Practice problems for first midterm, Spring '98

midterm to be held Wednesday, February 25, 1998, in class

Dave Bayer, Modern Algebra

All rings are assumed to be commutative with identity, as in our text.

[**1**] Prove that if a ring R has no ideals other than (0) and (1), then R is a field.

[**2**] Prove that if an integral domain R has only finitely many elements, then R is a field.

[**3**] Consider the ideal $I = (5, x^2) \subset \mathbb{Z}[x]$.
(a) Describe the ideal I in words. How would you tell at a glance if a given polynomial $f(x) \in \mathbb{Z}[x]$ belongs to I?
(b) List the elements of the quotient ring $R = \mathbb{Z}[x]/I$, and describe the multiplication rule for R.

[**4**] Repeat problem 3 for $I = (x^3, xy, y^3) \subset \mathbb{F}_2[x, y]$, where $\mathbb{F}_2$ is the finite field with two elements.

[**5**] The ideal $I = (2, 1+3i) \subset \mathbb{Z}[i]$ is principal, where $\mathbb{Z}[i]$ are the Gaussian integers. Find a single generator for I. (Repeat for $I = (3, 1+i)$, and $I = (6, 3+5i)$.)

[**6**] Let F be a field. The ideal $I = (1-x^2, 1-3x+3x^2-x^3) \subset F[x]$ is principal. Find a single generator for I.

[**7**] Consider the polynomial ring $R = \mathbf{C}[x, y]$, where $\mathbf{C}$ is the field of complex numbers.
(a) What are the maximal ideals of R?
(b) Which maximal ideals contain the ideal $I = (xy) \subset R$?
(c) What are the maximal ideals of the quotient ring R/I?

[**8**] Let $\mathbf{X} \subset \mathbb{R}^3$ be the union of the x-axis, y-axis, and z-axis in $\mathbb{R}^3$, where $\mathbb{R}$ denotes the real numbers. Define $I \subset \mathbb{R}[x, y, z]$ to be the set of all polynomials $f(x, y, z)$ that vanish on every point of $\mathbf{X}$. That is,

$$I \quad = \quad \{\ f(x,y,z) \in \mathbb{R}[x,y,z] \ \mid \ f(a,b,c) = 0 \ \text{ for every point } \ (a,b,c) \in X\ \}.$$

(a) Prove that I is an ideal. (Your proof only needs to use the fact that $\mathbf{X}$ is *some* subset of $\mathbb{R}^3$.)

(a) Give a set of generators for I.

[9]

(a) Prove that every positive integer can be uniquely factored into primes, up to the order of the primes.

(b) How do you need to modify this proof so it works for a polynomial ring in one variable over a field?

[10] Let R be a principal ideal domain, and let

$$I_1 \quad \subset \quad I_2 \quad \subset \quad I_3 \quad \subset \quad \cdots \quad \subset \quad I_n \quad \subset \quad \cdots$$

be an infinite ascending chain of ideals in R. Show that this chain *stabilizes*, i.e.

$$I_N \quad = \quad I_{N+1} \quad = \quad I_{N+2} \quad = \quad \cdots$$

for some N.