., Monomial ideals and duality, Lecture notes, Berkeley 1995-96, available by anonymous ftp from `ftp://math.columbia.edu/pub/bayer/monomials_duality/`.

[BaSt87] Bayer, D., Stillman, M., A criterion for detecting m-regularity. *Invent. Math.* **87**, (1987), no. 1, 1–11.

[Mac] Bayer, D., Stillman, M., Macaulay: A system for computation in algebraic geometry and commutative algebra. Source and object code available for Unix and Macintosh computers. Contact the authors, or download from `ftp://math.harvard.edu/`.

[Br76] Bruns, W., "Jede" endliche freie Auflösung ist freie Auflösung eines von drei Elementen erzeugten Ideals. *J. Algebra* **39**, (1976), no. 2, 429–439.

[BH93] Bruns, W., Herzog, J., *Cohen-Macaulay Rings*, Cambridge Studies in advanced mathematics, **39**, Cambridge University Press 1993.

[Ea95] Eagon, J., Minimal resolutions of ideals associated to homology triangulated manifolds, preprint 1995.

[ER96] Eagon, J., Reiner, V., Resolutions of Stanley-Reisner rings and Alexander duality, preprint 1996.

[Ei95] Eisenbud, D., *Commutative Algebra with a View Toward Algebraic Geometry*, Springer, 1995.

[EG85] Evans E.G., Griffith, P., *Syzygies*, London Mathematical Society LNS, **106**, Cambridge University Press, 1985.

[FT97] Frühbis, A., Terai, N., Bounds for the regularity of monomial ideals, preprint 1997

[Ho77] Hochster, M.,Cohen-Macaulay rings, combinatorics and simplicial complexes, in Ring theory II, McDonald B.R., Morris, R. A. (eds), *Lecture Notes in Pure and Appl. Math.*, **26**, M. Dekker 1977.

[Mun84] Munkres, J. R., *Elements of algebraic topology*, Benjamin/Cummings, Menlo Park CA, 1984.

[Rei76] Reisner, G.A., Cohen-Macaulay quotients of polynomial rings, *Adv. in Math.*, **21**, (1976) 30–49.

[Sta77] Stanley, R., Cohen-Macaulay complexes, in *Higher Combinatorics*, (M. Aigner, ed.), Reidel, Dordrecht and Boston, 1977.

[Sta96] Stanley, R., Combinatorics and Commutative Algebra, Second edition, Progress in Math. **41**, Birkhäuser, 1996.

[Te97] Terai, N., Generalization of Eagon-Reiner theorem and h-vectors of graded rings, preprint 1997.

[Wh] White, A., Graphs, Groups and Surfaces, *North-Holland Mathematics Studies*, **8**, North Holland 1984.

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