First Midterm Exam

Modern Algebra, Dave Bayer, February 17, 1999

Name: ____________________

ID: ____________________ **School:** ____________________

[**1**] (6 pts)	[**2**] (6 pts)	[**3**] (6 pts)	[**4**] (6 pts)	[**5**] (6 pts)	**TOTAL**

Each problem is worth 6 points, for a total of 30 points. Please work only one problem per page, and label all continuations in the spaces provided. Extra pages are available.

[**1**] Consider the ideal $I = (x^3, 2x, 4) \subset \mathbb{Z}[x]$.

(a) List representatives for the elements of the quotient ring $R = \mathbb{Z}[x]/I$, and describe the multiplication rule in R for these representatives.

(b) Is R an integral domain?

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[2] Prove that if a ring R has no ideals other than (0) and (1), then R is a field.

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[3] Prove that if an integral domain R has only finitely many elements, then R is a field. Prove any lemmas that you use.

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[4] Let $\mathbb{F}_3$ be the finite field with 3 elements. Find a polynomial $f(x)$ in the polynomial ring $\mathbb{F}_3[x]$ such that the quotient ring $\mathbb{F}_3[x]/(f(x))$ is a field with 27 elements.

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[5] Let $\mathbf{X} \subset \mathbb{R}^2$ be the union of the parabola $y = x^2$ and the point $(0, 1)$. Define $I \subset \mathbb{R}[x, y]$ to be the set of all polynomials $f(x, y)$ that vanish on every point of $\mathbf{X}$. That is,

$$I \quad = \quad \{ \ f(x, y) \in \mathbb{R}[x, y] \ \mid \ f(a, b) = 0 \ \text{ for every point } \ (a, b) \in X \ \}.$$

(a) Prove that I is an ideal.
(b) Give a set of generators for I.

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