Final Examination

Dave Bayer, Modern Algebra, May 13, 1998

[1] Which of the following rings are integral domains? Explain your reasoning.
(a) $\mathbb{F}_2[x]/(x^4 + x^2 + 1)$
(b) $\mathbb{Q}[x]/(x^4 + 2x^2 + 2)$

[2] Consider the ideal $I = (2, 3x^2) \subset \mathbb{Z}[x]$, and let $R = \mathbb{Z}[x]/I$.
(a) Does $f(x) = 7x^4 + 5x^3 + 3x^2 + 2x$ belong to I?
(b) How many elements are there in R?
(c) List representatives for the elements of R, and describe the multiplication rule in R for these representatives.

[3] Let R be a principal ideal domain, and let

$$I_1 \subset I_2 \subset I_3 \subset \cdots \subset I_n \subset \cdots$$

be an infinite ascending chain of ideals in R. Show that this chain *stabilizes*, i.e.

$$I_N = I_{N+1} = I_{N+2} = \cdots$$

for some N.

[4] Working in the Gaussian integers $\mathbb{Z}[i]$, factor 2 into primes.

[5]
(a) Give a presentation of the finite field with 27 elements $\mathbb{F}_{27}$, of the form $\mathbb{F}_3[x]/(f(x))$.
(b) In terms of this presentation, guess a generator α of the multiplicative group $\mathbb{F}_{27}^*$. Compute the odds that such a guess is correct. Explain your reasoning.

[6] Prove that $\alpha = e^{2\pi i/11} + 3$ is not constructible.

[7] What is the minimal polynomial of $\alpha = \sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$?

[8] Let $f(x, y, z) = x^3 + y^3 + z^3$. Express $f(x)$ as a polynomial $g(\sigma_1, \sigma_2, \sigma_3)$ where σ_1, σ_2, σ_3 are the elementary symmetric functions

$$\sigma_1 = x + y + z, \quad \sigma_2 = xy + xz + yz, \quad \sigma_3 = xyz.$$

Recall that the discriminant of $f(x) = x^2 + bx + c$ is $D = b^2 - 4c$, and that the discriminant of $f(x) = x^3 + px + q$ is $D = -4p^3 - 27q^2$.

[**9**] Let $f(x) = x^3 - 12$.
(a) What is the degree of the splitting field K of f over $\mathbb{Q}$?
(b) What is the Galois group $G = G(K/\mathbb{Q})$ of f?
(c) List the subfields L of K, and the corresponding subgroups $H = G(K/L)$ of G.

[**10**] Prove the primitive element theorem (14.4.1, p. 552): Let K be a finite extension of a field F of characteristic zero. There is an element $\gamma \in K$ such that $K = F(\gamma)$.