Final Exam

Modern Algebra, Dave Bayer, May 12, 1999

Name: ______________________________

ID: ____________________ **School:** ____________________

[**1**] (5 pts)	[**2**] (5 pts)	[**3**] (5 pts)	[**4**] (5 pts)	
[**5**] (5 pts)	[**6**] (5 pts)	[**7**] (5 pts)	[**8**] (5 pts)	**TOTAL**

Please work only one problem per page, and label all continuations in the spaces provided. Extra pages are available. Check your work, where possible.

[**1**] Let $\mathbf{X} = \{a_1, \ldots, a_n\} \subset \mathbb{C}$ be a finite set of points, where $\mathbb{C}$ is the complex numbers. Define $I \subset \mathbb{C}[x]$ to be the set of all polynomials $f(x)$ that vanish on every point of $\mathbf{X}$. That is,

$$I \quad = \quad \{\ f(x) \in \mathbb{C}[x] \ \mid\ f(a_i) = 0 \ \text{ for every point } \ a_i \in \mathbf{X}\ \}.$$

(a) Prove that I is an ideal.

(b) Give a set of generators for I.

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[2] What is the minimal polynomial of $\alpha = \sqrt{3} + \sqrt{5}$ over $\mathbb{Q}$?

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[3] Let $f(x,y,z) = x^3yz + xy^3z + xyz^3$. Express $f(x)$ as a polynomial $g(\sigma_1, \sigma_2, \sigma_3)$ where σ_1, σ_2, σ_3 are the elementary symmetric functions

$$\sigma_1 = x + y + z, \quad \sigma_2 = xy + xz + yz, \quad \sigma_3 = xyz.$$

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[4] Prove the primitive element theorem: Let K be a finite extension of a field F of characteristic zero. There is an element $\gamma \in K$ such that $K = F(\gamma)$.

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[5] Prove the following theorem about Kummer extensions: Let F be a subfield of $\mathbb{C}$ which contains the pth root of unity ζ for a prime p, and let K/F be a Galois extension of degree p. Then K is obtained by adjoining a pth root to F.

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Recall that the discriminant of $f(x) = x^3 + px + q$ is $D = -4p^3 - 27q^2$.

[**6**] Let $f(x) = x^3 + 2$.

(a) What is the degree of the splitting field K of f over $\mathbb{Q}$?

(b) What is the Galois group $G = G(K/\mathbb{Q})$ of f?

(c) List the subfields L of K, and the corresponding subgroups $H = G(K/L)$ of G.

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[**7**] Let $f(x) = x^3 + x - 2$.

(a) What is the degree of the splitting field K of f over $\mathbb{Q}$?

(b) What is the Galois group $G = G(K/\mathbb{Q})$ of f?

(c) List the subfields L of K, and the corresponding subgroups $H = G(K/L)$ of G.

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[**8**] Let $f(x) = x^5 - 1$.

(a) What is the degree of the splitting field K of f over $\mathbb{Q}$?

(b) What is the Galois group $G = G(K/\mathbb{Q})$ of f?

(c) List the subfields L of K, and the corresponding subgroups $H = G(K/L)$ of G.

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